FURTHER CONSTRUCTIONS OF STRICT LYAPUNOV FUNCTIONS FOR TIME-VARYING SYSTEMS

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NOTATION and DEFINITIONS

Given System:

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_o \qquad (\Sigma)$$

f assumed locally Lipschitz on $[0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$ and time periodic.

$t \mapsto \phi(t; x_o, t_o, \mathbf{u})$: max traj. for Σ and $\mathbf{u} \in \mathcal{U} := \mathcal{MEB}([0, \infty), \mathbb{R}^m)$.

Goal: Design explicit ISS Lyapunov functions for Σ.

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Comparison and UPPD Functions:

$\mathcal{K}_\infty$: cts. strictly incr. unbdd $\rho : [0, \infty) \to [0, \infty)$ s.t. $\rho(0) = 0$.

$\mathcal{KL}$: cts. $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ s.t. (a) $\beta(\cdot, t) \in \mathcal{K}_\infty$ $\forall t$,
(b) $\beta(s, \cdot)$ nonincr. $\forall s$, and (c) $\forall s$, $\beta(s, t) \to 0$ as $t \to +\infty$.

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UPPD: C^1 fcns. $V : [0, \infty) \times \mathbb{R}^n \to [0, \infty)$ s.t. $\exists \alpha_i \in \mathcal{K}_\infty$ such that $\alpha_1(|x|) \le V(t, x) \le \alpha_2(|x|)$ and $|\nabla V(t, x)| \le \alpha_3(|x|)$ everywhere.

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$\mathcal{P}(\tau, \varepsilon, \bar{p})$: cts. $p : \mathbb{R} \to [0, \infty)$ s.t. $\int_{t-\tau}^{t} p(s)ds \geq \varepsilon$ and $p(t) \leq \bar{p}\ \forall t$.

$\mathcal{P}$: All decay rates $p(t)$; i.e., $\mathcal{P} := \cup\{\mathcal{P}(\tau, \varepsilon, \bar{p}) : \tau, \varepsilon, \bar{p} > 0\}$

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ISS(p) Lyap. Fcn. for Σ: $V \in$ UPPD s.t. $\exists\ \chi \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}_\infty \cap C^1$ s.t.

$$|x| \geq \chi(|u|) \Rightarrow \dot{V}(t,x,u) \leq -p(t)\mu(|x|) \quad \text{everywhere.}$$

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ISS(p) Property for Σ: $\exists\ \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ s.t. everywhere

$$|\phi(t_o + h; x_o, t_o, \mathbf{u})| \leq \beta\left(|x_o|, \int_{t_o}^{t_o+h} p(s)ds\right) + \gamma\left(|\mathbf{u}|_{[t_o, t_o+h]}\right).$$

Called ISS with decay rate p. Sontag's ISS property is ISS(p) for $p \equiv 1$.
ISS $\Leftrightarrow$ [$\dot{x} = F(t, x, d) := f(t, x, d\Delta(|x|))$ GAS on $[0, \infty) \times \mathbb{R}^n \times \mathcal{B}$ for appropriate $\Delta \in \mathcal{K}_\infty$].

STATEMENT and SIGNIFICANCE of MAIN RESULT

Main Theorem: [*M&Mazenc, Automatica, in press*] Let $p \in \mathcal{P}$ and f be as above. Then the following six conditions are equivalent:

(C_1) f admits an ISS(p) Lyapunov function.

(C_2) f admits a strict ISS Lyapunov function.

(C_3) f admits a DIS(p) Lyapunov function.

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DIS(p) Lyap. Fcn. for Σ: $V \in$ UPPD s.t. $\exists\, \Omega \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}_\infty \cap C^1$ s.t. $\forall\, t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m$, we have $\dot{V}(t,x,u) \ \leq -p(t)\mu(|x|) + \Omega(|u|)$.
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Significance: Proof provides explicit constructions of ISS Lyap. fcns. in terms of given ISS(p) or DIS(p) Lyap. fcn. Valuable: ISS(p) and DIS(p) Lyap. functions are often already available e.g. from backstepping.

EXPLICIT CONSTRUCTION #1

Let $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$, V be a DIS(p) Lyap. fcn. for Σ, and $\alpha_2, \mu \in \mathcal{K}_\infty \cap C^1$ satisfy the UPPD and DIS(p) requirements. Define $V^\sharp$ by

$$V^\sharp(t,x) = V(t,x) + \left[\int_{t-\tau}^{t} \left(\int_s^t p(r)\,dr\right) ds\right] w(V(t,x)), \qquad \textbf{(\#)}$$

where

$$w = \frac{1}{4\tau}\mu \circ \tilde{\alpha}_2^{-1},\ \tilde{\alpha}_2(s) = \max\left\{\frac{\tau\bar{p}}{2}, 1\right\}(\alpha_2(s) + \mu(s) + s).$$

Then $V^\sharp$ is an ISS Lyap. fcn. for Σ. It has period τ in t if V and p do.

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Motivation: Double integral in (#) is motivated by Lyap. function theory for delay systems.

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N.B.: Any ISS(p) Lyap. fcn. for Σ is also a DIS(p) Lyap. fcn. for Σ, so construction strictifies ISS(p) Lyap. fcns.

EXPLICIT CONSTRUCTION #2

Control-Affine Σ_{ca}: $\dot{x} = f(t,x,u) := h(t,x) + g(t,x)u, \quad x(t_0) = x_o$,
where we assume h, g are locally Lipschitz and time-periodic.

