Asymptotic Stabilization for Feedforward Systems with Delayed Feedbacks

Michael Malisoff, Associate Professor of Mathematics
Holder of Roy P. Daniels Professorship #3

JOINT WITH FREDERIC MAZENC, CR1, INRIA DISCO, L2S CNRS–SUPÉLEC

Summary of Forthcoming Paper in Automatica

LSU Applied Analysis Seminar
November 19, 2012

What Do We Mean By Delayed Control Systems?

What Do We Mean By Delayed Control Systems?

These are *doubly* parameterized families of ODEs of the form

$$Y'(t) = \mathcal{F}\big(t, Y(t), u(t, Y(t-\tau)), \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (1)$$

What Do We Mean By Delayed Control Systems ?

These are *doubly* parameterized families of ODEs of the form

$$Y'(t) = \mathcal{F}\big(t, Y(t), u(t, Y(t-\tau)), \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (1)$$

$\mathcal{Y} \subseteq \mathbb{R}^n$.

What Do We Mean By Delayed Control Systems?

These are *doubly* parameterized families of ODEs of the form

$$Y'(t) = \mathcal{F}\big(t, Y(t), u(t, Y(t-\tau)), \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (1)$$

$\mathcal{Y} \subseteq \mathbb{R}^n$. We have freedom to choose the control function u.

What Do We Mean By Delayed Control Systems ?

These are *doubly* parameterized families of ODEs of the form

$$Y'(t) = \mathcal{F}\big(t, Y(t), u(t, Y(t-\tau)), \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (1)$$

$\mathcal{Y} \subseteq \mathbb{R}^n$. We have freedom to choose the control function u.
The functions $\delta : [0, \infty) \to \mathcal{D}$ represent uncertainty.

What Do We Mean By Delayed Control Systems?

These are *doubly* parameterized families of ODEs of the form

$$Y'(t) = \mathcal{F}\big(t, Y(t), u(t, Y(t-\tau)), \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (1)$$

$\mathcal{Y} \subseteq \mathbb{R}^n$. We have freedom to choose the control function u.
The functions $\delta : [0, \infty) \to \mathcal{D}$ represent uncertainty. $\mathcal{D} \subseteq \mathbb{R}^m$.

What Do We Mean By Delayed Control Systems ?

These are *doubly* parameterized families of ODEs of the form

$$Y'(t) = \mathcal{F}\big(t, Y(t), u(t, Y(t-\tau)), \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (1)$$

$\mathcal{Y} \subseteq \mathbb{R}^n$. We have freedom to choose the control function u.
The functions $\delta : [0, \infty) \to \mathcal{D}$ represent uncertainty. $\mathcal{D} \subseteq \mathbb{R}^m$.

$Y_t(\theta) = Y(t + \theta)$.

What Do We Mean By Delayed Control Systems?

These are *doubly* parameterized families of ODEs of the form

$$Y'(t) = \mathcal{F}\big(t, Y(t), u(t, Y(t-\tau)), \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (1)$$

$\mathcal{Y} \subseteq \mathbb{R}^n$. We have freedom to choose the control function u.
The functions $\delta : [0, \infty) \to \mathcal{D}$ represent uncertainty. $\mathcal{D} \subseteq \mathbb{R}^m$.

$Y_t(\theta) = Y(t + \theta)$. Specify u to get a singly parameterized family

$$Y'(t) = \mathcal{G}(t, Y_t, \delta(t)), \quad Y(t) \in \mathcal{Y}, \qquad (2)$$

where $\mathcal{G}(t, Y_t, d) = \mathcal{F}(t, Y(t), u(t, Y(t-\tau)), d)$.

What Do We Mean By Delayed Control Systems ?

These are *doubly* parameterized families of ODEs of the form

$$Y'(t) = \mathcal{F}\big(t, Y(t), u(t, Y(t-\tau)), \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (1)$$

$\mathcal{Y} \subseteq \mathbb{R}^n$. We have freedom to choose the control function u.
The functions $\delta : [0, \infty) \to \mathcal{D}$ represent uncertainty. $\mathcal{D} \subseteq \mathbb{R}^m$.

$Y_t(\theta) = Y(t+\theta)$. Specify u to get a singly parameterized family

$$Y'(t) = \mathcal{G}(t, Y_t, \delta(t)), \quad Y(t) \in \mathcal{Y}, \qquad (2)$$

where $\mathcal{G}(t, Y_t, d) = \mathcal{F}(t, Y(t), u(t, Y(t-\tau)), d)$.

Typically we construct u such that all trajectories of (2) for all possible choices of δ satisfy some control objective.

Motivation for Delayed System

Motivation for Delayed System

Engineering applications:

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)
- population dynamics (e.g., delayed chemostats)

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)
- population dynamics (e.g., delayed chemostats)

Causes:

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)
- population dynamics (e.g., delayed chemostats)

Causes:

- transport phenomena

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)
- population dynamics (e.g., delayed chemostats)

Causes:

- transport phenomena
- time consuming information gathering

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)
- population dynamics (e.g., delayed chemostats)

Causes:

- transport phenomena
- time consuming information gathering
- sensor designs

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)
- population dynamics (e.g., delayed chemostats)

Causes:

- transport phenomena
- time consuming information gathering
- sensor designs

Literature:

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)
- population dynamics (e.g., delayed chemostats)

Causes:

- transport phenomena
- time consuming information gathering
- sensor designs

Literature:

- Krstic

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)
- population dynamics (e.g., delayed chemostats)

Causes:

- transport phenomena
- time consuming information gathering
- sensor designs

Literature:

- Krstic
- Jankovic

Motivation for Delayed System

Engineering applications:

- aerospace models (e.g., UAVs)
- networked control (e.g., communication systems)
- population dynamics (e.g., delayed chemostats)

Causes:

- transport phenomena
- time consuming information gathering
- sensor designs

Literature:

- Krstic
- Jankovic
- Karafyllis..

What is One Possible Control Objective?

What is One Possible Control Objective?

Input-to-state stability generalizes global asymptotic stability.

What is One Possible Control Objective?

Input-to-state stability generalizes global asymptotic stability.

$$Y'(t) = \mathcal{G}(t, Y_t), \quad Y(t) \in \mathcal{Y}. \qquad (\Sigma)$$

What is One Possible Control Objective?

