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STRICT LYAPUNOV FUNCTIONS AND FEEDBACK CONTROLS FOR SIR MODELS WITH QUARANTINE AND VACCINATION

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Abstract. We provide a new global strict Lyapunov function construction for a susceptible, infected, and recovered (or SIR) disease dynamics that includes quarantine of infected individuals and mass vaccination. We use the Lyapunov function to design feedback controls to asymptotically stabilize a desired endemic equilibrium, and to prove input-to-state stability for the dynamics with a suitable restriction on the disturbances. Our simulations illustrate the potential of our feedback controls to reduce peak levels of infected individuals.

1. **Introduction.**

The recent COVID-19 pandemic has motivated the development of significant new control theoretic methods for disease dynamics, e.g. [2, 23, 33] to name a few. While such models may enjoy asymptotic convergence to states in which the disease is no longer present in a population even if no controls are used, it is of interest to apply feedback design in such models, to reduce peak levels of infection, and thereby reduce the numbers of fatalities and reduce the burden on the medical community. Feedback design entails comparing the effects of different state dependent parameters in dynamical systems, with a view towards choosing state dependent parameters that produce desirable asymptotic stability properties for the systems. Such state dependent parameters are called feedback controls, and they differ from open loop controls that are typically used in optimal control theory, which depend on time but not on the state. Feedback controls are useful for representing possible mediation efforts that can be used during a pandemic, such

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as quarantining or vaccination, subject to physical constraints that can arise from factors like limited availability of vaccines or other medical resources and logistical considerations.

Feedback design is usually done in conjunction with the construction of a strict Lyapunov function for the dynamics on the entire state space of the system. A strict Lyapunov function is a positive definite and radially unbounded function whose time derivative along all trajectories of the system is upper bounded by a negative definite function of the state [16, 21]. This decay condition ensures asymptotic convergence to a desired equilibrium vector. Starting from a candidate Lyapunov function for a controlled dynamical system, feedback design usually involves choosing the feedback control in order to make the time derivative of the Lyapunov function satisfy a desired decay condition along solutions of the feedback controlled system. Strict Lyapunov functions are also useful when one needs to study robustness properties with respect to uncertainties in the model. In engineering, one important robustness property is input-to-state stability (or ISS) [27], which implies that bounded uncertainties produce bounded states and which coincides with global asymptotic stability when the uncertainty is the zero function. One typically proves ISS by constructing a special type of strict Lyapunov function, called an ISS Lyapunov function. For linear time invariant systems, constructing strict Lyapunov functions is often an elementary task that involves linear matrix inequalities. However, for nonlinear systems, the construction of ISS Lyapunov functions is not always easy.

While there are works on constructing strict Lyapunov functions for time-varying linear or nonlinear systems [21, 31, 32], we believe that the problem of constructing strict Lyapunov functions for SIR models with quarantine and vaccination on their entire spaces was open, owing to their bilinearities involving products of states. Here, we solve this problem in a recursive way. First, we build a strict Lyapunov function for a basic two-dimensional SI model. In the second step, we modify the strict Lyapunov function from the first step to cover a more general model with vaccination. Finally, we transform the Lyapunov function from the second step into a strict Lyapunov function for cases with vaccination and isolation. The last step uses the triangular structure of the dynamics. The augmented Lyapunov function and its time derivative contain all the state variables as desired. Our strict Lyapunov functions are ISS ones with explicit expressions, which enable us to prove ISS properties and design stabilizing feedback controls. Our simulations illustrate how our new feedback controls can reduce peak levels of infected populations in our models.

A key ingredient in our strict Lyapunov function constructions is the non-classical use of logarithmic functions that had been used to build nonstrict global Lyapunov functions (meaning, Lyapunov functions whose time derivatives along solutions of the dynamics are only required to be nonpositive) [18, 26, 29]. For a given controller, nonstrict Lyapunov functions can sometimes verify the asymptotic convergence of trajectories to an equilibrium, with the help of LaSalle's invariance principle. However, nonstrict Lyapunov functions generally only lead to heuristic ways to find controllers. Moreover, the nonstrictness property cannot quantify the effects of uncertainties, even if the uncertainty magnitude is arbitrarily small. More importantly, in prior literature, it is commonly assumed that the inflow is fixed to keep the total population constant [17], which precludes the possibility of considering the uncertainties in ISS. To achieve ISS and remove the assumption of constant total population, a strict Lyapunov function was proposed in [9] for a simpler three state

SIR model. However, the Lyapunov function obtained in [9] is semi-global and not differentiable, and it leads to discontinuities in controllers which make them less amenable to implementation than more standard continuous feedback controls.

The major drawback of semi-global Lyapunov function constructions is that the negativity of their time derivatives is only on a subset of the state space, instead of being on the entire state space. The dependency of the negativity condition on the domain size makes the Lyapunov function inconvenient, insofar that it cannot be directly used in feedback control design because the time derivative is not conducive to indicating the performance of the controls. This work improves on the semi-global results [9, 10, 11, 12] for two- and three-dimensional models, by providing global strict ISS Lyapunov functions for higher dimensional systems which are conducive to ensuring ISS and to constructing continuous feedback controls. Therefore, [9, 11, 12] motivate our global strict Lyapunov function constructions in this work that are more conducive to control design.

The SIR model with quarantine is sometimes called the SIQR model. It has been used widely for prediction and interpretation of infectious diseases [8, 14]. Recently, the model was used to estimate the basic reproduction number and to interpret statistical figures of the COVID-19 outbreak in Brazil [6]. The SIQR model was also used to describe the COVID-19 outbreak in Japan, and to compare the effectiveness of quarantine versus lockdown measures [24]. The work [1] focused on numerical techniques to compute solutions to epidemic models. It also applied the Routh-Hurwitz criterion to the Jacobian approximation of the SIQR model to numerically detect bifurcations. Its local stability analysis applies under constant inflow (i.e., constant immigration and newborn rates). The work [8] on the SIQR model constructed a Lyapunov function in the so-called feasible region that is widely used in quasi-steady-state stability analysis under the assumption of constant inflow. The Lyapunov function contains only partial state measurements, and only leads to a nonpositive time derivative for the Lyapunov function. Hence, LaSalle's invariance principle was used in [8], and then it was combined with a stability analysis for the remaining variables to complete the stability analysis. A similar approach was pursued in [20] by incorporating culling (i.e., elimination) into the SIQR model to study diseases in animals and quarantining for humans, under fixed constant values for the vaccination, quarantine, and culling rates. By contrast, our novel construction of a strict ISS Lyapunov function for the entire four-dimensional SIQR model on its entire state space combined with our feedback control approach enables us to quantify the effects of perturbations of the immigration/newborn rates using ISS, while also quantifying the effects of using different vaccination rates as state-dependent feedback controls. This has the potential to make our treatment more amenable to more realistic cases where the immigration/newborn rates are uncertain, and where a comparison is called for to compare the effects of different vaccination rates. Also, since [1], [14], and [20] are not based on strict Lyapunov function constructions for the full SIQR model, they are not amenable to proving ISS results.

We use the following standard definitions and notation, which we simplify when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary unless we indicate otherwise. We use $|f|_\infty$ (resp., $|f|_J$) to denote the usual sup norm of a bounded function f over its entire domain (resp., a subset J of its domain). Let $\mathcal{K}$ denote the set of all strictly increasing continuous functions $\alpha : [0, +\infty) \to [0, +\infty)$ such that $\alpha(0) = 0$; if, in addition, α is unbounded, then we say that α is of class $\mathcal{K}_\infty$. We say that a continuous function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$

is of class $\mathcal{KL}$ provided for each fixed $s > 0$, the function $\beta(\cdot, s)$ belongs to class $\mathcal{K}$, and for each fixed $r \geq 0$, the function $\beta(r, \cdot)$ is non-increasing and $\beta(r, s) \to 0$ as $s \to +\infty$. A system of the form $\dot{x}(t) = f(x(t), \varepsilon(t))$ with a state space $\mathcal{X} \subseteq \mathbb{R}^n$ is called *input-to-state stable (or ISS)* [16] on $\mathcal{X}$ with respect to a disturbance set $\mathcal{S}$ provided: There are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for each initial state $x(0) \in \mathcal{X}$ and each locally bounded piecewise continuous function ε that is valued in $\mathcal{S}$, the unique solution $x(t)$ satisfies $|x(t)| \leq \beta(|x(0)|, t) + \gamma(|\varepsilon|_{[0,t]})$ for all $t \geq 0$.