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Added Assumption: $\exists$ time-independent $W \in$ UPPD and $\bar{g} > 0$ such that

$$|\nabla W(x)h(t,x)| \leq \frac{\varepsilon}{\tau^2 \bar{p}} W(x) \quad , \qquad |\nabla W(x)g(t,x)| \leq \bar{g}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$ is given.

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Construction: Assume that $V \in$ UPPD and $\chi \in \mathcal{K}_\infty$ are such that

$$|x| \geq \chi(|u|) \Rightarrow V_t(t,x) + V_x(t,x)\left[h(t,x) + g(t,x)u\right] \leq -p(t)W(x)$$

for all $t \geq 0$, where W is as above. Then

$$U(t,x) := V(t,x) + \frac{1}{\tau}\left[\int_{t-\tau}^{t} \left(\int_s^t p(r)\,dr\right) ds\right] W(x)$$

is an ISS Lyap. fcn. for Σ_{ca}. It has period τ in t if p and V do.

EXTENSION for SYSTEMS with OUTPUTS

Given System: When x is not available to measure, we instead look at

$$\dot{x} = f(t, x, u), \quad y = H(x) \qquad (\Sigma_H)$$

We assume H is locally Lipschitz and f is forward complete.

Outputs: $y(t_o + h; x_o, t_o, \mathbf{u}) = H(\phi(t_o + h; x_o, t_o, \mathbf{u}))$.

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IOS of Σ_H: $\exists\, \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ s.t. $\forall\, t_o \geq 0$, $x_o \in \mathbb{R}^n$, $\mathbf{u} \in \mathcal{U}$ and $h \geq 0$, we have $|y(t_o + h; x_o, t_o, \mathbf{u})| \leq \beta\,(|x_o|, h)\; + \; \gamma\left(|\mathbf{u}|_{[t_o, t_o+h]}\right)$.

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IOS Lyap Fcn. for Σ_H: C^1 fcn. $V : [0, \infty) \times \mathbb{R}^n \to [0, \infty)$ s.t. $\exists\, \alpha_1$, $\alpha_2, \chi \in \mathcal{K}_\infty$ and $\kappa \in \mathcal{KL}$ s.t. (i) $\alpha_1(|H(x)|) \leq V(t, x) \leq \alpha_2(|x|)$ and (ii) $V(t, x) \geq \chi(|u|) \;\Rightarrow\; \dot{V}(t, x, u) \;\leq\; -\kappa(V(t, x), |x|)$ everywhere.

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Sontag/Wang, SICON 2K: Shows equivalence of IOS to the existence of an IOS Lyapunov function for time-invariant systems.

EXPLICIT CONSTRUCTION #3

Assume we are given a system Σ_H, a decay rate $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$, a C^1 fcn. $U : [0, \infty) \times \mathbb{R}^n \to [0, \infty)$, and $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\chi} \in \mathcal{K}_\infty$ and $\hat{\kappa} \in C^1 \cap \mathcal{K}_\infty$ s.t.

$$\hat{\alpha}_1(|H(x)|) \leq U(t,x) \leq \hat{\alpha}_2(|x|) \;\; \forall x \in \mathbb{R}^n \tag{1}$$

$$U(t,x) \geq \hat{\chi}(|u|) \;\Rightarrow\; \dot{U}(t,x,u) \;\leq\; -p(t)\hat{\kappa}(U(t,x)) \tag{2}$$

for all $t \geq 0$. Define $w : [0, \infty) \to [0, \infty)$ by

$$w(r) = \frac{1}{\tau^2 \bar{p} + 2\tau} \int_0^r \mathrm{sat}\{\hat{\kappa}'(s)\} ds,$$

where $\mathrm{sat}\{x\} = \mathrm{sign}\{x\} \min\{1, |x|\}$. Then

$$V(t,x) = U(t,x) + \left[\int_{t-\tau}^{t} \left(\int_s^t p(l) dl \right) ds \right] w(U(t,x))$$

is an IOS Lyapunov function for Σ_H.

See [*M&Mazenc, ACC05 Proc., in press*] for proof.

APPLICATION to ROTATING RIGID BODY

Dynamics of Velocities: $\dot{\omega}_1 = \delta_1 + u_1 \ , \ \ \dot{\omega}_2 = \delta_2 + u_2 \ , \ \ \dot{\omega}_3 = \omega_1 \omega_2$.

Track the reference trajectory $\omega_{1r}(t) = \sin(t)$, $\omega_{2r}(t) = \omega_{3r}(t) = 0$.

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Error Equations and Feedback: Setting $\tilde{\omega}_i(t) = \omega_i(t) - \omega_{ir}(t)$ gives

$$\dot{\tilde{\omega}}_1 = \delta_1 + u_1 - \cos(t), \ \ \dot{\tilde{\omega}}_2 = \delta_2 + u_2, \ \ \dot{\tilde{\omega}}_3 = (\tilde{\omega}_1 + \sin(t))\tilde{\omega}_2$$

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DIS(p) Lyap. Fcn.: $V(t,\tilde{\omega}) = \frac{1}{2}\left[\tilde{\omega}_1^2 + (\tilde{\omega}_2 + \sin(t)\tilde{\omega}_3)^2 + \tilde{\omega}_3^2\right]$.

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ISS Lyap. Fcn.: Using Explicit Construction #1, we get

$$\begin{aligned} V^\sharp(t, \tilde{\omega}) &= V(t, \tilde{\omega}) + \left[\int_{t-\tau}^{t} \left(\int_s^t p(r)dr\right) ds\right] w(V(t, \tilde{\omega})) \\ &= \left[1 + \tfrac{\pi}{32} - \tfrac{1}{32}\sin(2t)\right] V(t, \tilde{\omega}). \end{aligned}$$