Input-to-state stability generalizes global asymptotic stability.

$$Y'(t) = \mathcal{G}(t, Y_t), \quad Y(t) \in \mathcal{Y}. \tag{Σ}$$

$$|Y(t)| \le \gamma_1 \left(e^{t_0 - t} \gamma_2(|Y_{t_0}|_{[-\tau,0]}) \right) \tag{UGAS}$$

What is One Possible Control Objective?

Input-to-state stability generalizes global asymptotic stability.

$$Y'(t) = \mathcal{G}(t, Y_t), \quad Y(t) \in \mathcal{Y}. \qquad (\Sigma)$$

$$|Y(t)| \leq \gamma_1 \left(e^{t_0 - t}\gamma_2(|Y_{t_0}|_{[-\tau,0]})\right) \qquad \text{(UGAS)}$$

Our γ_i's are 0 at 0, strictly increasing, and unbounded.

What is One Possible Control Objective?

Input-to-state stability generalizes global asymptotic stability.

$$Y'(t) = \mathcal{G}(t, Y_t), \quad Y(t) \in \mathcal{Y}. \qquad (\Sigma)$$

$$|Y(t)| \leq \gamma_1 \left(e^{t_0 - t}\gamma_2(|Y_{t_0}|_{[-\tau,0]})\right) \qquad \text{(UGAS)}$$

Our γ_i's are 0 at 0, strictly increasing, and unbounded. $\gamma_i \in \mathcal{K}_\infty$.

What is One Possible Control Objective?

Input-to-state stability generalizes global asymptotic stability.

$$Y'(t) = \mathcal{G}(t, Y_t), \quad Y(t) \in \mathcal{Y}. \qquad (\Sigma)$$

$$|Y(t)| \leq \gamma_1 \left(e^{t_0 - t} \gamma_2(|Y_{t_0}|_{[-\tau,0]})\right) \qquad \text{(UGAS)}$$

Our γ_i's are 0 at 0, strictly increasing, and unbounded. $\gamma_i \in \mathcal{K}_\infty$.

$$Y'(t) = \mathcal{G}\big(t, Y_t, \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (\Sigma_{\rm pert})$$

What is One Possible Control Objective?

Input-to-state stability generalizes global asymptotic stability.

$$Y'(t) = \mathcal{G}(t, Y_t), \quad Y(t) \in \mathcal{Y}. \qquad (\Sigma)$$

$$|Y(t)| \le \gamma_1 \left(e^{t_0 - t}\gamma_2(|Y_{t_0}|_{[-\tau,0]})\right) \qquad \text{(UGAS)}$$

Our γ_i's are 0 at 0, strictly increasing, and unbounded. $\gamma_i \in \mathcal{K}_\infty$.

$$Y'(t) = \mathcal{G}\big(t, Y_t, \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (\Sigma_{\rm pert})$$

$$|Y(t)| \le \gamma_1 \left(e^{t_0 - t}\gamma_2(|Y_{t_0}|_{[-\tau,0]})\right) + \gamma_2(|\delta|_{[t_0,t]}) \qquad \text{(ISS)}$$

What is One Possible Control Objective?

Input-to-state stability generalizes global asymptotic stability.

$$Y'(t) = \mathcal{G}(t, Y_t), \quad Y(t) \in \mathcal{Y}. \qquad (\Sigma)$$

$$|Y(t)| \le \gamma_1\left(e^{t_0 - t}\gamma_2(|Y_{t_0}|_{[-\tau,0]})\right) \qquad \text{(UGAS)}$$

Our γ_i's are 0 at 0, strictly increasing, and unbounded. $\gamma_i \in \mathcal{K}_\infty$.

$$Y'(t) = \mathcal{G}\big(t, Y_t, \delta(t)\big), \quad Y(t) \in \mathcal{Y}. \qquad (\Sigma_{\rm pert})$$

$$|Y(t)| \le \gamma_1\left(e^{t_0 - t}\gamma_2(|Y_{t_0}|_{[-\tau,0]})\right) + \gamma_2(|\delta|_{[t_0,t]}) \qquad \text{(ISS)}$$

Find γ_i's by building certain LKFs for $Y'(t) = \mathcal{G}(t, Y_t, 0)$.

What is a Lyapunov-Krasovskii Functional (LKF) ?

What is a Lyapunov-Krasovskii Functional (LKF)?

Definition: We call $V^\sharp$ an ISS-LKF for $Y'(t) = \mathcal{G}(t, Y_t, \delta(t))$ provided there exist functions $\gamma_i \in \mathcal{K}_\infty$ such that:

What is a Lyapunov-Krasovskii Functional (LKF) ?

Definition: We call $V^\sharp$ an ISS-LKF for $Y'(t) = \mathcal{G}(t, Y_t, \delta(t))$ provided there exist functions $\gamma_i \in \mathcal{K}_\infty$ such that:

1. $\gamma_1(|\phi(0)|) \leq V^\sharp(t, \phi) \leq \gamma_2(|\phi|_{[-\tau,0]})$ for all $(t, \phi) \in [0, +\infty) \times \mathcal{C}([-\tau, 0], \mathbb{R}^n)$

What is a Lyapunov-Krasovskii Functional (LKF) ?

Definition: We call $V^\sharp$ an ISS-LKF for $Y'(t) = \mathcal{G}(t, Y_t, \delta(t))$ provided there exist functions $\gamma_i \in \mathcal{K}_\infty$ such that:

1. $\gamma_1(|\phi(0)|) \leq V^\sharp(t,\phi) \leq \gamma_2(|\phi|_{[-\tau,0]})$ for all $(t,\phi) \in [0,+\infty) \times \mathcal{C}([-\tau,0],\mathbb{R}^n)$ and
2. $\frac{d}{dt}\left[V^\sharp(t, Y_t)\right] \leq -\gamma_3(V^\sharp(t, Y_t)) + \gamma_4(|\delta(t)|)$ along all trajectories of the system

What is a Lyapunov-Krasovskii Functional (LKF) ?

Definition: We call $V^\sharp$ an ISS-LKF for $Y'(t) = \mathcal{G}(t, Y_t, \delta(t))$ provided there exist functions $\gamma_i \in \mathcal{K}_\infty$ such that:

1. $\gamma_1(|\phi(0)|) \leq V^\sharp(t, \phi) \leq \gamma_2(|\phi|_{[-\tau,0]})$ for all $(t, \phi) \in [0, +\infty) \times \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ and
2. $\frac{d}{dt}\left[V^\sharp(t, Y_t)\right] \leq -\gamma_3(V^\sharp(t, Y_t)) + \gamma_4(|\delta(t)|)$ along all trajectories of the system

Example:

What is a Lyapunov-Krasovskii Functional (LKF) ?

