NETWORKS AND HETEROGENEOUS MEDIA
Website: http://aimSciences.org

Volume **2**, Number **2**, June **2007**
pp. **313–331**

OPTIMAL CONTROL PROBLEMS ON STRATIFIED DOMAINS

Alberto Bressan

Department of Mathematics
Penn State University
University Park, PA 16802, USA

Yunho Hong

Department of Industrial and Manufacturing Engineering
Penn State University, University Park, PA 16802, USA

(Communicated by Benedetto Piccoli)

Abstract. We consider a class of optimal control problems defined on a stratified domain. Namely, we assume that the state space $\mathbb{R}^N$ admits a stratification as a disjoint union of finitely many embedded submanifolds $\mathcal{M}_i$. The dynamics of the system and the cost function are Lipschitz continuous restricted to each submanifold. We provide conditions which guarantee the existence of an optimal solution, and study sufficient conditions for optimality. These are obtained by proving a uniqueness result for solutions to a corresponding Hamilton-Jacobi equation with discontinuous coefficients, describing the value function. Our results are motivated by various applications, such as minimum time problems with discontinuous dynamics, and optimization problems constrained to a bounded domain, in the presence of an additional overflow cost at the boundary.

1. **Introduction.** The theory of viscosity solutions was initially developed in connection with continuous solutions of Hamilton-Jacobi equations, whose coefficients are also continuous.

Various authors have then extended the theory in cases where the value function is discontinuous [2, 15]. In particular, upper or lower solutions to a H-J equation can now be defined within a more general class of semicontinuous functions. In a different direction, motivated by problems in optimal control, sufficient conditions for the optimality of a feedback synthesis have been established in [12], under assumptions that do not require the continuity of the value function.

A further line of investigation, more recently pursued in [14, 5], is the case where the coefficients of the H-J equation are themselves discontinuous. The present paper represents a contribution in this direction, in a specific case. Namely, we study the value function for an infinite-horizon optimal control problem, on a structured domain. The space $\mathbb{R}^N$ is decomposed as the disjoint union of finitely many submanifolds of different dimensions, and we assume that the dynamics of the system as well as the running cost are sufficiently regular when restricted to each given manifold, but may well differ from one manifold to the other.

2000 *Mathematics Subject Classification.* Primary: 49L25; Secondary: 49J24.
Key words and phrases. Hamilton-Jacobi equation, Viscosity solution, Optimal control theory.

More precisely, we assume that there exists a decomposition

$$\mathbb{R}^N = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_M \tag{1}$$

with the following properties. Each $\mathcal{M}_j \subset \mathbb{R}^N$ is an embedded submanifold. If $j \neq k$, then $\mathcal{M}_j \cap \mathcal{M}_k = \emptyset$. In addition, if $\mathcal{M}_j \cap \overline{\mathcal{M}}_k \neq \emptyset$, then $\mathcal{M}_j \subset \overline{\mathcal{M}}_k$, where the upper bar denotes closure.

We call $d_k \doteq \dim(\mathcal{M}_k)$ so that $d_k = 0$ if $\mathcal{M}_k$ consists of a single point and $d_k = N$ if $\mathcal{M}_k$ is an open subset of $\mathbb{R}^N$. For example, in figure 1 we have

$$d_1 = d_2 = 2, \qquad d_3 = d_4 = d_5 = d_6 = 1, \qquad d_7 = d_8 = d_9 = d_{10} = 0.$$

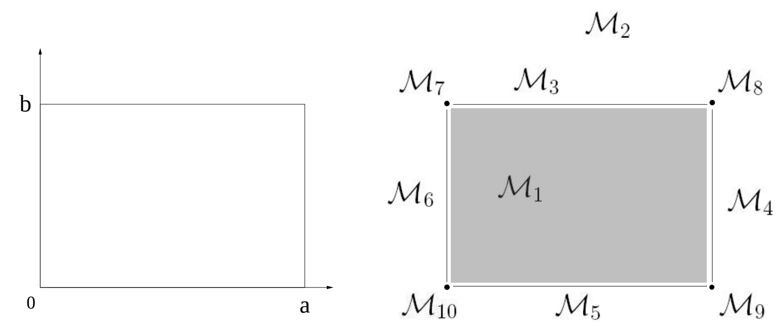

FIGURE 1. A stratification of $\mathbb{R}^2$ induced by a rectangle.

We now consider an optimal control problem with infinite horizon and exponentially discounted cost with $\beta > 0$:

$$\text{minimize:} \qquad J(\bar{x}, \alpha) \doteq \int_0^\infty e^{-\beta t} \ell(x(t), \alpha(t))\, dt \tag{2}$$

for a system with dynamics

$$\dot{x}(t) = f(x(t), \alpha(t)), \qquad\qquad x(0) = \bar{x} \in \mathbb{R}^N. \tag{3}$$

Here $t \mapsto \alpha(t)$ denotes the control function.