2. SIR Model with Quarantine and Vaccination.

Our main model for which we will construct our strict Lyapunov function and feedback controls is

$$\dot{S}(t) = B + \epsilon(t) - \rho(t)S(t) - \mu S(t) - \beta I(t)S(t), \tag{1a}$$

$$\dot{I}(t) = \beta S(t)I(t) - (\gamma + \nu + \mu)I(t), \tag{1b}$$

$$\dot{Q}(t) = \nu I(t) - (\tau + \mu)Q(t), \tag{1c}$$

$$\dot{R}(t) = \gamma I(t) + \tau Q(t) - \mu R(t) + \rho(t)S(t), \tag{1d}$$

whose positive valued states S, I, Q, and R are numbers of susceptible, infected, quarantined, and recovered individuals, respectively [14]. The positive parameters β, γ and μ are the contact/transmission rate, the recovery rate and the non-associated mortality rate, respectively. The parameter $\nu > 0$ is the rate at which infected individuals are isolated [14]. The parameter $\tau > 0$ is the reciprocal of the average time spent in isolation, and the constant $B > 0$ is the immigration/newborn rate. The piecewise continuous locally bounded function ϵ represents the immigration/newborn perturbation, and we assume that it satisfies

$$\epsilon(t) > -B \text{ for all } t \geq 0, \tag{2}$$

which ensures that the positive orthant $(0, +\infty)^4$ is a forwardly invariant set for (1), meaning, each state component stays positive for all $t \geq 0$ if the initial state for (1) is in $(0, +\infty)^4$. The vaccination rate ρ is

$$\rho(t) = \hat{\rho} + u(t), \tag{3}$$

where the control u (which will be specified in our theorem, and which will depend on time t through its dependence on state components of the system) is valued in $[-\hat{\rho}, +\infty)$ and $\hat{\rho}$ is a positive constant, which produces the system

$$\dot{S}(t) = B - (\hat{\rho} + \mu)S(t) - \beta I(t)S(t) - u(t)S(t) + \epsilon(t), \tag{4a}$$

$$\dot{I}(t) = \beta S(t)I(t) - (\gamma + \nu + \mu)I(t), \tag{4b}$$

$$\dot{Q}(t) = \nu I(t) - (\tau + \mu)Q(t), \tag{4c}$$

$$\dot{R}(t) = \gamma I(t) + \tau Q(t) - \mu R(t) + \hat{\rho}S(t) + u(t)S(t). \tag{4d}$$

We assume that

$$\beta B > (\hat{\rho} + \mu)(\gamma + \nu + \mu), \tag{5}$$

which is equivalent to the usual condition that the basic reproduction R_0 satisfies $R_0 > 1$; see also Remark 4 below for a discussion on R_0, and see Remark 1 for more on the derivation of the preceding model.

Let $\lambda = \gamma + \nu + \mu$ and $\chi = \hat{\rho} + \mu$. When $\epsilon = 0$ and $u = 0$, the system (4) admits the componentwise positive (endemic) equilibrium point

$$(S_\star, I_\star, Q_\star, R_\star) = \left(\frac{\lambda}{\beta}, \ \frac{B}{\lambda} - \frac{\chi}{\beta}, \ \frac{\nu}{\tau+\mu}\left(\frac{B}{\lambda} - \frac{\chi}{\beta}\right), \ \frac{1}{\mu}\left[\left(\gamma + \frac{\tau\nu}{\tau+\mu}\right)\left(\frac{B}{\lambda} - \frac{\chi}{\beta}\right) + \frac{\hat{\rho}\lambda}{\beta}\right]\right). \tag{6}$$

With the choices

$$\begin{aligned} &\xi = \ln(I), \ \ \xi_\star = \ln(I_\star), \ \ (\tilde{\xi}, \tilde{S}, \tilde{Q}, \tilde{R}) = (\xi - \xi_\star, S - S_\star, Q - Q_\star, R - R_\star) \\ &\text{and} \ \ \psi_\star = \lambda e^{\xi_\star}, \end{aligned} \tag{7}$$

we can then use the relation

$$B - \left(\chi + \beta e^{\xi_\star}\right) S_\star = B - \left[\chi + \beta\left(\tfrac{B}{\lambda} - \tfrac{\chi}{\beta}\right)\right]\tfrac{\lambda}{\beta} = B - \beta\tfrac{B}{\lambda}\tfrac{\lambda}{\beta} = 0 \tag{8}$$

to obtain

$$\dot{\tilde{\xi}}(t) = \beta\tilde{S}(t), \tag{9a}$$

$$\dot{\tilde{S}}(t) = -\left(\chi + \beta e^{\tilde{\xi}(t)+\xi_\star}\right)\tilde{S}(t) + \psi_\star\left(1 - e^{\tilde{\xi}(t)}\right) + \epsilon(t) - u(t)S(t), \tag{9b}$$

$$\dot{\tilde{Q}}(t) = \nu e^{\xi_\star}(e^{\tilde{\xi}(t)} - 1) - (\tau + \mu)\tilde{Q}(t), \tag{9c}$$

$$\dot{\tilde{R}}(t) = \gamma e^{\xi_\star}(e^{\tilde{\xi}(t)} - 1) + \tau\tilde{Q}(t) - \mu\tilde{R}(t) + \hat{\rho}\tilde{S}(t) + u(t)S(t) \tag{9d}$$

with $\tilde{\xi}(t) \in \mathbb{R}$, $\tilde{S}(t) \in (-S_\star, +\infty)$, $\tilde{Q}(t) \in (-Q_\star, +\infty)$ and $\tilde{R}(t) \in (-R_\star, +\infty)$ for all $t \geq 0$. Finally, we assume that

$$|\epsilon|_\infty \leq \frac{\psi_\star}{4}. \tag{10}$$

Our first goal in the next section is to find a strict Lyapunov function for (9) on its entire domain $\mathcal{X} = \mathbb{R} \times (-S_\star, +\infty) \times (-Q_\star, +\infty) \times (-R_\star, +\infty)$ when $u = 0$ and $\epsilon = 0$. By the change of variables that transformed (1) into (9), this is equivalent to finding a strict Lyapunov function for (1) and (6) on the positive orthant when $u = 0$ and $\epsilon = 0$. Then, we will use this strict Lyapunov function for (9) to show that, for a suitable class of control functions u that are valued in $[-\hat{\rho}, +\infty)$, the controlled system (9) satisfies ISS with respect to the immigration perturbation $\epsilon(t)$ with the disturbance set $\mathcal{S} = [-\min\{B, \psi_\star/4\}, \psi_\star/4]$.

Remark 1. In the special case where $\rho = \epsilon = 0$ and $B = \mu$, (1) agrees with the SIR model with quarantine in [14, Equation (8.6)] under the assumption in [14] that $S(t) + I(t) + Q(t) + R(t) = 1$ for all $t \geq 0$, which calls for our use of the coefficient $-(\tau + \mu)$ in the Q dynamics. As usual, model (1) employs the simple mechanism of massively vaccinating the susceptible population [14, Equation (8.6)] in the SIR model with isolation [14, Equation (8.24)]. Model (1) can be viewed as a four-dimensional core of the six-dimensional model for the study of controlling SARS outbreaks without vaccines, and it includes disease-associated death in the population R [14]. Importantly, this paper employs $B + \epsilon(t)$ for the newborn/immigration value to analyze the robustness of the nonlinear model with respect to its uncertainty ϵ. Hence, in addition to incorporating uncertainty and feedback control, a key difference between (1) and the popular models is that the model (1) does not require that the total population is $S(t)+I(t)+Q(t)+R(t) = 1$. The models in [14, Equations (8.6) and (8.24)]) use μ instead of $B + \epsilon(t)$ in order ensure that the total population at all times is $S(t)+I(t)+Q(t)+R(t) = 1$. In fact, the normalization excludes the idea of globalness and perturbation in the robustness analysis. Model (1) removes the constant unity assumption on the population size for the study of global stability and robustness. In the special case where the perturbation is $\epsilon = 0$ and when the vaccination rate $\rho(t)$ is replaced by zero, the model (1) is identical to the model in [8, Equation (6)]

3. **Strict Lyapunov Function for (9).** In terms of the constants from the preceding section, any constants $c > 0$ and $g > 0$, the constants

$$k_1 = \max\left\{\frac{1+2c}{\chi}, \frac{2}{c\psi_\star e^{\xi_\star}}\left[2ce^{2\xi_\star} + \frac{(c+1)\psi_\star}{2\beta}\right]\right\}, \tag{11}$$

$$k_2 = k_1 + \left(\frac{4c\chi^2}{\beta} + (c+1)\psi_\star\right)\frac{4}{c\psi_\star\left(2\chi + \beta e^{\xi_\star}\right)}, \tag{12}$$

$$k_3 = \left(\frac{2c\chi^2}{\beta} + \frac{(c+1)\psi_\star}{2}\right)\frac{16\beta}{c^2\psi_\star^2\chi^2}, \tag{13}$$

$$k_4 = \frac{k_2^2}{4k_3^2}, \quad c_\sharp = \frac{\ln(2)c\chi}{\beta} + ce^{\xi_\star}, \quad c_\flat = \frac{(1+c)^2}{2\chi}, \quad \text{and} \quad c_\diamondsuit = \frac{(\tau+\mu)c\psi_\star}{2\nu^2 e^{\xi_\star}}, \tag{14}$$

and the functions

$$U_c(\tilde{\xi}, \tilde{S}) = \frac{1}{2}\tilde{S}^2 + \frac{c}{2}\left[\tilde{S} + \frac{\chi}{\beta}\tilde{\xi} + e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right)\right]^2 + \frac{(c+1)\psi_\star}{\beta}\left(e^{\tilde{\xi}} - 1 - \tilde{\xi}\right), \tag{15}$$

$$J_c(\tilde{\xi}, \tilde{S}) = -\left[(1+c)\tilde{S} + \frac{c\chi}{\beta}\tilde{\xi} + ce^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right)\right], \tag{16}$$

and

$$\mathcal{N}_c(r) = \frac{1}{2}\left[\sqrt{k_4 + \min\left\{\frac{1}{k_3}, 4\sqrt{k_4}\mu\right\}r} - \sqrt{k_4}\right], \tag{17}$$

we prove the following, where we write the controls u as functions of t alone to keep the notation simple but where u will later depend on the state of (9), and where part (b) implies ISS of (9) with the controls u (by standard results from [16, Chapter 4] on the sufficiency of the existence of the ISS Lyapunov V_c to have ISS):