Definition: We call $V^\sharp$ an ISS-LKF for $Y'(t) = \mathcal{G}(t, Y_t, \delta(t))$ provided there exist functions $\gamma_i \in \mathcal{K}_\infty$ such that:

1. $\gamma_1(|\phi(0)|) \leq V^\sharp(t, \phi) \leq \gamma_2(|\phi|_{[-\tau,0]})$ for all $(t, \phi) \in [0, +\infty) \times \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ and
2. $\frac{d}{dt}\left[V^\sharp(t, Y_t)\right] \leq -\gamma_3(V^\sharp(t, Y_t)) + \gamma_4(|\delta(t)|)$ along all trajectories of the system

Example: The function $V(Y) = \frac{1}{2}|Y|^2$ is an ISS-LKF for $Y'(t) = -Y(t) + \frac{1}{4}Y(t) + \delta(t)$ for any $\mathcal{D}$.

What is a Lyapunov-Krasovskii Functional (LKF)?

Definition: We call $V^\sharp$ an ISS-LKF for $Y'(t) = \mathcal{G}(t, Y_t, \delta(t))$ provided there exist functions $\gamma_i \in \mathcal{K}_\infty$ such that:

1. $\gamma_1(|\phi(0)|) \le V^\sharp(t, \phi) \le \gamma_2(|\phi|_{[-\tau,0]})$ for all $(t, \phi) \in [0, +\infty) \times \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ and
2. $\frac{d}{dt}\left[V^\sharp(t, Y_t)\right] \le -\gamma_3(V^\sharp(t, Y_t)) + \gamma_4(|\delta(t)|)$ along all trajectories of the system

Example: The function $V(Y) = \frac{1}{2}|Y|^2$ is an ISS-LKF for $Y'(t) = -Y(t) + \frac{1}{4}Y(t) + \delta(t)$ for any $\mathcal{D}$. Fix $\tau > 0$.

What is a Lyapunov-Krasovskii Functional (LKF)?

Definition: We call $V^\sharp$ an ISS-LKF for $Y'(t) = \mathcal{G}(t, Y_t, \delta(t))$ provided there exist functions $\gamma_i \in \mathcal{K}_\infty$ such that:

1. $\gamma_1(|\phi(0)|) \le V^\sharp(t,\phi) \le \gamma_2(|\phi|_{[-\tau,0]})$ for all $(t,\phi) \in [0,+\infty) \times \mathcal{C}([-\tau,0],\mathbb{R}^n)$ and
2. $\frac{d}{dt}\left[V^\sharp(t, Y_t)\right] \le -\gamma_3(V^\sharp(t, Y_t)) + \gamma_4(|\delta(t)|)$ along all trajectories of the system

Example: The function $V(Y) = \frac{1}{2}|Y|^2$ is an ISS-LKF for $Y'(t) = -Y(t) + \frac{1}{4}Y(t) + \delta(t)$ for any $\mathcal{D}$. Fix $\tau > 0$.

$$V^\sharp(Y_t) = V(Y(t)) + \frac{1}{4}\int_{t-\tau}^{t} |Y(\ell)|^2 \mathrm{d}\ell + \frac{1}{8\tau}\int_{t-\tau}^{t} \left[\int_s^t |Y(r)|^2 \mathrm{d}r\right] \mathrm{d}s$$

is an ISS-LKF for $Y'(t) = -Y(t) + \frac{1}{4}Y(t-\tau) + \delta(t)$.

Linear Feedforward Systems

Linear Feedforward Systems

Consider the set of all systems having the feedforward form

$$\left\{ \begin{array}{rcl} \dot{x} & = & h_1(z) + h_2(z)v(t-\tau) \\ \dot{z} & = & f(z) + g(z)v(t-\tau) \, . \end{array} \right. \tag{3}$$

Linear Feedforward Systems

Consider the set of all systems having the feedforward form

$$\left\{ \begin{array}{rcl} \dot{x} & = & h_1(z) + h_2(z)v(t-\tau) \\ \dot{z} & = & f(z) + g(z)v(t-\tau) \; . \end{array} \right. \tag{3}$$

The state space is $\mathbb{R}^n \times \mathbb{R}^p$.

Linear Feedforward Systems

Consider the set of all systems having the feedforward form

$$\left\{ \begin{array}{rcl} \dot{x} & = & h_1(z) + h_2(z)v(t-\tau) \\ \dot{z} & = & f(z) + g(z)v(t-\tau) \; . \end{array} \right. \tag{3}$$

The state space is $\mathbb{R}^n \times \mathbb{R}^p$. Linearizing (3) around period τ reference trajectories produces a system of the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & C(t)z(t) + D(t)u(t-\tau) \\ \dot{z}(t) & = & A(t)z(t) + B(t)u(t-\tau) \; , \end{array} \right. \tag{4}$$

where A, B, C, and D are C^1 matrix valued functions of period τ.

Linear Feedforward Systems

Consider the set of all systems having the feedforward form

$$\left\{ \begin{array}{rcl} \dot{x} & = & h_1(z) + h_2(z)v(t-\tau) \\ \dot{z} & = & f(z) + g(z)v(t-\tau) \, . \end{array} \right. \quad (3)$$

The state space is $\mathbb{R}^n \times \mathbb{R}^p$. Linearizing (3) around period τ reference trajectories produces a system of the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & C(t)z(t) + D(t)u(t-\tau) \\ \dot{z}(t) & = & A(t)z(t) + B(t)u(t-\tau) \, , \end{array} \right. \quad (4)$$

where A, B, C, and D are C^1 matrix valued functions of period τ.

We focus on (4), and cases where uncertainties δ are added to u.

Preliminary Technical Results

Assumption 1. *The system*

$$\dot{\theta}(t) = A(t)\theta(t) \tag{5}$$

is UGAS. The matrices A, B, C, and D are C^1 and have period τ.

Preliminary Technical Results

Assumption 1. *The system*

$$\dot{\theta}(t) = A(t)\theta(t) \tag{5}$$

is UGAS. The matrices A, B, C, and D are C^1 and have period τ.

Hence, (5) admits a Lyapunov function $V(t, \theta) = \theta^\top P(t)\theta$ such that $\dot{V} \leq -|\theta|^2$ along all trajectories of (5) and P has period τ.