The value function is defined as

$$V(\bar{x}) \doteq \inf_{\alpha \in \mathcal{A}} J(\bar{x}, \alpha), \tag{4}$$

where $\mathcal{A}$ is the set of all admissible control functions.

Our key assumption is that both the field f and the cost ℓ are sufficiently regular when restricted to each of the manifolds $\mathcal{M}_j$. More precisely

(H1) For each $i = 1, \ldots, M$ there exists a compact set of controls $A_i \subset \mathbb{R}^m$, a continuous map $f_i : \mathcal{M}_i \times A_i \mapsto \mathbb{R}^N$, and a cost function ℓ_i with the following properties

(a) At each point $x \in \mathcal{M}_i$, all vectors $f_i(x, a)$, $a \in A_i$ are tangent to the manifold $\mathcal{M}_i$.

(b) $|f_i(x, a) - f_i(y, a)| \leq Lip(f_i)\, |x - y|$, for all $x, y \in \mathcal{M}_i$, $a \in A_i$.

(c) Each cost function $\ell_i(x, a)$ is non-negative and continuous.

(d) We have $f(x, a) = f_i(x, a)$ and $\ell(x, a) = \ell_i(x, a)$ whenever $x \in \mathcal{M}_i$, $i = 1, \ldots, M$.

By $Lip(f_i)$ we denote here a Lipschitz constant for the function f_i w.r.t. the first variable. In the following, for any $x \in \mathbb{R}^N$, the index $i(x) \in \{1, \ldots, M\}$ identifies the manifold which contains the point x. In other words,

$$i(x) \doteq k \qquad \text{if} \qquad x \in \mathcal{M}_k .$$

The assumption (d) can now be written as

$$f(x,a) = f_{i(x)}(x,a), \qquad \ell(x,a) = \ell_{i(x)}(x,a), \qquad a \in A_{i(x)} .$$

We recall that the tangent cone $T_{\mathcal{M}_i}(x)$ to the manifold $\mathcal{M}_i$ at the point x is

$$T_{\mathcal{M}_i}(x) \doteq \left\{ y \in \mathbb{R}^N ;\ \lim_{h \to 0} \frac{d(x+hy; \mathcal{M}_i)}{h} = 0 \right\}, \tag{5}$$

where $d(x; \mathcal{M}_i) \doteq \inf_{z \in \mathcal{M}_i} |z - x|$. The tangency condition in (H1-a) can thus be restated as

$$f_i(x,a) \in T_{\mathcal{M}_i}(x) \qquad \forall x \in \mathcal{M}_i ,\ a \in A_i .$$

Since the functions f_i are Lipschitz continuous w.r.t. x and the sets of controls A_i are assumed to be compact, it follows that trajectories of the control system cannot approach infinity in finite time. Indeed, all solutions of (3) satisfy the a-priori bounds

$$\big|\dot{x}(t)\big| \leq C\big(1 + |x(t)|\big), \tag{6}$$

$$\big|x(t)\big| \leq e^{Ct}\big(1 + |x(0)|\big), \tag{7}$$

for some constant C.

In the above setting, our main interest is in the existence of optimal controls, and in the characterization of the value function as the unique solution to the corresponding H-J equation, in an appropriate sense. In Section 3 we discuss a simple example, showing that the standard definition of the viscosity solution is not adequate in the case of discontinuous dynamics and cost functions. Indeed, in addition to the value function, one can now have infinitely many other Lipschitz continuous admissible solutions to the H-J equation. We then show how to modify the definition of a solution, in connection with the stratification (1), in order to uniquely characterize the value function.

In the case of an upper solution v, the comparison result relies on an invariance property of the epigraph of v, as in [15]. To analyze lower solutions, our techniques resemble those used in [7] and [12] to prove the optimality of a regular feedback synthesis. The main technical difficulty encountered here is due to the stratification (1). In particular, the case of an optimal trajectory that enters and exits infinitely many times from the same manifold $\mathcal{M}_i$ cannot be ruled out a priori, and requires a more careful study.

Example 1 (Minimum time problem with discontinuous coefficients). Consider a minimum time problem on $\mathbb{R}^2$, assuming that the speed can be much higher along "highways", described by a finite number of curves in the plane. As admissible velocity sets one can then take, for example

$$F_0(x) = \{y \in \mathbb{R}^2 ;\ |y| \leq c_0(x)\}$$

outside the highways, and

$$F_i(x) = \{y \in \mathbb{R}^2 ;\ y \in T_{\mathcal{M}_i}(x),\ |y| \leq c_i(x)\}$$

along the highway $\mathcal{M}_i$, for some speeds $0 < c_0(x) << c_i(x)$. Given a target point $x^\dagger \in \mathbb{R}^2$, consider the cost function $\ell(x, \cdot) = 1$ if $x \neq x^\dagger$ while $\ell(x^\dagger, \cdot) = 0$. Then value function for the problem

$$\min_{x(\cdot)} \int_0^\infty e^{-\beta t} \ell(x(t), \dot{x}(t))\, dt$$

subject to

$$x(0) = \bar{x}\,, \qquad\qquad \dot{x}(t) \in F(x(t))$$

is given by

$$V(\bar{x}) = \frac{1 - e^{-\beta T(\bar{x})}}{\beta}.$$

Here $T(\bar{x})$ is the minimum time needed to steer the system from $\bar{x}$ to the target point $x^\dagger$. This provides a simple example of a minimum time problem with discontinuous velocities, which can be recast in the form (1).