Theorem 3.1. *The following conclusions hold: (a) The time derivative of*

$$V_c(\tilde{\xi}, \tilde{S}, \tilde{Q}, \tilde{R}) = U_c(\tilde{\xi}, \tilde{S}) + \frac{g}{2}\left[\tilde{S} + e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right) + \tilde{Q} + \tilde{R}\right]^2 + \frac{c_\diamondsuit}{2}\tilde{Q}^2 \tag{18}$$

along all trajectories of (9) satisfies

$$\begin{aligned}\dot{V}_c(t) \le & -\mathcal{N}_c(V_c(\tilde{\xi}(t), \tilde{S}(t), \tilde{Q}(t), \tilde{R}(t))) + J_c(\tilde{\xi}(t), \tilde{S}(t))(\tilde{S}(t) + S_\star)u(t) \\ & + c_\sharp|\epsilon(t)| + \left[c_\flat + \frac{g}{2\mu}\right]\epsilon(t)^2 \end{aligned} \tag{19}$$

for all $t \ge 0$, all piecewise continuous functions

$$\epsilon : [0, +\infty) \to [-\min\{B, \psi_\star/4\}, \psi_\star/4], \tag{20}$$

and all control functions u. (b) For each feedback control $u(t)$ such that

$$J_c(\tilde{\xi}(t), \tilde{S}(t))(\tilde{S}(t) + S_\star)u(t) \le 0 \tag{21}$$

for all $t \ge 0$, the function V_c is an ISS Lyapunov function for (9) on its state space

$$\mathcal{X} = \mathbb{R} \times (-S_\star, +\infty) \times (-Q_\star, +\infty) \times (-R_\star, +\infty) \tag{22}$$

for the disturbance set $\mathcal{S} = [-\min\{B, \psi_\star/4\}, \psi_\star/4]$. □

Remark 2. Condition (21) provides a systematic procedure for feedback control design, namely, we choose any u that is bounded below by $-\hat{\rho}$ and that satisfies (21) along all solutions of (9); see for instance (76). Since the constant $c > 0$ arises in (21), different choices of c produce different feasible stabilizing feedback controls. We illustrate the effects of changing c (and the benefits of using nonzero choices of u) in our simulations below. The presence of g in the V_c formula implies that different choices of g lead to different rates of convergence of V_c to zero.

4. **Proof of Theorem 3.1.**

The proof has three parts. In the first part, we build a strict Lyapunov function for the SI dynamics corresponding to (1) (i.e., where the R and Q variables are not present), using three key lemmas that we prove in the appendix. The first of these lemmas provides nonstrict Lyapunov functions which we later transform into a strict Lyapunov function for the SI dynamics using a novel variant of the strictification approach [21]. In the second part, we build a strict Lyapunov function for the SIR dynamics corresponding to (1) (i.e., where Q is not present), using the first part of the proof. In the final part, we apply a cascade argument to the result from the second part to prove the theorem. Since the strict Lyapunov functions for the SI and SIR models that we construct in the proof of the theorem are of independent interest from both the mathematical and practical points of view, we state these two constructions as additional lemmas.

4.1. **SI Model.**

We consider the system

$$\dot{S}(t) = B - \chi S(t) - \beta S(t)I(t) + \delta(t), \tag{23a}$$

$$\dot{I}(t) = \beta S(t)I(t) - \lambda I(t), \tag{23b}$$

where S and I are valued in $(0, +\infty)$, and $B > 0$, $\chi > 0$, $\beta > 0$, and $\lambda > 0$ are constants, and the piecewise continuous locally bounded function δ represents uncertainty. In this subsection, we use δ instead of ϵ to represent the uncertainty, because when we apply this work from this subsection to later subsections, we will choose

$$\delta = \delta_1 + \delta_2, \text{ where } \delta_1 = \epsilon \text{ and } \delta_2 = -Su \tag{24}$$

for a suitable control u and the ϵ from our theorem. Throughout this subsection, we assume that $(0, +\infty)^2$ is a forward invariant set for (23), which will be the case if

$$\delta(t) \geq -B \tag{25}$$

for all $t \geq 0$. We assume that the inequality

$$\beta B > \chi\lambda \tag{26}$$

is satisfied. The inequality (26) ensures that (23) admits the componentwise positive equilibrium

$$(S_\star, I_\star) = \left(\frac{\lambda}{\beta}, \frac{B}{\lambda} - \frac{\chi}{\beta}\right) \tag{27}$$

when $\delta = 0$. Changing coordinates using the variables (7) as in the previous section transforms (23) into

$$\dot{\tilde{\xi}}(t) = \beta\tilde{S}(t), \tag{28a}$$

$$\dot{\tilde{S}}(t) = -\left(\chi + \beta e^{\tilde{\xi}(t)+\xi_\star}\right)\tilde{S}(t) + \psi_\star\left(1 - e^{\tilde{\xi}(t)}\right) + \delta(t) \tag{28b}$$

with $\psi_\star$ defined by (7) as before, and with $\tilde{\xi}(t) \in \mathbb{R}$ and $\tilde{S}(t) \in (-S_\star, +\infty)$ for all $t \geq 0$.

In terms of the function U_c we defined in (15), and the functions

$$V_1(\tilde{\xi},\tilde{S}) = \tfrac{1}{2}\tilde{S}^2 + \tfrac{\psi_\star}{\beta}\left(e^{\tilde{\xi}} - 1 - \tilde{\xi}\right), \tag{29}$$

$$V_2(\tilde{\xi},\tilde{S}) = \tfrac{1}{2}\left[\tilde{S} + \tfrac{\chi}{\beta}\tilde{\xi} + e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right)\right]^2 + \tfrac{\psi_\star}{\beta}\left(e^{\tilde{\xi}} - 1 - \tilde{\xi}\right), \tag{30}$$

and

$$W_c(a,b) = \left(\chi + \beta e^{a+\xi_\star}\right) b^2 + c\psi_\star\left[\tfrac{\chi}{\beta}a + e^{\xi_\star}\left(e^a - 1\right)\right]\left(e^a - 1\right), \tag{31}$$

for any constant $c > 0$, our first three lemmas are as follows, where the first lemma can be interpreted to mean that V_1 and V_2 are weak (or nonstrict) Lyapunov functions for (28) when $\delta = 0$ in the sense of [21]:

Lemma 4.1. *The time derivative of the functions V_1 and V_2 defined in* (29) *and* (30) *satisfy*

$$\dot{V}_1(t) = -\left(\chi + \beta e^{\tilde{\xi}(t)+\xi_\star}\right)\tilde{S}(t)^2 + \tilde{S}(t)\delta(t) \tag{32}$$

and

$$\begin{aligned}\dot{V}_2(t) = &-\tfrac{\psi_\star}{\beta}\left(e^{\tilde{\xi}(t)} - 1\right)\left[\chi\tilde{\xi}(t) + \beta e^{\xi_\star}\left(e^{\tilde{\xi}(t)} - 1\right)\right] \\ &+ \left[\tilde{S}(t) + \tfrac{\chi}{\beta}\tilde{\xi}(t) + e^{\xi_\star}\left(e^{\tilde{\xi}(t)} - 1\right)\right]\delta(t)\end{aligned} \tag{33}$$

respectively along all trajectories of the system (28) for all $t \geq 0$.

Lemma 4.2. *For all $(a,b) \in \mathbb{R}^2$, the inequality*

$$\frac{4\beta^2}{c^2\psi_\star^2\chi^2}W_c(a,b)^2 + \frac{2}{c\psi_\star\left(\frac{\chi}{\beta} + \frac{e^{\xi_\star}}{2}\right)}W_c(a,b) \geq a^2 \tag{34}$$

is satisfied.

Lemma 4.3. *The constants k_3 and k_4 defined in* (13)-(14) *are such that*

$$\sqrt{k_4 + \tfrac{1}{k_3}U_c(a,b)} - \sqrt{k_4} \leq \tfrac{1}{2}W_c(a,b) \tag{35}$$

holds for all $(a,b) \in \mathbb{R}^2$.