Preliminary Technical Results

Assumption 1. *The system*

$$\dot{\theta}(t) = A(t)\theta(t) \tag{5}$$

is UGAS. The matrices A, B, C, and D are C^1 and have period τ.

Hence, (5) admits a Lyapunov function $V(t, \theta) = \theta^\top P(t)\theta$ such that $\dot{V} \leq -|\theta|^2$ along all trajectories of (5) and P has period τ.

Let ψ_a be the inverse of the fundamental matrix for (5).

Preliminary Technical Results

Assumption 1. *The system*

$$\dot{\theta}(t) = A(t)\theta(t) \tag{5}$$

is UGAS. The matrices A, B, C, and D are C^1 and have period τ.

Hence, (5) admits a Lyapunov function $V(t,\theta) = \theta^\top P(t)\theta$ such that $\dot{V} \leq -|\theta|^2$ along all trajectories of (5) and P has period τ.

Let ψ_a be the inverse of the fundamental matrix for (5).

$$\left\{ \begin{array}{rcl} \frac{\partial \psi_a}{\partial t}(t,m) & = & -\psi_a(t,m)A(t) \\ \psi_a(m,m) & = & \mathrm{I} \end{array} \right. \tag{6}$$

for all $t \in \mathbb{R}$ and $m \in \mathbb{R}$.

Preliminary Technical Results

Lemma

Let Assumption 1 hold. Then the function $\mathrm{I} - \psi_a(\ell, \ell - \tau)$ *is invertible for all* $\ell \in \mathbb{R}$. *Also, the function* $q : \mathbb{R} \to \mathbb{R}^{n \times p}$ *defined by*

$$q(t) = -\int_{t-\tau}^{t} C(\ell)[\mathrm{I} - \psi_a(\ell, \ell - \tau)]^{-1}\psi_a(t, \ell)\mathrm{d}\ell \qquad (7)$$

has period τ, *and* $\dot{q}(t) + q(t)A(t) + C(t) = 0$ *for all* $t \in \mathbb{R}$. □

Preliminary Technical Results

Lemma

Let Assumption 1 hold. Then the function $\mathrm{I} - \psi_a(\ell, \ell - \tau)$ *is invertible for all* $\ell \in \mathbb{R}$*. Also, the function* $q : \mathbb{R} \to \mathbb{R}^{n \times p}$ *defined by*

$$q(t) = -\int_{t-\tau}^{t} C(\ell)[\mathrm{I} - \psi_a(\ell, \ell - \tau)]^{-1} \psi_a(t, \ell) \mathrm{d}\ell \qquad (7)$$

has period τ*, and* $\dot{q}(t) + q(t)A(t) + C(t) = 0$ *for all* $t \in \mathbb{R}$*.* □

Assumption 2. *There exists a constant* $c > 0$ *such that the matrix* $R(t) = q(t)B(t) + D(t)$ *satisfies*

$$\int_{t-\tau}^{t} R(m)R(m)^{\top} \mathrm{d}m \ \geq \ c\mathrm{I} \qquad (8)$$

for all $t \in \mathbb{R}$*. (That means* I *is the* $n \times n$ *identity matrix.)*

Main Result

Our coordinate change $\xi(t) = x(t) + q(t)z(t)$ gave the system

$$\begin{cases} \dot{\xi}(t) &= R(t)u(t-\tau) \\ \dot{z}(t) &= A(t)z(t) + B(t)u(t-\tau) \end{cases} \tag{9}$$

where $R(t) = q(t)B(t) + D(t)$ and q is from the lemma.

Main Result

Our coordinate change $\xi(t) = x(t) + q(t)z(t)$ gave the system

$$\begin{cases} \dot{\xi}(t) &= R(t)u(t-\tau) \\ \dot{z}(t) &= A(t)z(t) + B(t)u(t-\tau) \end{cases} \tag{9}$$

where $R(t) = q(t)B(t) + D(t)$ and q is from the lemma.

Theorem

Let Assumptions 1 and 2 hold. Then for all constants $\tau > 0$ and $\epsilon \in (0, \frac{1}{1+4\tau||R||^2})$, the controller

$$u(t-\tau) = -\epsilon \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \tag{10}$$

renders (9) UGAS.

Proof of Theorem

Proof of Theorem

Show that the closed loop system (9) admits the LKF

$$V^{\sharp}(t,\xi_t,z(t)) \;=\; z^{\top}(t)P(t)z(t)+21\beta_1 W_3(t,\xi_t), \;\; \text{where}$$

$$W_3(t,\xi_t)=W_2(t,\xi_t)+k\left[(1+2U(\xi_t))^{3/2}-1\right],$$

$$W_2(t,\xi_t)=W_1(t,\xi_t)+\beta_0\int_{t-\tau}^{t}\left|\frac{R(m)^{\top}\xi(m)}{\sqrt{1+|\xi(m)|^2}}\right|^2\mathrm{d}m,$$

$$W_1(t,\xi_t)=\xi(t)^{\top}\left[\int_{t-\tau}^{t}\int_{m}^{t}R(\ell)R(\ell)^{\top}\mathrm{d}\ell\mathrm{d}m\right]\xi(t),$$

$$U(\xi_t)=\frac{1}{2}|\xi|^2+\frac{1}{4\tau}\int_{t-2\tau}^{t}\int_{m}^{t}\frac{\epsilon|R(\ell)^{\top}\xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}}\,\mathrm{d}\ell\mathrm{d}m,$$

$$\beta_0=\frac{1}{2c}||R||^6\tau^4\epsilon^2, \quad k=\frac{4\sqrt{2}}{3\epsilon}(\tau+\beta_0),$$

$$\beta_1=\max\{v_1,v_2\}, \quad v_1=\frac{2}{c}[4||P||^2\,||B||^2\,||R||^2+1],$$

$$\text{and} \;\; v_2=\frac{16\sqrt{2}\tau}{3\epsilon k}(1+8\tau||P||^2\,||B||^2\,||R||^4).$$

ISS Result

ISS Result

Allowing additive uncertainties on the control gives

$$\begin{cases} \dot{\xi}(t) & = & R(t)\big[u(t-\tau)+\delta(t)\big] \\ \dot{z}(t) & = & A(t)z(t)+B(t)\big[u(t-\tau)+\delta(t)\big]\ . \end{cases} \tag{11}$$