Example 2 (Optimization problem with reflecting boundary). Consider an open domain $\Omega \subset \mathbb{R}^N$ whose closure consists of finitely many smooth manifolds such as $\bar{\Omega} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_M$. Typically, Ω could be a polytope in $\mathbb{R}^n$. We assume that its dynamics is described by the equation

$$\dot{x}(t) = \pi_{\bar{\Omega}(x)}\big(g(x(t), \alpha(t))\big) = g(x(t), \alpha(t)) - n(x(t), \alpha(t))\,, \qquad x \in \bar{\Omega}, \tag{8}$$

where $n(x(t), \alpha(t))$ is a vector in the outer normal cone $N_{\bar{\Omega}}(x)$ to $\bar{\Omega}$ at the point x, i.e.,

$$N_{\bar{\Omega}}(x) \doteq \Big\{ p \in \mathbb{R}^N\,;\ \langle p, v\rangle \leq 0 \quad \forall v \in T_{\bar{\Omega}}(x) \Big\}.$$

The map $v \mapsto \pi_{\bar{\Omega}(x)}(v)$ here denotes the perpendicular projection of a vector v on the tangent space to $\bar{\Omega}$ at the point $x \in \bar{\Omega}$. The measurable map, $\alpha : [0, \infty) \mapsto A$, is the control function, where A is a compact subset of $\mathbb{R}^m$. The map $g : \bar{\Omega} \times A \mapsto \mathbb{R}^N$ is Lipschitz continuous in the first variable.

In connection with (8), we consider the problem of minimizing a functional of the discounted sum of a running cost plus an additional cost due to the boundary reflection:

$$J(\bar{x}, \alpha) \doteq \int_0^\infty e^{-\beta t} \Big\{ c(x(t), \alpha(t)) + b(x(t), n(x(t), \alpha(t))) \Big\} dt, \tag{9}$$

subject to the initial condition and the constraint

$$x(0) = \bar{x}\,, \qquad\qquad x(t) \in \bar{\Omega}, \qquad \forall t > 0. \tag{10}$$

Let $b(x, 0) = 0$ and $b(x, n) \geq 0$ for $x \in \partial\Omega$, $n \in N_{\bar{\Omega}}(x)$.

In the case of a piecewise smooth boundary, this type of dynamics fits naturally within our framework. Suppose $\Omega = \mathcal{M}_1$ and $\partial\Omega = \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_M$. On the submanifold $\mathcal{M}_1$,

$$F_1(x) = \{\ g(x, a)\ ;\ a \in A\ \}.$$

On $\mathcal{M}_i$, $i = 2, \cdots, M$.

$$F_i(x) = \{\ g(x, a) - n(x, a)\ ;\ a \in A_i\ \}.$$

where $A_i = \{\ a \in A\ ;\ g(x, a) - n(x, a) \in T_{\mathcal{M}_i},\ \forall x \in \mathcal{M}_i\ \}$.

In order to retain the whole space $\mathbb{R}^N$ as the domain for the control system, it suffices to choose a cost $c(x, a)$ very large when $x \notin \bar{\Omega}$. This will force the solution of the optimization problem to remain inside $\bar{\Omega}$ at all times. Notice that the reflecting (or overflow) cost, $b(x(t), n(x(t), \alpha(t))$ is not present at points in the interior of Ω.

The case where boundary reflection occurs at no additional cost, i.e. $b \equiv 0$, has been studied in the literature as the Skorokhod problem [9, 13].

A related differential inclusion. To study certain aspects of the optimization problem, it is convenient to reformulate it as a differential inclusion, leaving aside the parametrization of the velocity sets in terms of the control values.