See the appendix below for proofs of Lemmas 4.1-4.3. We next use the preceding lemmas to provide our strict Lyapunov function construction for the SI model (28). In terms of the function J_c from (16) and the constants $c_\sharp$ and $c_\flat$ that we defined in (14), our strict Lyapunov function for (28) is provided by the following lemma, which shows that U_c is a strict Lyapunov function for (28) on its state space when $\delta = 0$:

Lemma 4.4. *With the choices of U_c, J_c, and W_c in* (15), *(16), and* (31), *the time derivative of the function $U_c(\tilde{\xi},\tilde{S})$ along all trajectories of the system (28) satisfies*

$$\dot{U}_c(t) = -W_c(\tilde{\xi}(t),\tilde{S}(t)) - J_c(\tilde{\xi}(t),\tilde{S}(t))\delta(t) \tag{36}$$

for all $t \geq 0$. Also, when δ has the form

$$\delta(t) = \delta_1(t) + \delta_2(t) \tag{37}$$

where δ_1 is a piecewise continuous function such that

$$|\delta_1|_\infty < \tfrac{\psi_\star}{4}, \tag{38}$$

then, with the choices of $c_\sharp$, $c_\flat$, k_3, *and* k_4 *defined in*(13)-(14)*, the inequalities*

$$\dot U_c(t) \le -\frac{1}{2}W_c(\tilde\xi(t),\tilde S(t)) - J_c(\tilde\xi(t),\tilde S(t))\delta_2(t) + c_\sharp|\delta_1(t)| + c_\flat\delta_1(t)^2 \tag{39}$$

and

$$\begin{aligned}\dot U_c(t) \le& -\left(\sqrt{k_4+\tfrac{1}{k_3}U_c(\tilde\xi(t),\tilde S(t))}-\sqrt{k_4}\right)\\ &- J_c(\tilde\xi(t),\tilde S(t))\delta_2(t) + c_\sharp|\delta_1(t)| + c_\flat\delta_1(t)^2\end{aligned} \tag{40}$$

hold along all solutions of (28) *for all* $t \ge 0$.

Proof. Since

$$U_c(\tilde\xi,\tilde S) = V_1(\tilde\xi,\tilde S) + cV_2(\tilde\xi,\tilde S), \tag{41}$$

we deduce from (32) and (33) that (36) is satisfied. Then, when $\delta(t) = \delta_1(t) + \delta_2(t)$, we have

$$\dot U_c(t) = -W_c(\tilde\xi,\tilde S) - J_c(\tilde\xi,\tilde S)\delta_1 - J_c(\tilde\xi,\tilde S)\delta_2. \tag{42}$$

Here, and in the rest of the proof, time derivatives of functions are along all solutions of (28) for all $t \ge 0$. To complete the proof of the lemma, we first consider the case where $\delta_2 = 0$. Then

$$\begin{aligned}\dot U_c(t) \;=\;& -\left(\chi+\beta e^{\tilde\xi+\xi_\star}\right)\tilde S^2 - \left[\tfrac{c\psi_\star\chi}{\beta}\tilde\xi\left(e^{\tilde\xi}-1\right)+c\psi_\star e^{\xi_\star}\left(e^{\tilde\xi}-1\right)^2\right]\\ &+\left[(1+c)\tilde S+\tfrac{c\chi}{\beta}\tilde\xi+ce^{\xi_\star}\left(e^{\tilde\xi}-1\right)\right]\delta_1.\end{aligned} \tag{43}$$

Using the triangle inequality to get

$$(1+c)\tilde S\delta_1 \le \tfrac{\chi}{2}\tilde S^2 + \tfrac{(1+c)^2}{2\chi}\delta_1^2, \tag{44}$$

we obtain

$$\begin{aligned}\dot U_c(t) \;\le\;& -\left(\tfrac{\chi}{2}+\beta e^{\tilde\xi+\xi_\star}\right)\tilde S^2 - \left[\tfrac{c\psi_\star\chi}{\beta}\tilde\xi\left(e^{\tilde\xi}-1\right)+c\psi_\star e^{\xi_\star}\left(e^{\tilde\xi}-1\right)^2\right]\\ &+\left[\tfrac{c\chi}{\beta}\tilde\xi+ce^{\xi_\star}\left(e^{\tilde\xi}-1\right)\right]\delta_1 + c_\flat\delta_1^2\end{aligned} \tag{45}$$

where $c_\flat$ is the constant defined in (14). Next, we distinguish between two cases.
1) $|\tilde\xi| \le \ln(2)$. Then (45) gives

$$\begin{aligned}\dot U_c(t) \le& -\left(\tfrac{\chi}{2}+\beta e^{\tilde\xi+\xi_\star}\right)\tilde S^2 - \left[\tfrac{c\psi_\star\chi}{\beta}\tilde\xi\left(e^{\tilde\xi}-1\right)+c\psi_\star e^{\xi_\star}\left(e^{\tilde\xi}-1\right)^2\right]\\ &+\left[\tfrac{c\chi}{\beta}\ln(2)+ce^{\xi_\star}\right]|\delta_1| + c_\flat\delta_1^2.\end{aligned} \tag{46}$$

2) $|\tilde\xi| \ge \ln(2)$. Then $|e^{\tilde\xi}-1| \ge \frac{1}{2}$. Consequently, since

$$\tilde\xi(e^{\tilde\xi}-1) = |\tilde\xi||e^{\tilde\xi}-1|, \tag{47}$$

we can use (45) to get

$$\begin{aligned}\dot U_c(t) \le& -\left(\tfrac{\chi}{2}+\beta e^{\tilde\xi+\xi_\star}\right)\tilde S^2 - \tfrac{1}{2}\left[\tfrac{c\psi_\star\chi}{\beta}\tilde\xi\left(e^{\tilde\xi}-1\right)+c\psi_\star e^{\xi_\star}\left(e^{\tilde\xi}-1\right)^2\right]\\ &-\tfrac{1}{4}\left(\tfrac{c\psi_\star\chi}{\beta}|\tilde\xi|+c\psi_\star e^{\xi_\star}\left|e^{\tilde\xi}-1\right|\right)+\left(\tfrac{c\chi}{\beta}|\tilde\xi|+ce^{\xi_\star}\left|e^{\tilde\xi}-1\right|\right)|\delta_1|\\ &+c_\flat\delta_1^2.\end{aligned} \tag{48}$$

From (38), we deduce that in case 2), we have

$$\dot{U}_c(t) \leq -\left(\tfrac{\chi}{2}+\beta e^{\tilde{\xi}+\xi_\star}\right)\tilde{S}^2 - \tfrac{1}{2}\left[\tfrac{c\psi_\star\chi}{\beta}\tilde{\xi}\left(e^{\tilde{\xi}}-1\right)+c\psi_\star e^{\xi_\star}\left(e^{\tilde{\xi}}-1\right)^2\right] + c_\flat\delta_1^2. \tag{49}$$

We deduce that in both cases, $\dot{U}_c(t) \leq -\frac{1}{2}W_c(\tilde{\xi}(t),\tilde{S}(t)) + c_\sharp|\delta_1(t)| + c_\flat\delta_1(t)^2$. It follows that (39) is satisfied when $\delta_2 = 0$. To check that (39) is also satisfied when δ_2 is not necessarily 0, it suffices to notice that $\partial U_c/\partial\tilde{S} = -J_c$. Finally, Lemma 4.3 ensures that (40) is satisfied. □

4.2. SIR Model.

We next consider the more sophisticated model

$$\dot{S}(t) = B + \epsilon(t) - \rho(t)S(t) - \mu S(t) - \beta I(t)S(t), \tag{50a}$$

$$\dot{I}(t) = \beta S(t)I(t) - (\gamma+\mu)I(t), \tag{50b}$$

$$\dot{R}(t) = \gamma I(t) - \mu R(t) + \rho(t)S(t). \tag{50c}$$

where S, I and R are valued in $(0,+\infty)$, and where B, μ, β, and γ are positive constants. The variables S and I and constants have the same interpretations as in the preceding subsections, and R is the number of recovered or resistant individuals, as a result of mass vaccination of susceptible individuals. The piecewise continuous locally bounded function ϵ represents uncertainty as before.