ISS Result

Allowing additive uncertainties on the control gives

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & R(t)\left[u(t-\tau)+\delta(t)\right] \\ \dot{z}(t) & = & A(t)z(t)+B(t)\left[u(t-\tau)+\delta(t)\right]. \end{array} \right. \tag{11}$$

$$\overline{\delta} = \frac{c}{9k||R||(1+2\overline{u})^{1/2}}, \quad \text{where} \quad k = \frac{4\sqrt{2}}{3\epsilon}\left(\tau + \frac{1}{2c}||R||^6\tau^4\epsilon^2\right)$$
$$\text{and} \quad \overline{u} = \max\left\{\frac{1}{2} + \frac{\epsilon||R||^2\tau}{4\sqrt{2}}, \frac{\epsilon||R||^2\tau}{4\sqrt{2}}\left(1 + 2\epsilon||R||^2\tau\right)\right\}. \tag{12}$$

ISS Result

Allowing additive uncertainties on the control gives

$$\left\{\begin{array}{rcl}\dot{\xi}(t) & = & R(t)\big[u(t-\tau)+\delta(t)\big] \\ \dot{z}(t) & = & A(t)z(t)+B(t)\big[u(t-\tau)+\delta(t)\big]\,.\end{array}\right. \tag{11}$$

$$\overline{\delta}=\frac{c}{9k||R||(1+2\overline{u})^{1/2}},\ \ \text{where}\ \ k=\frac{4\sqrt{2}}{3\epsilon}\left(\tau+\frac{1}{2c}||R||^6\tau^4\epsilon^2\right)$$
$$\text{and}\quad \overline{u}=\max\left\{\frac{1}{2}+\frac{\epsilon||R||^2\tau}{4\sqrt{2}},\frac{\epsilon||R||^2\tau}{4\sqrt{2}}\left(1+2\epsilon||R||^2\tau\right)\right\}\ . \tag{12}$$

Theorem

Under the preceding assumptions, (11) in closed loop with

$$u(t-\tau)=-\epsilon\frac{R(t-\tau)^\top\xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \tag{13}$$

is ISS with respect to the set of all disturbances δ bounded by $\overline{\delta}$.

Application to UAV Dynamics

We study the UAV with standard autopilots which is first order for heading and Mach hold and second order for the altitude hold.

Application to UAV Dynamics

We study the UAV with standard autopilots which is first order for heading and Mach hold and second order for the altitude hold.

$$\begin{cases} \dot{x} &= v\cos(\theta) \\ \dot{y} &= v\sin(\theta) \\ \dot{\theta} &= \alpha_\theta(\theta_c(t-\tau)-\theta) \\ \dot{v} &= \alpha_v(v_c(t-\tau)-v), \end{cases} \tag{14}$$

where we omit the altitude subdynamics $\ddot{h} = -\alpha_h \dot{h} + \alpha_h(h^c - h)$.

Application to UAV Dynamics

We study the UAV with standard autopilots which is first order for heading and Mach hold and second order for the altitude hold.

$$\begin{cases} \dot{x} &= v\cos(\theta) \\ \dot{y} &= v\sin(\theta) \\ \dot{\theta} &= \alpha_\theta(\theta_c(t-\tau)-\theta) \\ \dot{v} &= \alpha_v(v_c(t-\tau)-v), \end{cases} \tag{14}$$

where we omit the altitude subdynamics $\ddot{h} = -\alpha_h\dot{h} + \alpha_h(h^c - h)$.

Key Model : Underactuated kino-dynamic representation that is justifiable for high-level formation flight control.

Application to UAV Dynamics

We study the UAV with standard autopilots which is first order for heading and Mach hold and second order for the altitude hold.

$$\begin{cases} \dot{x} &= v\cos(\theta) \\ \dot{y} &= v\sin(\theta) \\ \dot{\theta} &= \alpha_\theta(\theta_c(t-\tau)-\theta) \\ \dot{v} &= \alpha_v(v_c(t-\tau)-v), \end{cases} \tag{14}$$

where we omit the altitude subdynamics $\ddot{h} = -\alpha_h\dot{h} + \alpha_h(h^c - h)$.

Key Model : Underactuated kino-dynamic representation that is justifiable for high-level formation flight control.

See e.g. 2004 IEEE-TCST paper by Ren and Beard.

Application to UAV Dynamics

We are given a C^1 reference trajectory $(x_r, y_r, \theta_r, v_r) : \mathbb{R} \to \mathbb{R}^4$, so there is a reference input $(\theta_{cr}, v_{cr}) : \mathbb{R} \to \mathbb{R}^2$ such that

$$\left\{ \begin{array}{rcl} \dot{x}_r(t) & = & v_r(t)\cos(\theta_r(t)) \\ \dot{y}_r(t) & = & v_r(t)\sin(\theta_r(t)) \\ \dot{\theta}_r(t) & = & \alpha_\theta(\theta_{cr}(t) - \theta_r(t)) \\ \dot{v}_r(t) & = & \alpha_v(v_{cr}(t) - v_r(t)) \end{array} \right. \tag{15}$$

holds for all $t \in \mathbb{R}$.

Application to UAV Dynamics

We are given a C^1 reference trajectory $(x_r, y_r, \theta_r, v_r) : \mathbb{R} \to \mathbb{R}^4$, so there is a reference input $(\theta_{cr}, v_{cr}) : \mathbb{R} \to \mathbb{R}^2$ such that

$$\left\{ \begin{array}{rcl} \dot{x}_r(t) & = & v_r(t)\cos(\theta_r(t)) \\ \dot{y}_r(t) & = & v_r(t)\sin(\theta_r(t)) \\ \dot{\theta}_r(t) & = & \alpha_\theta(\theta_{cr}(t) - \theta_r(t)) \\ \dot{v}_r(t) & = & \alpha_v(v_{cr}(t) - v_r(t)) \end{array} \right. \tag{15}$$

holds for all $t \in \mathbb{R}$.

Assumption 3 : The functions $\cos(\theta_r(t))$ and $\sin(\theta_r(t))$ have period τ, there exists a constant $t_c \in [0, \tau]$ such that $\dot{\theta}_r(t_c) \neq 0$, and v_r is bounded.