For each $x \in \mathbb{R}^N$, define the set of admissible velocities

$$F(x) \doteq \Big\{ f_{i(x)}(x,a)\,;\ a \in A_{i(x)} \Big\} \subset \mathbb{R}^N. \tag{11}$$

Sometimes, we will use $F_{i(x)}(x)$ for $F(x)$ in order to show $i(x)$ explicitly. Moreover, define the extended multifunction

$$\widehat{F}(x) \doteq \Big\{ (y,\eta)\,;\ y = f_{i(x)}(x,a)\,,\ \eta \geq \ell_{i(x)}(x,a)\,, \quad a \in A_{i(x)} \Big\} \subset \mathbb{R}^{N+1}\,. \tag{12}$$

Denoting by $\overline{\text{co}}\, S$ the closed convex hull of a set S, we shall also consider the upper semicontinuous, convex-valued regularization

$$G(x) \doteq \bigcap_{\varepsilon>0} \overline{\text{co}} \Big\{ (y,\eta) \in \widehat{F}(x')\,;\ |x'-x| < \varepsilon \Big\} \subset \mathbb{R}^{N+1}\,. \tag{13}$$

To achieve the existence of an optimal control for the problem (2)-(3), we shall use the following assumption.

(H2) For every $x \in \mathbb{R}^N$ one has

$$\Big\{ (y,\eta) \in G(x)\,;\ y \in T_{\mathcal{M}_{i(x)}}(x) \Big\} = \widehat{F}(x)\,. \tag{14}$$

In particular, (H2) implies

(H2′) For each fixed $x \in \mathbb{R}^N$, the set $F(x) \subset \mathbb{R}^N$ is convex. Moreover, the function

$$p \mapsto L(x,p) \doteq \min \left\{ \ell_{i(x)}(x,a)\,;\ f_{i(x)}(x,a) = p, \quad a \in A_{i(x)} \right\},$$

defined for $p \in F(x)$, is convex.

The Hamilton-Jacobi equation. Besides proving the existence of an optimal control, we wish to characterize the value function as the unique solution of the corresponding Hamilton-Jacobi (H-J) equation

$$\beta u(x) + H(x, Du(x)) = 0\,. \tag{15}$$

Here the Hamiltonian function is defined as

$$H(x,p) \doteq \sup_{(f,\eta)\in G(x)} \Big\{ -f \cdot p\, -\, \eta \Big\}. \tag{16}$$

We mention here some relations of the present work with earlier literature. The type of stratified control system which we consider in (H1) is reminiscent of the definition of regular synthesis by Boltyanskii [3] and by Brunovský [4]. However, in their case the stratification referred to the structure of the value function, while here the stratification is a property of the control system. In certain ways, our framework is similar to a hybrid control system [11, 16], where the state can jump within a finite set of manifolds. The main differences here are that (i) the times t_i where the state moves from one manifold to another are determined by the position of the system itself, and not directly by the controller, and (ii) there is no cost associated to the transition from one manifold to another. As a result, an optimal trajectory may well leave and re-enter a given manifold $\mathcal{M}_i$ infinitely many times.

Various studies on H-J equations with discontinuous coefficients have appeared in recent years, due to a growing recognition of the importance of these equations.

Newcomb and Su [10] introduced the Monge solution for an equation of eikonal type

$$H(Du) = n(x),$$

where H is assumed to be convex and $n(x)$ is a positive, measurable function. In [14], Soravia studied a class of optimal control problems with discontinuous Lagrangian. The H-J equations take the special form

$$\beta u(x) + \sup_{a\in A}\{-f(x,a)\cdot Du(x) - h(x,a)\} = g(x), \tag{17}$$

where f and h are locally Lipschitz continuous and g is a Borel measurable function.

In a paper by Camilli and Siconolfi, a general class of H-J equations with measurable coefficients is considered. In [5], they propose a definition of a solution which disregards sets of measure zero. This is very different from our approach, where the form of the control system on submanifolds $\mathcal{M}_j$ of dimension $d_j < N$ (hence of measure zero) plays a key role in the optimization problem.

In Section 2 we prove a theorem on the existence of optimal controls. The main ingredients of the proof are the same as in the standard case, with continuous dynamic and cost functionals. The convexity assumption (H2) here provides the key tool for passing to the limit in a minimizing sequence.

In the remaining sections we seek conditions which imply the optimality of a given trajectory. Toward this goal, in Section 3, we introduce suitable notions of upper and lower solutions to the corresponding H-J equation with discontinuous coefficients (15)-(16), valid in connection with the given stratification. We then prove that the value function V in (4) is an admissible solution. In Section 4, its uniqueness, within the class of admissible solutions, is proved by showing that

$$u(x) \le V(x) \le v(x) \qquad \text{for all } x\in\mathbb{R}^N.$$

where u and v denote respectively a lower and an upper solution. These comparison results require some minimum regularity assumptions. Namely, the value function V should be globally Hölder continuous of exponent $1/2$, and its restriction to each submanifold $\mathcal{M}_k$ should be a.e. differentiable (almost everywhere w.r.t. the d_k-dimensional measure). By Rademacher's theorem, this last condition certainly holds if V is locally Lipschitz continuous in a neighborhood of a.e. point $x\in\mathcal{M}_k$. For example, the function $V(x,y) = \sqrt{|x|} + \sqrt{|y|}$ satisfies the above requirements, for a stratification with $\mathcal{M}_1 = \{(x,y)\,;\ y=0\}$, $\mathcal{M}_2 = \mathbb{R}^2\setminus\mathcal{M}_1$.