Let the vaccination rate $\rho(t)$ be represented by

$$\rho(t) = \hat{\rho} + u(t), \tag{51}$$

where $\hat{\rho} \geq 0$ is a constant, and the control u satisfies $u(t) \in [-\hat{\rho},+\infty)$ for all $t \geq 0$. We assume that the analog

$$\beta B > (\hat{\rho}+\mu)(\gamma+\mu) \tag{52}$$

of (26) is satisfied. We also use the notation $\lambda = \gamma+\mu$ and $\chi = \hat{\rho}+\mu$ and $S_\star$ and $\xi_\star$ from (6) and (7). Let

$$R_\star = \tfrac{1}{\mu}\left(\gamma e^{\xi_\star} + \hat{\rho}S_\star\right). \tag{53}$$

Notice that

$$R_\star = \tfrac{\gamma B}{\mu(\gamma+\mu)} + \tfrac{\hat{\rho}-\gamma}{\beta}. \tag{54}$$

Also, (52) implies that (26) is satisfied. The inequality (52) ensures that with the choices in (6), the componentwise positive vector

$$(S,I,R) = (S_\star,I_\star,R_\star) \tag{55}$$

is the endemic equilibrium for a given constant $B>0$ when ϵ and u are the zero function. Then, with $\tilde{S}$ and $\tilde{\xi}$ defined in the previous section, and with $\tilde{R} = R - R_\star$ and $\psi_\star$ defined as in (7), the reasoning that led to (9) produces the system

$$\dot{\tilde{\xi}}(t) = \beta\tilde{S}(t), \tag{56a}$$

$$\dot{\tilde{S}}(t) = -\left(\chi+\beta e^{\tilde{\xi}(t)+\xi_\star}\right)\tilde{S}(t) + \psi_\star\left(1-e^{\tilde{\xi}(t)}\right) + \epsilon(t) - u(t)S(t), \tag{56b}$$

$$\dot{\tilde{R}}(t) = \gamma e^{\xi_\star}(e^{\tilde{\xi}(t)}-1) - \mu\tilde{R}(t) + \hat{\rho}\tilde{S}(t) + S(t)u(t) \tag{56c}$$

with $\tilde{\xi}$ valued in $\mathbb{R}$, and with $\tilde{S}(t) \in (-S_\star,+\infty)$ and $\tilde{R}(t) \in (-R_\star,+\infty)$ for all $t \geq 0$.

As in the preceding subsection, we assume that $\epsilon(t)$ is a piecewise continuous function that is valued in $\mathcal{S} = [-\min\{B,\psi_\star/4\},\psi_\star/4]$ which ensures the forward invariance of the state space as before. In terms of the notation from (14), the function U_c from (15), the function J_c from (16), and the functions

$$F_1(\tilde{\xi}, \tilde{S}, \tilde{R}) = \frac{1}{2}\left[\tilde{S} + e^{\xi_\star}(e^{\tilde{\xi}} - 1) + \tilde{R}\right]^2 \tag{57}$$

and

$$W_{U,c}(\tilde{\xi}, \tilde{S}, \tilde{R}) = \sqrt{k_4 + \frac{1}{k_3}U_c(\tilde{\xi}, \tilde{S})} - \sqrt{k_4} + \mu F_1(\tilde{\xi}, \tilde{S}, \tilde{R}), \tag{58}$$

we then have the following analog of Theorem 3.1 for the SIR dynamics, which implies the ISS property of (56) on its state space $\mathcal{X}_U = \mathbb{R} \times (-R_\star, +\infty) \times (-S_\star, +\infty)$ when ϵ is restricted to $\mathcal{S} = [-\min\{B, \psi_\star/4\}, \psi_\star/4]$ and when u satisfies the requirements of part (b) of the lemma, and where the class of feasible controls u satisfying (21) depends on the parameter $c > 0$:

Lemma 4.5. *The following conclusions hold: (a) The time derivative of the function*

$$V_{U,c}(\tilde{\xi}, \tilde{S}, \tilde{R}) = U_c(\tilde{\xi}, \tilde{S}) + F_1(\tilde{\xi}, \tilde{S}, \tilde{R}) \tag{59}$$

along all trajectories of (56) satisfies

$$\begin{aligned}\dot{V}_{U,c}(t) \leq & - W_{U,c}(\tilde{\xi}(t), \tilde{S}(t), \tilde{R}(t)) + J_c(\tilde{\xi}(t), \tilde{S}(t))S(t)u(t) \\ & + c_\sharp|\epsilon(t)| + \left(c_\flat + \tfrac{1}{2\mu}\right)\epsilon(t)^2\end{aligned} \tag{60}$$

for all $t \geq 0$. (b) For any choice of the control u such that (21) is satisfied for all $t \geq 0$, the function $V_{U,c}$ is an ISS Lyapunov function for (56) on its state space

$$\mathcal{X}_U = \mathbb{R} \times (-R_\star, +\infty) \times (-S_\star, +\infty) \tag{61}$$

for the disturbance set $\mathcal{S} = [-\min\{B, \psi_\star/4\}, \psi_\star/4]$.

Proof. We deduce from (40) (applied with $\delta_1 = \epsilon$ and $\delta_2 = -uS$) and the definition of J_c in (16) that

$$\dot{U}_c(t) \quad \leq \quad -\left(\sqrt{k_4 + \tfrac{1}{k_3}U_c(\tilde{\xi}, \tilde{S})} - \sqrt{k_4}\right) + J_c(\tilde{\xi}, \tilde{S})Su + c_\sharp|\epsilon| + c_\flat\epsilon^2. \tag{62}$$

On the other hand, since $\hat{\rho} - \chi = \gamma - \lambda = -\mu$, it follows from our formula $\psi_\star = \lambda e^{\xi_\star}$ that with the choice

$$\tilde{S}^\sharp = \tilde{S} + e^{\xi_\star}(e^{\tilde{\xi}} - 1) + \tilde{R}, \tag{63}$$

we have

$$\begin{aligned}\dot{F}_1(t) \quad &= \quad \tilde{S}^\sharp\left[-\left(\chi + \beta e^{\tilde{\xi}+\xi_\star}\right)\tilde{S} + \psi_\star\left(1 - e^{\tilde{\xi}}\right) - uS + \beta e^{\xi_\star}e^{\tilde{\xi}}\tilde{S}\right. \\ & \qquad \left. + \gamma e^{\xi_\star}(e^{\tilde{\xi}} - 1) - \mu\tilde{R} + \hat{\rho}\tilde{S} + Su\right] + \tilde{S}^\sharp\epsilon \\ &= \quad \tilde{S}^\sharp\left[(\hat{\rho} - \chi)\tilde{S} + \left(\gamma e^{\xi_\star} - \psi_\star\right)(e^{\tilde{\xi}} - 1) - \mu\tilde{R}\right] + \tilde{S}^\sharp\epsilon \\ &= \quad -\mu\left[\tilde{S} + e^{\xi_\star}(e^{\tilde{\xi}} - 1) + \tilde{R}\right]^2 + \left[\tilde{S} + e^{\xi_\star}(e^{\tilde{\xi}} - 1) + \tilde{R}\right]\epsilon \\ &\leq \quad -\tfrac{\mu}{2}\left[\tilde{S} + e^{\xi_\star}(e^{\tilde{\xi}} - 1) + \tilde{R}\right]^2 + \tfrac{1}{2\mu}\epsilon^2 \quad = \quad -\mu F_1(\tilde{\xi}, \tilde{S}, \tilde{R}) + \tfrac{1}{2\mu}\epsilon^2,\end{aligned} \tag{64}$$

where the last inequality in (64) used Young's inequality. It follows from adding (62) and (64) that conclusion (a) of the lemma holds.

To check part (b) of the lemma, note that our formulas (14) give

$$\begin{aligned} W_{U,c}(\tilde{\xi},\tilde{S},\tilde{R}) &\geq \sqrt{k_4 + \frac{1}{k_3}U_c(\tilde{\xi},\tilde{S}) + 2\mu\sqrt{k_4}F_1(\tilde{\xi},\tilde{S},\tilde{R})} - \sqrt{k_4} \\ &\geq \sqrt{k_4 + \chi_0 V_{U,c}(\tilde{\xi},\tilde{S},\tilde{R})} - \sqrt{k_4}, \end{aligned} \tag{65}$$

where

$$\chi_0 = \min\left\{\tfrac{1}{k_3}, 2\mu\sqrt{k_4}\right\}. \tag{66}$$

The function $V_{U,c}$ is positive definite and radially unbounded. Thus $V_{U,c}$ is a strict Lyapunov function for the system (56) on its state space when ϵ and u are zero, from which an ISS inequality can be deduced when u satisfies the requirements of part (b) of the lemma. This completes the proof of part (b) of the lemma. □

Remark 3. The added function F_1 in the formula (59) for $V_{U,c}$ was used to transform the strict Lyapunov function U_c for the lower dimensional system (28) into a strict Lyapunov function for (56). This is necessary because U_c is not a proper positive definite function of the state of the three-dimensional system (56), and also because the time derivative of U_c lacks the required negative definiteness requirement for (56), because the right side of the decay condition (62) for U_c could be zero without $\tilde{R}$ being zero. Therefore, U_c lacks the two basic properties for being a strict Lyapunov function for (56) when ϵ and u are zero, namely, the shape requirement (of being a proper and positive definite function of the three-dimensional state) and the decay condition (on its time derivative along solutions of (56)). On the other hand, when $u = 0$, the sum of the right sides of (62) and (64) can only be zero when all components of $(\tilde{\xi},\tilde{S},\tilde{R})$ are zero. Therefore, adding the function F_1 to U_c plays the dual role of providing the required proper and positive definiteness conditions for $V_{U,c}$ while also acting as the auxiliary function in the Matrosov approach to strict Lyapunov function constructions (e.g., from [21]), by providing the required negative definiteness of the decay condition on the strict Lyapunov function $V_{U,c}$. From (50) it is clear that the increase of the total population $S+I+R$ is exactly the inflow $B+\epsilon(t)$, and the decrease is the outflow $-\mu(S+I+R)$ since (50) is a compartmental system inheriting conservation. The function F_1 makes use of the conserved variable in terms of the deviation from the equilibrium.