Application to UAV Dynamics

We are given a C^1 reference trajectory $(x_r, y_r, \theta_r, v_r) : \mathbb{R} \to \mathbb{R}^4$, so there is a reference input $(\theta_{cr}, v_{cr}) : \mathbb{R} \to \mathbb{R}^2$ such that

$$\left\{ \begin{array}{rcl} \dot{x}_r(t) & = & v_r(t)\cos(\theta_r(t)) \\ \dot{y}_r(t) & = & v_r(t)\sin(\theta_r(t)) \\ \dot{\theta}_r(t) & = & \alpha_\theta(\theta_{cr}(t) - \theta_r(t)) \\ \dot{v}_r(t) & = & \alpha_v(v_{cr}(t) - v_r(t)) \end{array} \right. \tag{15}$$

holds for all $t \in \mathbb{R}$.

Assumption 3 : The functions $\cos(\theta_r(t))$ and $\sin(\theta_r(t))$ have period τ, there exists a constant $t_c \in [0, \tau]$ such that $\dot{\theta}_r(t_c) \neq 0$, and v_r is bounded.

Tracking Error : $(\bar{x}, \bar{y}, \bar{\theta}, \bar{v}) = (x - x_r, y - y_r, \theta - \theta_r, v - v_r)$.

Application to UAV Dynamics

After a preliminary change of feedbacks, the tracking dynamics are

$$\begin{cases} \dot{\bar{x}} &= \cos(\theta_r(t))\bar{v} \\ & \quad + [\bar{v} + v_r(t)][\cos(\bar{\theta} + \theta_r(t)) - \cos(\theta_r(t))] \\ \dot{\bar{y}} &= \sin(\theta_r(t))\bar{v} \\ & \quad + [\bar{v} + v_r(t)][\sin(\bar{\theta} + \theta_r(t)) - \sin(\theta_r(t))] \\ \dot{\bar{v}} &= -\alpha_v \bar{v} + u(t-\tau) \\ \dot{\bar{\theta}} &= -\alpha_\theta \bar{\theta} \, . \end{cases} \tag{16}$$

Application to UAV Dynamics

After a preliminary change of feedbacks, the tracking dynamics are

$$\begin{cases} \dot{\overline{x}} &= \cos(\theta_r(t))\overline{v} \\ & \quad + [\overline{v} + v_r(t)][\cos(\overline{\theta} + \theta_r(t)) - \cos(\theta_r(t))] \\ \dot{\overline{y}} &= \sin(\theta_r(t))\overline{v} \\ & \quad + [\overline{v} + v_r(t)][\sin(\overline{\theta} + \theta_r(t)) - \sin(\theta_r(t))] \\ \dot{\overline{v}} &= -\alpha_v \overline{v} + u(t-\tau) \\ \dot{\overline{\theta}} &= -\alpha_\theta \overline{\theta}\ . \end{cases} \tag{16}$$

We apply our theory to the $(\bar{x}, \bar{y}, \bar{v})$ dynamics obtained by setting $\overline{\theta} = 0$, and then we reincorporate the $\overline{\theta}$ dynamics to get θ_c and v_c.

Application to UAV Dynamics

After a preliminary change of feedbacks, the tracking dynamics are

$$\begin{cases} \dot{\overline{x}} &= \cos(\theta_r(t))\overline{v} \\ & \quad + [\overline{v} + v_r(t)][\cos(\overline{\theta} + \theta_r(t)) - \cos(\theta_r(t))] \\ \dot{\overline{y}} &= \sin(\theta_r(t))\overline{v} \\ & \quad + [\overline{v} + v_r(t)][\sin(\overline{\theta} + \theta_r(t)) - \sin(\theta_r(t))] \\ \dot{\overline{v}} &= -\alpha_v \overline{v} + u(t-\tau) \\ \dot{\overline{\theta}} &= -\alpha_\theta \overline{\theta} \, . \end{cases} \tag{16}$$

We apply our theory to the $(\bar{x}, \bar{y}, \bar{v})$ dynamics obtained by setting $\bar{\theta} = 0$, and then we reincorporate the $\bar{\theta}$ dynamics to get θ_c and v_c.

$$\begin{aligned} \theta_c(t-\tau) &= \theta_{cr}(t) \text{ and} \\ v_c(t-\tau) &= v_{cr}(t) - \tfrac{\epsilon}{\alpha_v} \tfrac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} . \end{aligned} \tag{17}$$

Conclusions

Conclusions

- Input delays naturally occur in many engineering applications and preclude the use of standard control designs.

Conclusions

- Input delays naturally occur in many engineering applications and preclude the use of standard control designs.
- Our controllers provide UGAS under arbitrarily long input delays and ISS via a LKF.

Conclusions

- Input delays naturally occur in many engineering applications and preclude the use of standard control designs.
- Our controllers provide UGAS under arbitrarily long input delays and ISS via a LKF.
- Using a tuning parameter ϵ, we can satisfy arbitrarily small input magnitude constraints.

Conclusions

- Input delays naturally occur in many engineering applications and preclude the use of standard control designs.
- Our controllers provide UGAS under arbitrarily long input delays and ISS via a LKF.
- Using a tuning parameter ϵ, we can satisfy arbitrarily small input magnitude constraints.
- Our work applies to a broad class of feedforward linear systems including a key model for UAVs.

Conclusions

- Input delays naturally occur in many engineering applications and preclude the use of standard control designs.
- Our controllers provide UGAS under arbitrarily long input delays and ISS via a LKF.
- Using a tuning parameter ϵ, we can satisfy arbitrarily small input magnitude constraints.
- Our work applies to a broad class of feedforward linear systems including a key model for UAVs.
- It would be interesting to extend the analysis to allow nonlinear analogs and systems governed by PDEs.

Main Result

Our coordinate change $\xi(t) = x(t) + q(t)z(t)$ gave the system

$$\begin{cases} \dot{\xi}(t) = R(t)u(t-\tau) \\ \dot{z}(t) = A(t)z(t) + B(t)u(t-\tau) \end{cases} \quad (18)$$

where $R(t) = q(t)B(t) + D(t)$ and q is from the lemma.

Theorem

Let Assumptions 1 and 2 hold. Then for all constants $\tau > 0$ and $\epsilon \in (0, \frac{1}{1+4\tau||R||^2})$, the controller

$$u(t-\tau) = -\epsilon \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \quad (19)$$

renders (18) UGAS.