2. **Existence of an optimal control.** The aim of this section is to prove a theorem on the existence of optimal controls. This will be achieved by a suitable modification of Filippov's argument [6], to account for the discontinuities in the dynamics and in the cost functions.

Theorem 1. *Consider the optimization problem* (2)*, for the control system* (3) *on a stratified domain. Let the assumptions (H1), (H2) hold. If there exists at least one trajectory having finite cost, then the minimization problem admits an optimal solution.*

Proof. The proof will be given in several steps.

1. (Existence of a minimizing sequence). By assumption, there exists a sequence of admissible controls $\alpha_k(\cdot)$ with corresponding trajectories $x_k(\cdot)$ such that

$$\dot{x}_k(t) = f(x_k(t), \alpha_k(t)), \qquad x_k(0) = \bar{x}$$

$$\lim_{k\to\infty} \int_0^\infty e^{-\beta t}\, \ell(x_k(t), \alpha_k(t))\, dt = \inf_{\alpha\in\mathcal{A}} J(\bar{x}, \alpha) \doteq m < +\infty\,. \tag{18}$$

2. (Compactness $\Longrightarrow$ existence of a convergent subsequence). By the continuity assumption in (H1), the cost function ℓ is locally bounded. We can thus find a continuous function $x \mapsto K^\dagger(x)$ such that

$$\ell(x, a) < K^\dagger(x) \qquad \text{for all } x \in \mathbb{R}^N\,,\ a \in A_{i(x)}\,. \tag{19}$$

Recalling (13), we define the truncated, time dependent multifunction

$$G^\dagger(t, x) \doteq \Big\{(y\,,\ e^{-\beta t}\eta)\,;\ (y, \eta) \in G(x)\,,\ \eta \le K^\dagger(x)\Big\} \subset \mathbb{R}^{N+1}. \tag{20}$$

We observe that $G^\dagger$ is upper semicontinuous with convex, compact values. Define

$$\gamma_k(t) \doteq \int_0^t e^{-\beta s} \ell\big(x_k(s),\, \alpha_k(s)\big)\, ds\,.$$

Then for each $k \ge 1$ the map

$$t \mapsto \big(x_k(t)\,,\ \gamma_k(t)\big)$$

provides a solution to the differential inclusion

$$\frac{d}{dt}(x(t), \gamma(t)) \in G^\dagger(t, x(t))\,, \qquad \big(x(0),\, \gamma(0)\big) = (\bar{x},\, 0). \tag{21}$$

The Lipschitz continuity of the functions f_i, and the compactness of the sets of controls A_i, imply that all solutions of (3) satisfy the a-priori bounds (6), (7). In particular, on any given time interval $[0, T]$, all values $|x_k(t)|$ as well as all derivatives $|\dot{x}_k(t)|$ remain uniformly bounded. Because of (19), the cost functions $\ell(x_k, \alpha_k)$ are also uniformly bounded. By the Ascoli-Arzelà compactness theorem, by possibly taking a subsequence, we can assume the convergence

$$x_k(t) \to x^*(t)\,, \qquad \gamma_k(t) \to \gamma^*(t)$$

for some limit functions $x^*(\cdot)$, $\gamma^*(\cdot)$, uniformly for t in bounded sets.

3. (The limit trajectory is admissible). By the theory of differential inclusions [1], the upper semicontinuity and convexity properties of the multifunction $G^\dagger$ imply that the limit trajectory satisfies

$$\frac{d}{dt}\big(x^*(t),\, \gamma^*(t)\big) \in G^\dagger\big(t,\, x^*(t)\big)\,, \qquad \big(x^*(0),\, \gamma^*(0)\big) = (\bar{x},\, 0).$$

For $i = 1, \ldots, M$, consider the set of times

$$J_i \doteq \Big\{t \ge 0\,;\ x^*(t) \in \mathcal{M}_i\Big\}\,. \tag{22}$$

Each J_i is a Borel measurable subset of the real line. Moreover,

$$\dot{x}^*(t) \in T_{\mathcal{M}_i}(x^*(t)) \qquad \text{for a.e. } t \in J_i\,.$$

We can thus use the assumption (H2$'$) and deduce that, for a.e. time $t \geq 0$,

$$\dot{x}^*(t) \in F(x^*(t)),$$

$$\dot{\gamma}^*(t) \geq \min \left\{ e^{-\beta t} \ell_{i(x^*(t))}\big(x^*(t),\, a\big)\,;\ f_{i(x^*(t))}\big(x^*(t)\,,\ a\big) = \dot{x}^*(t),\ a \in A_{i(x^*(t))} \right\}.$$