4.3. Proof of Theorem 3.1.

To simplify, we first consider the case of (9) where u is the zero function. Let us introduce the functions

$$\tilde{\kappa} = \tilde{S} + e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right) + \tilde{Q} + \tilde{R},\ F_2(\tilde{\kappa}) = \tfrac{1}{2}\tilde{\kappa}^2, \text{ and } F_3(\tilde{Q}) = \tfrac{1}{2}\tilde{Q}^2. \tag{67}$$

Since $\hat{\rho} - \chi = -\mu$ and $-\lambda + \nu + \gamma = -\mu$ and $\psi_\star = \lambda e^{\xi_\star}$, simple calculations give $\dot{\tilde{\kappa}}(t) = -\mu\tilde{\kappa}(t) + \epsilon(t)$. Here and in the sequel, all equalities and inequalities are along solutions of (9) for all $t \geq 0$. Hence,

$$\dot{F}_2(t) = -\mu\tilde{\kappa}^2 + \tilde{\kappa}\epsilon \leq -\tfrac{\mu}{2}\tilde{\kappa}^2 + \tfrac{1}{2\mu}\epsilon^2 \text{ and} \tag{68}$$

$$\dot{F}_3(t) = -(\tau+\mu)\tilde{Q}^2 + \nu e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right)\tilde{Q} \leq -\tfrac{\tau+\mu}{2}\tilde{Q}^2 + \tfrac{\nu^2 e^{2\xi_\star}}{2(\tau+\mu)}\left(e^{\tilde{\xi}} - 1\right)^2 \tag{69}$$

follow from Young's inequality. Now, we observe that

$$V_c(\tilde{\xi},\tilde{S},\tilde{Q},\tilde{R}) = U_c(\tilde{\xi},\tilde{S}) + gF_2(\tilde{\kappa}) + c_\diamond F_3(\tilde{Q}). \tag{70}$$

Since the $(\tilde{\xi},\tilde{S})$ dynamics in (9) is the same as (28) when $u=0$ (with ϵ in (9) replaced by δ in (28)), it follows from from (39) (with $\delta_2=0$ and $\delta_1=\epsilon$), (68) and (69) that

$$\begin{aligned}
\dot{V}_c(t) \leq & -\tfrac{1}{2}W_c(\tilde{\xi},\tilde{S}) + c_\sharp|\epsilon| + c_\flat\epsilon^2 - \tfrac{g\mu}{2}\tilde{\kappa}^2 + \tfrac{g}{2\mu}\epsilon^2 \\
& - c_\diamondsuit\tfrac{\tau+\mu}{2}\tilde{Q}^2 + c_\diamondsuit\tfrac{\nu^2 e^{2\xi_\star}}{2(\tau+\mu)}\left[e^{\tilde{\xi}}-1\right]^2 \\
= & -\left(\tfrac{\chi}{2}+\tfrac{\beta}{2}e^{\tilde{\xi}+\xi_\star}\right)\tilde{S}^2 - \tfrac{c\psi_\star}{2}\left[\tfrac{\chi}{\beta}\tilde{\xi} + e^{\xi_\star}\left(e^{\tilde{\xi}}-1\right)\right]\left(e^{\tilde{\xi}}-1\right) - \tfrac{g\mu}{2}\tilde{\kappa}^2 \\
& - c_\diamondsuit\tfrac{\tau+\mu}{2}\tilde{Q}^2 + c_\diamondsuit\tfrac{\nu^2 e^{2\xi_\star}}{2(\tau+\mu)}\left(e^{\tilde{\xi}}-1\right)^2 + c_\sharp|\epsilon| + \left(c_\flat+\tfrac{g}{2\mu}\right)\epsilon^2 \\
\leq & -\left(\tfrac{\chi}{2}+\tfrac{\beta}{2}e^{\tilde{\xi}+\xi_\star}\right)\tilde{S}^2 - \tfrac{c\psi_\star}{2}\left[\tfrac{\chi}{\beta}\tilde{\xi} + \tfrac{e^{\xi_\star}}{2}\left(e^{\tilde{\xi}}-1\right)\right]\left(e^{\tilde{\xi}}-1\right) - \tfrac{g\mu}{2}\tilde{\kappa}^2 \\
& - c_\diamondsuit\tfrac{\tau+\mu}{2}\tilde{Q}^2 + c_\sharp|\epsilon| + \left(c_\flat+\tfrac{g}{2\mu}\right)\epsilon^2 \\
\leq & -\tfrac{1}{4}W_c(\tilde{\xi},\tilde{S}) - \tfrac{g\mu}{2}\tilde{\kappa}^2 - c_\diamondsuit\tfrac{\tau+\mu}{2}\tilde{Q}^2 + c_\sharp|\epsilon| + \left(c_\flat+\tfrac{g}{2\mu}\right)\epsilon^2,
\end{aligned} \tag{71}$$

where the second to last inequality followed from our formula for $c_\diamondsuit$ from (14). Hence, Lemma 4.3 gives

$$\begin{aligned}
\dot{V}_c(t) \leq & -\tfrac{1}{2}\left[\sqrt{k_4+\tfrac{1}{k_3}U_c(\tilde{\xi},\tilde{S})}-\sqrt{k_4}\right] - \tfrac{g\mu}{2}\tilde{\kappa}^2 - c_\diamondsuit\tfrac{\tau+\mu}{2}\tilde{Q}^2 + c_\sharp|\epsilon| \\
& + \left(c_\flat+\tfrac{g}{2\mu}\right)\epsilon^2 \\
\leq & -\tfrac{1}{2}\left[\sqrt{k_4+\tfrac{1}{k_3}U_c(\tilde{\xi},\tilde{S})+4\sqrt{k_4}\left(\tfrac{g\mu}{2}\tilde{\kappa}^2+c_\diamondsuit\tfrac{\tau+\mu}{2}\tilde{Q}^2\right)}-\sqrt{k_4}\right] \\
& + c_\sharp|\epsilon| + \left(c_\flat+\tfrac{g}{2\mu}\right)\epsilon^2,
\end{aligned} \tag{72}$$

where the second inequality in (72) used the relation

$$\sqrt{k_4+s}+r \geq \sqrt{k_4+s+2\sqrt{k_4}r} \tag{73}$$

for suitable nonnegative values of r and s. It follows that (19) is satisfied when $u=0$. Therefore, due to the way u enters the dynamics (9), the general case of the theorem where u is not necessarily the zero function follows because our choices of J_c and V in (16) and (18) give

$$\tfrac{\partial V_c}{\partial \tilde{R}}(\tilde{\xi},\tilde{S},\tilde{Q},\tilde{R}) - \tfrac{\partial V_c}{\partial \tilde{S}}(\tilde{\xi},\tilde{S},\tilde{Q},\tilde{R}) = -\tfrac{\partial U_c}{\partial \tilde{S}}(\tilde{\xi},\tilde{S}) = J_c(\tilde{\xi}(t),\tilde{S}(t)). \tag{74}$$

This allows us to conclude.

Remark 4. An alternative expression of (5) is $R_0>1$, where

$$R_0 = \frac{\beta B}{(\hat{\rho}+\mu)(\gamma+\nu+\mu)} \tag{75}$$

is called the basic reproduction number [14]. The condition (5) in Theorem 3.1 has no conservativeness since this strict inequality is not only necessary for the component-wise positiveness of the equilibrium (6), but is also needed for the local asymptotic stability of the equilibrium via Jacobian analysis. For the SIRQ model (1), the function

$$F_2(\tilde{\kappa}) = \tfrac{1}{2}\tilde{\kappa}^2 = \tfrac{1}{2}\left[\tilde{S}+e^{\xi_\star}\left(e^{\tilde{\xi}}-1\right)+\tilde{Q}+\tilde{R}\right]^2$$

used in the formula (18) for the strict Lyapunov function V_c replaces the role F_1 played in adding R for the lower dimensional SIR model (50). For obtaining a strict Lyapunov function of the four state variables for (1), adding the two variables R and Q to the function U_c cannot be completed by the single function F_2. This is why the new function $F_3(\tilde{Q}) = \frac{1}{2}\tilde{Q}^2$ is also incorporated into the construction process for the strict Lyapunov function (18) by using a cascade argument. Since the SI dynamics drives Q, the time derivative of F_3 is allowed to consume a portion of the decay provided by U_c appropriately, as seen in (69) and (71).

5. Comparison of Controlled and Uncontrolled Cases.

In addition to providing robustness to model uncertainty through ISS, Lemma 4.5 (for the SIR model) and Theorem 3.1 (for the full model with quarantines and vaccination) provide a framework for comparing the performance of different possible controls u, namely, different choices of u's that satisfy the ISS requirements from (21) in part (b) of each of the two results. We next illustrate this point for the full model (1), but analogous reasoning applies to the special case of the SIR model (50).