Proof of Theorem

Show that the closed loop system (18) admits the LKF

$$V^\sharp(t,\xi_t,z(t)) \; = \; z^\top(t)P(t)z(t) + 21\beta_1 W_3(t,\xi_t), \;\; \text{where}$$

$$W_3(t,\xi_t) = W_2(t,\xi_t) + k\left[(1+2U(\xi_t))^{3/2} - 1\right],$$

$$W_2(t,\xi_t) = W_1(t,\xi_t) + \beta_0 \int_{t-\tau}^{t} \left| \frac{R(m)^\top \xi(m)}{\sqrt{1+|\xi(m)|^2}} \right|^2 \mathrm{d}m,$$

$$W_1(t,\xi_t) = \xi(t)^\top \left[\int_{t-\tau}^{t} \int_m^t R(\ell)R(\ell)^\top \mathrm{d}\ell \mathrm{d}m \right] \xi(t),$$

$$U(\xi_t) = \tfrac{1}{2}|\xi|^2 + \tfrac{1}{4\tau} \int_{t-2\tau}^{t} \int_m^t \frac{\epsilon |R(\ell)^\top \xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}} \, \mathrm{d}\ell \mathrm{d}m,$$

$$\beta_0 = \tfrac{1}{2c} ||R||^6 \tau^4 \epsilon^2, \quad k = \tfrac{4\sqrt{2}}{3\epsilon}(\tau + \beta_0),$$

$$\beta_1 = \max\{v_1, v_2\}, \quad v_1 = \tfrac{2}{c}[4||P||^2 \, ||B||^2 \, ||R||^2 + 1],$$

$$\text{and} \;\; v_2 = \tfrac{16\sqrt{2}\tau}{3\epsilon k}(1 + 8\tau ||P||^2 \, ||B||^2 \, ||R||^4).$$

Key Ideas of Proof

The ξ-subsystem $\dot{\xi}(t) = R(t)u(t-\tau)$ is

$$\dot{\xi}(t) = -\epsilon \frac{R(t)R(t)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}}, \quad (20)$$

since R has period τ.

Key Ideas of Proof

The ξ-subsystem $\dot{\xi}(t) = R(t)u(t-\tau)$ is

$$\dot{\xi}(t) \quad = \quad -\epsilon \frac{R(t)R(t)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \,, \tag{20}$$

since R has period τ. For all $t \geq \tau$, we have

$$\xi(t-\tau) = \xi(t) + \epsilon \int_{t-\tau}^{t} \frac{R(m)R(m)^\top \xi(m-\tau)}{\sqrt{1+|\xi(m-\tau)|^2}} \mathrm{d}m. \tag{21}$$

Key Ideas of Proof

The ξ-subsystem $\dot{\xi}(t) = R(t)u(t-\tau)$ is

$$\dot{\xi}(t) = -\epsilon \frac{R(t)R(t)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}}, \tag{20}$$

since R has period τ. For all $t \geq \tau$, we have

$$\xi(t-\tau) = \xi(t) + \epsilon \int_{t-\tau}^{t} \frac{R(m)R(m)^\top \xi(m-\tau)}{\sqrt{1+|\xi(m-\tau)|^2}} \mathrm{d}m. \tag{21}$$

Hence, for all $t \geq \tau$, we have

$$\begin{aligned} \dot{\xi}(t) &= -\epsilon \frac{R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}} \xi(t) \\ &\quad -\epsilon^2 \frac{R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}} \int_{t-\tau}^{t} \frac{R(m)R(m)^\top \xi(m-\tau)}{\sqrt{1+|\xi(m-\tau)|^2}} \mathrm{d}m. \end{aligned} \tag{22}$$

Key Ideas of Proof

If the orange term were not present, the ξ subsystem would be

$$\dot{\xi}(t) \quad = \quad -\epsilon \frac{R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}}\xi(t). \tag{23}$$

Key Ideas of Proof

If the orange term were not present, the ξ subsystem would be

$$\dot{\xi}(t) = -\epsilon \frac{R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}}\xi(t). \tag{23}$$

That admits the strict Lyapunov function

$$U(\xi_t) = \frac{1}{2}|\xi(t)|^2 + \frac{1}{4\tau}\int_{t-2\tau}^{t}\int_{m}^{t} \frac{\epsilon|R(\ell)^\top\xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}}\,\mathrm{d}\ell\mathrm{d}m.$$

Key Ideas of Proof

If the orange term were not present, the ξ subsystem would be

$$\dot{\xi}(t) = -\epsilon \frac{R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}}\xi(t). \tag{23}$$

That admits the strict Lyapunov function

$$U(\xi_t) = \frac{1}{2}|\xi(t)|^2 + \frac{1}{4\tau}\int_{t-2\tau}^{t}\int_{m}^{t} \frac{\epsilon|R(\ell)^\top \xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}}\,\mathrm{d}\ell\mathrm{d}m.$$

Then we must take the orange term for $\dot{\xi}$ into account.

Key Ideas of Proof

If the orange term were not present, the ξ subsystem would be

$$\dot{\xi}(t) = -\epsilon \frac{R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}} \xi(t). \tag{23}$$

That admits the strict Lyapunov function

$$U(\xi_t) = \frac{1}{2}|\xi(t)|^2 + \frac{1}{4\tau}\int_{t-2\tau}^{t}\int_{m}^{t} \frac{\epsilon|R(\ell)^\top \xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}} \,\mathrm{d}\ell \mathrm{d}m.$$

Then we must take the orange term for $\dot{\xi}$ into account.

When the asymptotic stability of the ξ-subsystem is established, we can (more) easily prove the UGAS result with the additional component $\dot{z}(t) = A(t)z(t) + B(t)u(t-\tau)$ from the dynamics.

Key Ideas of Proof

If the orange term were not present, the ξ subsystem would be

$$\dot{\xi}(t) = -\epsilon \frac{R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}}\xi(t). \tag{23}$$

That admits the strict Lyapunov function

$$U(\xi_t) = \frac{1}{2}|\xi(t)|^2 + \frac{1}{4\tau}\int_{t-2\tau}^{t}\int_{m}^{t} \frac{\epsilon|R(\ell)^\top \xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}}\,\mathrm{d}\ell\mathrm{d}m.$$

Then we must take the orange term for $\dot{\xi}$ into account.

When the asymptotic stability of the ξ-subsystem is established, we can (more) easily prove the UGAS result with the additional component $\dot{z}(t) = A(t)z(t) + B(t)u(t-\tau)$ from the dynamics.

Benefit of LKF: Leads to robustness to actuator errors, using ISS.