4. (The limit trajectory is optimal). By the previous step, and by Filippov's measurable selection theorem [6], we can select control functions $\alpha_i^* : J_i \mapsto A_i$ such that

$$\ell_i(x^*(t), \alpha_i^*(t)) = \min \Big\{ \ell_i(x^*(t), a)\,;\ a \in A_i\,,\ f_i(x^*(t), a) = \dot{x}^*(t) \Big\}$$

for a.e. $t \in J_i$. Defining

$$\alpha^*(t) = \alpha_i^*(t) \qquad\qquad \text{for } t \in J_i\,,$$

we obtain

$$\dot{x}^*(t) = f_{i(x^*(t))}\big(x^*(t),\, \alpha^*(t)\big)\,. \tag{23}$$

Moreover, for every fixed $T > 0$,

$$\int_0^T e^{-\beta t}\, \ell\big(x^*(t), \alpha^*(t)\big)\, dt \leq \gamma^*(T) = \lim_{k\to\infty} \int_0^T e^{-\beta t}\, \ell\big(x_k(t), \alpha_k(t)\big)\, dt \leq m\,.$$

Letting $T \to \infty$ we obtain

$$\int_0^\infty e^{-\beta t}\, \ell\big(x^*(t), \alpha^*(t)\big)\, dt = \sup_{T>0} \int_0^T e^{-\beta t}\, \ell\big(x^*(t), \alpha^*(t)\big)\, dt \leq m\,. \tag{24}$$

Together, (23) and (24) yield the result. □

3. **Viscosity solutions of the Hamilton-Jacobi equation.** For optimal control problems with continuous dynamics, it is well known that the value function provides a solution of a corresponding Hamilton-Jacobi equation, in a viscosity sense [2, 8]. In the remainder of this paper, we show how the definition of the viscosity solution can be adapted to the case of stratified control system, and extend the well known comparison and uniqueness results to this case.

3.1. **Upper and lower solutions.** We now introduce the definitions of upper and lower solution for (15)-(16), relative to the stratified domain (1).

Definition 1. We say that a continuous function w is an **upper solution** of (15)-(16) relative to the stratification (1) if the following holds. If $w - \varphi$ has a local minimum at $\bar{x}$ for some $\varphi \in \mathcal{C}^1$, then

$$\beta w(\bar{x}) + \sup_{(y,\eta)\in G(\bar{x})} \Big\{ -y \cdot D\varphi(\bar{x}) - \eta \Big\} \geq 0. \tag{25}$$

Definition 2. We say that a continuous function w is a **lower solution** of (15)-(16) relative to the stratification (1) if the following condition holds. If $\bar{x} \in \mathcal{M}_i$ and the restriction of $w - \varphi$ to $\mathcal{M}_i$ has a local maximum at $\bar{x}$ for some $\varphi \in \mathcal{C}^1$, then

$$\beta w(\bar{x}) + \sup_{(y,\eta)\in G(\bar{x})} \Big\{ -y \cdot D\varphi(\bar{x}) - \eta \Big\} \leq 0. \tag{26}$$

Definition 3. A continuous function, which is at the same time an upper and a lower solution relative to the stratification (1), will be called a **viscosity solution**.

Notice that, in the definition of a lower solution, we restrict the analysis to the manifold $\mathcal{M}_{i(\bar{x})}$. This is motivated by the following example.

Example 3. Consider the problem of reaching the origin in minimum time, for the system of $\mathbb{R}^2$ described by

$$\frac{d}{dt}(x_1,x_2) \in F(x_1,x_2) \doteq \begin{cases} \big\{(y_1,0)\,;\ |y_1|\le 3\big\} & \text{if } x_2 = 0\,, \\ \big\{(y_1,y_2)\,;\ |y_1|+|y_2|\le 1\big\} & \text{if } x_2 \neq 0\,. \end{cases}$$

In this case, the optimal trajectories are easy to describe: To reach the origin starting from $(\bar{x}_1,\bar{x}_2)$ we first move vertically toward the point $(\bar{x}_1,0)$ with speed 1, then move horizontally to the origin, with speed 3. The minimum time function is thus

$$V(x_1,x_2) = \frac{|x_1|}{3} + |x_2|\,.$$

This provides a viscosity solution on $\mathbb{R}^2\setminus\{0\}$ to the corresponding H-J equation

$$\sup_{y\in F(x)} \big\{ -y\cdot\nabla v(x)\big\} - 1 = 0\,. \tag{27}$$

However, if we use the standard notion of a viscosity solution, then also the function

$$U(x_1,x_2) = \frac{|x_1|}{2} + |x_2|\,.$$

provides a solution. Indeed, at any point $\bar{x} = (\bar{x}_1,0)$ there is no $\mathcal{C}^1$ function φ such that $u-\varphi$ has a local maximum at $\bar{x}$. Therefore, the usual definition of a viscosity subsolution does not pose any requirement at these points.