Consider the feedback control law

$$u = \max\left\{-\hat{\rho}, -\omega S J_c(\tilde{\xi}, \tilde{S})\right\} \tag{76}$$

which depends on time through its dependence on the states $\tilde{\xi}$ and $\tilde{S}$, for constants $\omega \geq 0$ and $\hat{\rho} > 0$, which satisfies the requirements from part (b) of Theorem 3.1. The choice $\omega = 0$ removes the control u from the vaccination ρ in dynamics (1). In Figures 1-3, we compare the performance of (1) using $u = 0$ (in Figure 1), the control (76) with $\omega = 0.01$ and $c = 1$ (in Fig. 2), and (76) with $\omega = 0.1$ and $c = 0.1$ (in Fig. 3). In each case, we chose $\epsilon = 0$, $\beta = 0.45/6.5$, $\mu = 0.000034$, $\gamma = 0.0416$, $B = 221 \times 10^{-6}$, $\tau = 0.0454$, $\nu = 0.03$ and $\hat{\rho} = 0.0001$. The peak of the infected population I is reduced by (76) when $\omega > 0$, so this illustrates the value of our feedback control. The two controlled cases differ in the coefficient of $\tilde{S}$ appearing in the J_c formula from (76), according to (16). The more the susceptible individuals are removed when the population is large, the smaller the population of infected individuals becomes. The guarantee we proved is global, as illustrated by the convergence in Fig. 4 which is computed for a different set of initial populations. For the same parameters and the initial populations as in Figs. 1-3, simulations are performed and plotted in Fig. 5 for the non-zero immigration/newborn perturbation $\epsilon(t) = -20 \times 10^{-6}\cos(\pi t/150)$ million, which satisfies $\epsilon(t) \in [-\min\{B, \psi_\star/4\}, \psi_\star/4]$ in Theorem 3.1. With the 20% increase of immigrants (from $B + \epsilon(0) = 201 \times 10^{-6}$ to $B + \epsilon(150) = 241 \times 10^{-6}$), the reduction of the infection peak with (76) with $\omega = 0.1$ and $c = 0.1$ is larger than the reduction with the other two control inputs.

Remark 5. The values for the parameters μ, γ, B, and τ we chose above were based on the data reported in [7] for the outbreak of SARS in 2003, by combining four variables into two variables and incorporating the disease-associated death in the population R. The data is for Hong Kong, which has a population of 6.5 million. In the simulations, the unit of population is in millions, and the time t is in days. In our simulations, the transmission rate β triples 0.15/6.5 in [7] so that the basic reproduction number is increased to 6.282 since the transmission rate of COVID-19 has been reported as large as 7 or even higher numbers [19, 28, 30]. The initial conditions used in Fig. 1-4 are the populations obtained by simulation 25 days after March 1, 2003 which was the initial time in [7]. In other words, Figures 1-4 simulate

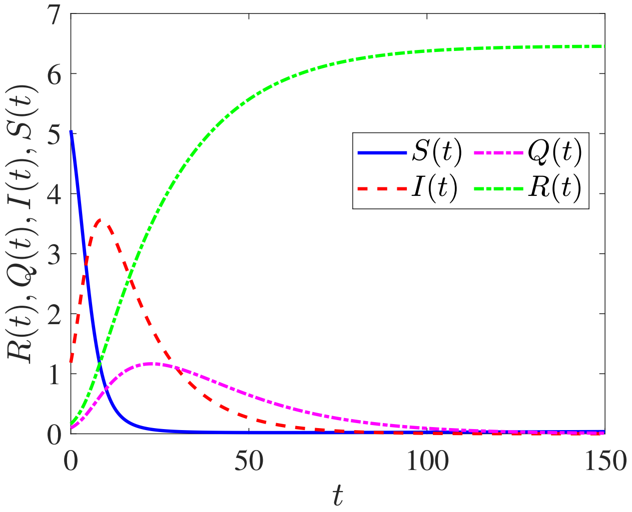

FIGURE 1. Populations of (1) with $u = 0$ and $\hat{\rho} = 0.0001$.

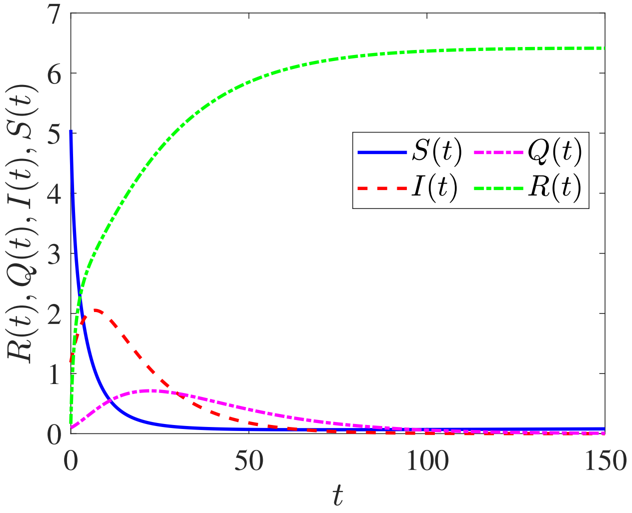

FIGURE 2. Populations of (1) under the control (76) with $\omega = 0.01$, $c = 1$, and $\hat{\rho} = 0.0001$.

the vaccine administration starting 30 days after the emergence of the disease in the region.

6. **Conclusion.** We provided new global strict Lyapunov function constructions for an SIR model that also includes quarantine and vaccination. Since our strict Lyapunov functions were also ISS Lyapunov functions, this made it possible to prove ISS properties with respect to piecewise continuous locally bounded uncertainties, under suitable bounds on the uncertainties. The ISS robustness property was beyond the scope of prior treatments of SIR models that did not provide ISS Lyapunov functions or that led to discontinuous feedbacks. Our strict Lyapunov

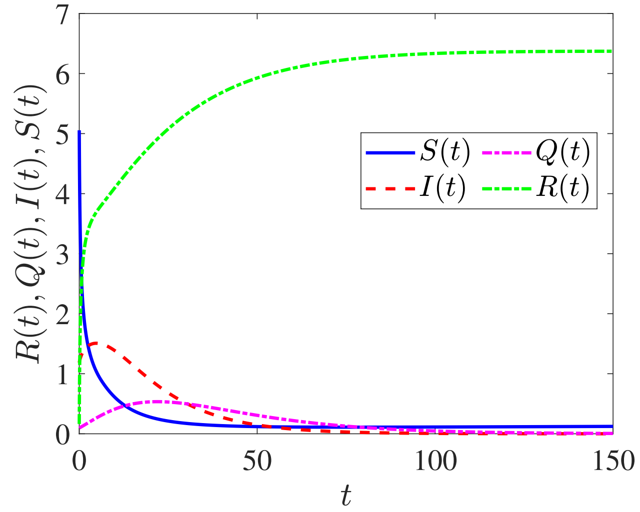

FIGURE 3. Populations of (1) under the control (76) with $\omega = 0.1$, $c = 0.1$, and $\hat{\rho} = 0.0001$.

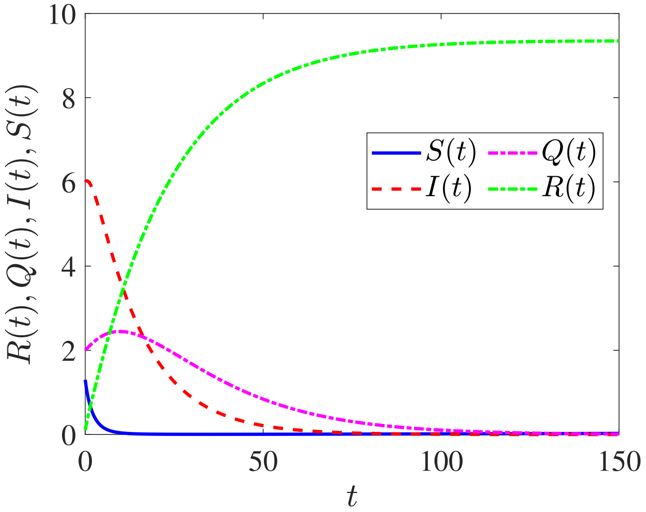

FIGURE 4. Populations of (1) under the control (76) with $\omega = 0.1$, $c = 0.1$, and $\hat{\rho} = 0.0001$.

function constructions also made it possible to directly design feedback controllers, and our simulations illustrated how nonzero choices of the feedback controls can have beneficial effects by reducing the peak infection levels. Our stepwise construction of ISS Lyapunov functions directly provided reasonable controllers that are independent of downstream populations and allowed us to concentrate only on susceptible and infected populations in achieving the ISS guarantee involving all four populations. In future work, we will study the effects of input delays [4, 13, 22] in our feedback controls, as well as delay compensation based on exact predictors, chain predictors [3, 5], or other dynamic extensions [25].