UAV Simulations

We simulated the UAV dynamics

$$\left\{ \begin{array}{rcl} \dot{x} & = & v\cos(\theta) \\ \dot{y} & = & v\sin(\theta) \\ \dot{\theta} & = & \alpha_\theta(\theta_c(t-\tau)-\theta) \\ \dot{v} & = & \alpha_v(v_c(t-\tau)+\delta(t)-v), \end{array} \right. \tag{24}$$

with our controllers

$$\begin{array}{l} \theta_c(t-\tau) = \theta_{cr}(t) \ \text{ and} \\ v_c(t-\tau) = v_{cr}(t) - \frac{\epsilon}{\alpha_v}\frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}}. \end{array} \tag{25}$$

UAV Simulations

Autopilot constants: $\alpha_v = 0.192$ and $\alpha_\theta = 0.55$.

Delay: $\tau = 2$

Reference trajectory: $(20 + 10\sin(\pi t)/\pi, 20 - 10\cos(\pi t)/\pi, \pi t, 10)$.

Reference control: $(\theta_{cr}(t), v_{cr}(t)) = (\pi(t + 1/\alpha_\theta), 10)$.

Controller parameter: $\epsilon = 0.257732$.

Disturbance: $\delta(t) = 0.1\sin(t)$ added to v_c.

Initial function: $(x_0, y_0, \theta_0, v_0) = (17, 22, -0.5, 8)$.

UAV Simulations

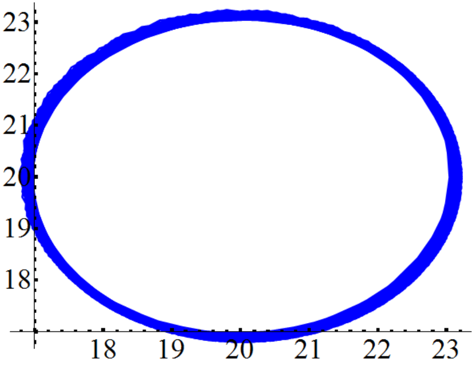

FIGURE : $(x(t), y(t))$ for Times $[480, 1000]$

UAV Simulations

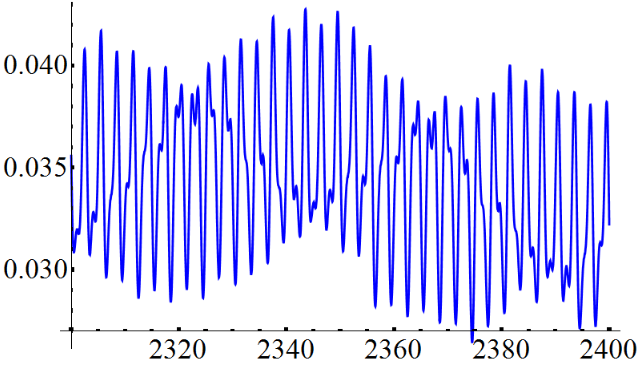

FIGURE : $x(t) - x_r(t)$

UAV Simulations

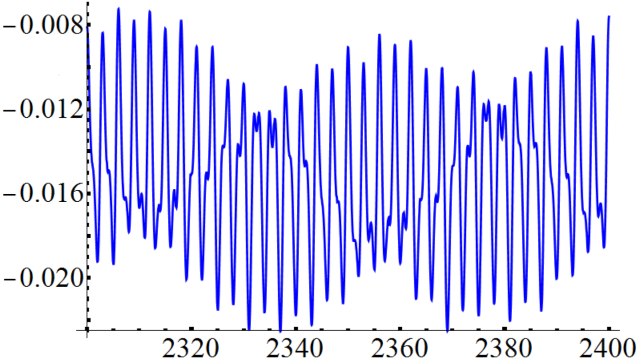

FIGURE : $y(t) - y_r(t)$

UAV Simulations

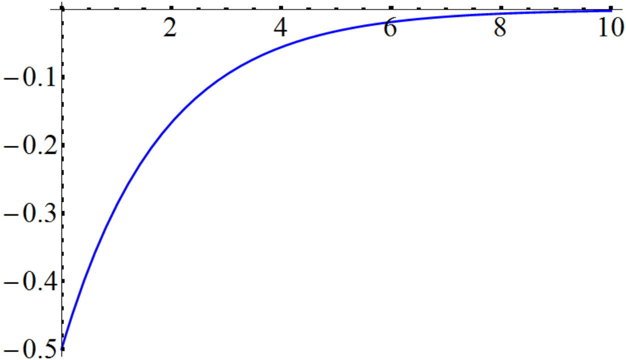

FIGURE : $\theta(t) - \theta_r(t)$

UAV Simulations

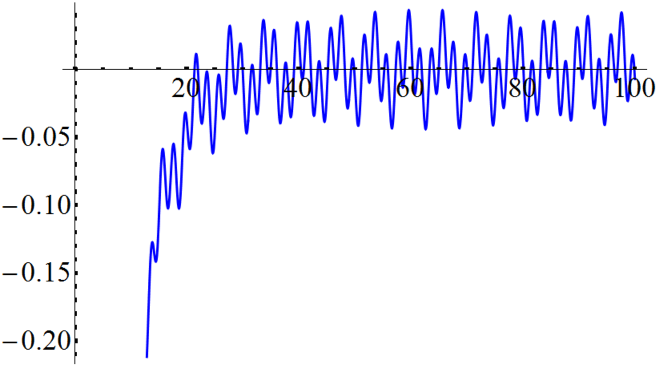

FIGURE : $v(t) - v_r(t)$

UAV Simulations

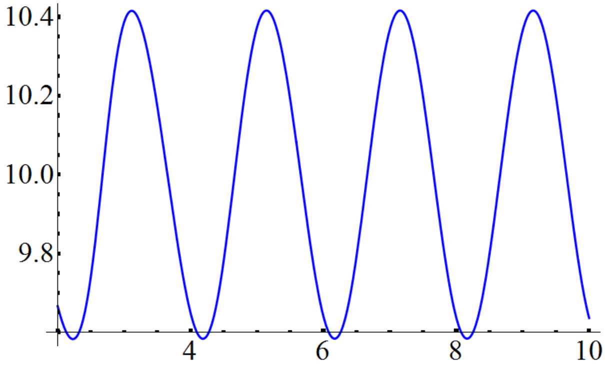

FIGURE : $v_c(t-\tau)$