Recalling Example 1, one checks that the functions

$$\widetilde{V}(x) \doteq 1 - e^{-V(x)}, \qquad\qquad \widetilde{U}(x) \doteq 1 - e^{-U(x)}$$

provide two distinct viscosity solutions (in the standard sense) to the same equation

$$u(x) + \sup_{y\in F(x)} \big\{ -y\cdot\nabla u(x)\big\} - 1 = 0 \tag{28}$$

on $\mathbb{R}^2\setminus\{0\}$. Notice however that $\widetilde{U}$ does not satisfy our present definition of a lower solution.

3.2. **The value function as a viscosity solution.**

Proposition 1. *Consider the optimal control problem* (2)*, for the control system* (3) *on a stratified domain. Let the assumptions (H1), (H2) hold and assume that the value function V is continuous. Then, V is a viscosity solution according to Definition 3.*

Proof. The argument naturally consists of two parts.

1. V **is an upper solution**. Let $\varphi\in\mathcal{C}^1$ and let $\bar{x}$ be a point where $V-\varphi$ attains a local minimum. We can assume that, for some $r>0$,

$$\varphi(\bar{x}) = V(\bar{x})\,, \qquad \varphi(x')\le V(x') \qquad \forall x'\in B(\bar{x},r). \tag{29}$$

Let $t\mapsto\alpha^*(t)$ and $t\mapsto x^*(t)$ be respectively an optimal control and a corresponding optimal trajectory, starting from the point $\bar{x}$. Their existence was proved in Theorem 1. For all $T\ge 0$ we now have

$$V(\bar{x}) = \int_0^T e^{-\beta t}\ell(x^*(t),\alpha^*(t))dt + e^{-\beta T}V(x^*(T))\,, \tag{30}$$

If we had $x^*(t) \in \mathcal{M}_j$ for a fixed index $j \in \{1,\ldots,M\}$ and all $t \in]0,\delta]$, $\delta > 0$, it would now be easy to conclude. However, we must consider the possibility that the optimal trajectory $x^*(\cdot)$ switches infinitely many times between different manifolds $\mathcal{M}_i$. To handle this more general situation, we consider the minimum dimension among these manifolds:

$$d^- \doteq \liminf_{t\to 0}\ d_{i(x^*(t))}\,.$$

We then choose a manifold $\mathcal{M}_k$ of minimum dimension d^- such that

$$x^*(T_n) \in \mathcal{M}_k$$

for a sequence of times $T_n \to 0$.

By possibly taking a subsequence, as $T_n \to 0$ we can assume that

$$\lim_{n\to\infty} \frac{x^*(T_n) - \bar{x}}{T_n} = \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} f\big(x^*(t),\, \alpha^*(t)\big)dt = \bar{f}\,, \tag{31}$$

$$\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} e^{-\beta t}\ell\big(x^*(t),\, \alpha^*(t)\big)\,dt = \bar{\eta}\,, \tag{32}$$

for some vector $\bar{f} \in T_{\overline{\mathcal{M}}_k}(\bar{x})$ and some $\bar{\eta} \geq 0$. Observing that

$$\begin{aligned}&\frac{d}{dt}\Big(x^*(t),\, -V\big(x^*(t)\big)\Big)\\ &\qquad = \Big(f_{i(x^*(t))}\big(x^*(t),\, \alpha^*(t)\big)\,,\ -\beta V\big(x^*(t)\big) + \ell_{i(x^*(t))}\big(x^*(t),\, \alpha^*(t)\big)\Big)\,,\end{aligned}$$

we have

$$\frac{d}{dt}\Big(x^*(t),\, -V\big(x^*(t)\big)\Big) + \Big(0,\, \beta\, V\big(x^*(t)\big)\Big) \in G\big(x^*(t)\big)\,. \tag{33}$$

By (30) and (32) it follows

$$\lim_{n\to\infty} \frac{V(\bar{x}) - V\big(x^*(T_n)\big)}{T_n} + \beta\, V(\bar{x}) = \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} e^{-\beta t}\ell\big(x^*(t),\, \alpha^*(t)\big)\,dt = \bar{\eta}\,. \tag{34}$$

The upper semicontinuity and the convexity of the multifunction G implies $(\bar{f},\, \bar{\eta}) \in G(\bar{x})$.

To prove that V is a supersolution, we need to show that

$$\beta\varphi(\bar{x}) + \sup_{(y,\eta)\in G(\bar{x})} \big\{ - y\cdot\nabla\varphi(\bar{x}) - \eta\big\} \geq \beta\varphi(\bar{x}) - \bar{f}\cdot\nabla\varphi(\bar{x}) - \bar{\eta} \geq 0\,. \tag{35}$$

From (34) and (29) it now follows

$$\bar{\eta} - \beta\, V(\bar{x}) = \lim_{n\to\infty}\frac{V(\bar{x}) - V\big(x^*(T_n)\big)}{T_n} \leq \lim_{n\to\infty}\frac{\varphi(\bar{x}) - \varphi\big(x^*(T_n)\big)}{T_n} = -\bar{f}\cdot\nabla\varphi(\bar{x})\,,$$

proving (35).