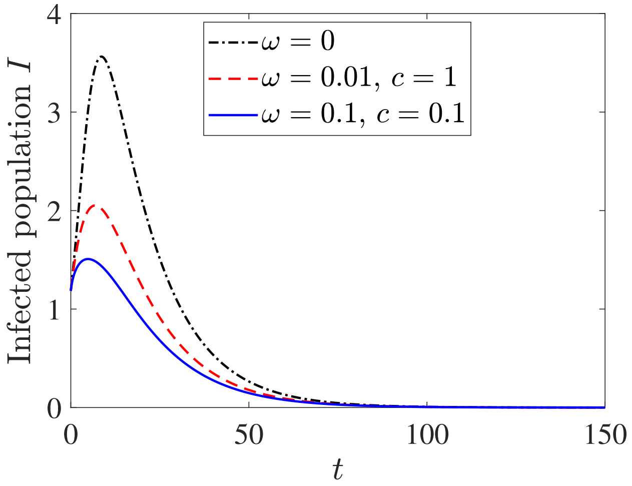

FIGURE 5. Infected population of (1) with three different controls in the presence of perturbation $\epsilon(t)$.

Appendix: Proofs of Lemmas 4.1-4.3. We provide proofs of our Lemmas 4.1-4.3 which we used to prove our result for the SI case in Section 4.1.

Proof of Lemma 4.1. The time derivative of V_1 along the trajectories of the system (28) satisfies

$$\dot{V}_1(t) = \tilde{S}\left[-\left(\chi + \beta e^{\tilde{\xi}+\xi_\star}\right)\tilde{S} + \psi_\star\left(1 - e^{\tilde{\xi}}\right)\right] + \frac{\psi_\star}{\beta}\left(e^{\tilde{\xi}} - 1\right)\beta\tilde{S} + \tilde{S}\delta. \tag{A.1}$$

We deduce that the equality (32) is satisfied. Next, let

$$\varpi = \tilde{S} + \frac{\chi}{\beta}\tilde{\xi} + e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right). \tag{A.2}$$

Then (28) can be rewritten as

$$\dot{\tilde{\xi}}(t) = \beta\left[\varpi(t) - \left(\frac{\chi}{\beta}\tilde{\xi}(t) + e^{\xi_\star}\left(e^{\tilde{\xi}(t)} - 1\right)\right)\right], \tag{A.3a}$$

$$\begin{aligned}\dot{\varpi}(t) = &-\left(\chi + \beta e^{\tilde{\xi}(t)+\xi_\star}\right)\tilde{S}(t) + \psi_\star\left(1 - e^{\tilde{\xi}(t)}\right)\\ &+ \frac{\chi}{\beta}\dot{\tilde{\xi}}(t) + e^{\tilde{\xi}(t)+\xi_\star}\dot{\tilde{\xi}}(t) + \delta(t),\end{aligned} \tag{A.3b}$$

which we rewrite as

$$\dot{\varpi}(t) = \psi_\star\left(1 - e^{\tilde{\xi}(t)}\right) + \delta(t), \tag{A.4a}$$

$$\dot{\tilde{\xi}}(t) = -\left[\chi\tilde{\xi}(t) + \beta e^{\xi_\star}\left(e^{\tilde{\xi}(t)} - 1\right)\right] + \beta\varpi(t) \tag{A.4b}$$

by using the fact that $\dot{\tilde{\xi}} = \beta\tilde{S}$. Using the equalities (A.4) and recalling that

$$V_2(\tilde{\xi}, \tilde{S}) = \frac{1}{2}\varpi^2 + \frac{\psi_\star}{\beta}\left(e^{\tilde{\xi}} - 1 - \tilde{\xi}\right) \tag{A.5}$$

we can easily prove that the time derivative of the function V_2 along the trajectories of (28) satisfies

$$\begin{aligned}\dot{V}_2(t) \;=&\; \left[\tilde{S} + \tfrac{\chi}{\beta}\tilde{\xi} + e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right)\right]\psi_\star\left(1 - e^{\tilde{\xi}}\right)\\ &- \tfrac{\psi_\star}{\beta}\left(e^{\tilde{\xi}} - 1\right)\left[\chi\tilde{\xi} + \beta e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right)\right]\\ &+ \tfrac{\psi_\star}{\beta}\left(e^{\tilde{\xi}} - 1\right)\beta\left[\tilde{S} + \tfrac{\chi}{\beta}\tilde{\xi} + e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right)\right]\\ &+ \left[\tilde{S} + \tfrac{\chi}{\beta}\tilde{\xi} + e^{\xi_\star}\left(e^{\tilde{\xi}} - 1\right)\right]\delta \end{aligned} \tag{A.6}$$

for all $t \geq 0$. We deduce that the equality (33) is satisfied.

Proof of Lemma 4.2. For all $(a, b) \in \mathbb{R}^2$ such that $|a| \geq \ln(2)$, the inequality

$$W_c(a, b) \geq \tfrac{c\psi_\star\chi}{\beta} a\left(e^a - 1\right) \geq \tfrac{c\psi_\star\chi}{2\beta}|a| \tag{A.7}$$

is satisfied. On the other hand, for all $(a, b) \in \mathbb{R}^2$ such that $|a| \leq \ln(2)$, we have $|e^a - 1| \geq \frac{1}{2}|a|$. Hence,

$$W_c(a, b) \geq c\psi_\star\left(\tfrac{\chi}{\beta}|a| + \tfrac{e^{\xi_\star}}{2}|a|\right)\tfrac{1}{2}|a| = \tfrac{c\psi_\star}{2}\left(\tfrac{\chi}{\beta} + \tfrac{e^{\xi_\star}}{2}\right)a^2 \tag{A.8}$$

in this case. From (A.7) and (A.8), we deduce that (34) holds.

Proof of Lemma 4.3. From (15), we deduce that for all $(a, b) \in \mathbb{R}^2$, we have

$$\begin{aligned}U_c(a, b) \;&\leq \tfrac{1+2c}{2}b^2 + c\left[\tfrac{\chi}{\beta}a + e^{\xi_\star}\left(e^a - 1\right)\right]^2 + \tfrac{(c+1)\psi_\star}{\beta}\left(e^a - 1 - a\right)\\ &\leq \tfrac{1+2c}{2}b^2 + \tfrac{2c\chi^2}{\beta^2}a^2 + 2ce^{2\xi_\star}\left(e^a - 1\right)^2 + \tfrac{(c+1)\psi_\star}{\beta}\left(e^a - 1 - a\right), \end{aligned} \tag{A.9}$$

by applying the relation $(q + m)^2 \leq 2q^2 + 2m^2$ for suitable real values of q and m. Hence, using the relation

$$e^a - 1 - a = \textstyle\int_0^a (e^\ell - 1)\mathrm{d}\ell \leq a\left(e^a - 1\right) \leq \tfrac{1}{2}a^2 + \tfrac{1}{2}\left(e^a - 1\right)^2 \tag{A.10}$$

(which follows, e.g., by separately considering the cases $a \geq 0$ and $a < 0$) we obtain

$$\begin{aligned}U_c(a, b) \;&\leq \frac{1+2c}{2}b^2 + \frac{2c\chi^2}{\beta^2}a^2 + 2ce^{2\xi_\star}\left(e^a - 1\right)^2\\ &\quad + \tfrac{(c+1)\psi_\star}{2\beta}a^2 + \tfrac{(c+1)\psi_\star}{2\beta}\left(e^a - 1\right)^2\\ &= \frac{1+2c}{2}b^2 + \left[2ce^{2\xi_\star} + \frac{(c+1)\psi_\star}{2\beta}\right]\left(e^a - 1\right)^2\\ &\quad + \left[\tfrac{2c\chi^2}{\beta^2} + \tfrac{(c+1)\psi_\star}{2\beta}\right]a^2\\ &\leq \frac{k_1}{2}W_c(a, b) + \left[\frac{2c\chi^2}{\beta^2} + \frac{(c+1)\psi_\star}{2\beta}\right]a^2 \end{aligned} \tag{A.11}$$

with k_1 defined in (11). From (34), we deduce that

$$\begin{aligned}U_c(a, b) \;&\leq \tfrac{k_1}{2}W_c(a, b)\\ &\quad + \left[\tfrac{2c\chi^2}{\beta^2} + \tfrac{(c+1)\psi_\star}{2\beta}\right]\left[\tfrac{4\beta^2}{c^2\psi_\star^2\chi^2}W_c(a, b)^2 + \tfrac{2}{c\psi_\star\left(\frac{\chi}{\beta} + \frac{e^{\xi_\star}}{2}\right)}W_c(a, b)\right]\\ &= \tfrac{k_2}{2}W_c(a, b) + \tfrac{k_3}{4}W_c(a, b)^2 \end{aligned} \tag{A.12}$$

with k_2 and k_3 defined in (12) and (13). It follows that

$$\begin{aligned}\frac{k_2^2}{4k_3} + U_c(a,b) &\leq \tfrac{k_2^2}{4k_3} + \tfrac{k_2}{2} W_c(a,b) + \tfrac{k_3}{4} W_c(a,b)^2 \\ &= k_3 \left(\tfrac{1}{2} W_c(a,b) + \tfrac{k_2}{2k_3} \right)^2 . \end{aligned} \tag{A.13}$$

By our formula for k_4 from (14), we deduce that (35) is satisfied.

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