2. V **is a lower solution**. Assume that $\varphi \in \mathcal{C}^1$ and that the function $V - \varphi$, restricted to $\mathcal{M}_i$, attains a strict local maximum at $\bar{x} \in \mathcal{M}_i$ We can assume that, for some $r > 0$,

$$\varphi(\bar{x}) = V(\bar{x})\,, \qquad\qquad \varphi(x') \geq V(x'), \qquad \forall x' \in B(\bar{x}, r)\cap\mathcal{M}_i\,. \tag{36}$$

Fix any $(y,\eta) \in G(\bar{x})$. We need to show that

$$\beta V(\bar{x}) - y\cdot D\varphi(\bar{x}) - \eta \leq 0. \tag{37}$$

By the assumption (H2), there exists a control value $a \in A_i$ such that

$$y = f_i(\bar{x}, a), \qquad\qquad \eta \geq \ell_i(\bar{x}, a). \tag{38}$$

Consider the trajectory $t \mapsto x(t)$ corresponding to the constant control $\alpha(t) \equiv a$. Our assumptions imply

$$x(t) \in \mathcal{M}_i \cap B(\bar{x}, r),$$

at least for a short time interval, say $t \in [0, T]$. Since

$$V(\bar{x}) \leq \int_0^t e^{-\beta s} \ell_i\big(x(s), a\big)\, ds + e^{-\beta t} V\big(x(t)\big),$$

we compute

$$\begin{aligned}
\lim_{t\to 0} \frac{\varphi(\bar{x}) - e^{-\beta t}\varphi\big(x(t)\big)}{t} &\leq \lim_{t\to 0} \frac{V(\bar{x}) - e^{-\beta t} V\big(x(t)\big)}{t} \\
&\leq \lim_{t\to 0} \frac{1}{t} \int_0^t e^{-\beta s} \ell_i\big(x(s), a\big)\, ds = \ell_i(\bar{x}, a).
\end{aligned}$$

Therefore, by (38),

$$\eta \geq \ell_i(\bar{x}, a) \geq \lim_{t\to 0} \frac{\varphi(\bar{x}) - e^{-\beta t}\varphi\big(x(t)\big)}{t} = \beta\, \varphi(\bar{x}) - f_i(\bar{x}, a) \cdot \nabla\varphi(\bar{x}) = \beta\, V(\bar{x}) - y \cdot \nabla\varphi(\bar{x}).$$

This establishes (37), completing the proof. □

4. **Uniqueness of the viscosity solution.** The aim of this section is to characterize the value function V as the unique solution to the Hamilton-Jacobi equation (15)-(16). Toward this goal, we shall establish comparison results stating that

$$u(x) \leq V(x) \leq v(x) \qquad \text{for all } x \in \mathbb{R}^N, \tag{39}$$

where V is the value function for the optimal control problem (2)-(3), while v and u are respectively an upper and a lower solution relative to the stratification (1), according to Definitions 1 and 2.

For an upper solution $v \geq 0$, a continuity assumption already suffices to achieve the comparison result. For lower solutions, a comparison theorem is valid under stronger assumptions, such as the Lipschitz continuity of the value function. An alternative set of assumptions, somewhat less restrictive than Lipschitz continuity, is the following.

(H3) The function u is Hölder continuous of exponent $1/2$. Moreover, the restriction of u to each manifold $\mathcal{M}_i$ is locally Lipschitz continuous outside a countable union of $\mathcal{C}^1$ sub-manifolds of strictly smaller dimension.

Still in connection with lower solutions, we shall need a bound on the growth of u as $|x| \to \infty$.

(H4) Either u is globally bounded or

$$|u(x)| \leq C_0\big(1 + |x|\big), \qquad\qquad \big|f_i(x, a)\big| \leq C_1\big(1 + |x|\big),$$

where C_0 and C_1 are some positive constants, with $C_1 < \beta$.

4.1. **The upper solution and the value function.**

Theorem 2. *Consider the optimal control problem* (2)*, for the control system* (3) *on a stratified domain. Let the assumptions (H1), (H2) hold. Let V be the value function and let v be a non-negative, continuous upper solution to the H-J equation* (15)-(16)*. Then*

$$V(x) \leq v(x) \qquad\qquad x \in \mathbb{R}^N. \tag{40}$$

Proof. Recalling (13), we introduce a new multifunction Γ on $\mathbb{R}^{N+1}$, defined as

$$\Gamma(x,z) \doteq \Big\{ (y,\xi)\,;\ (y\,,\ \beta z - \xi) \in G(x),\ \beta z - \xi \leq K^\dagger(x)\Big\}, \quad (x,z) \in \mathbb{R}^N \times \mathbb{R}\,. \tag{41}$$

By the properties of G it follows that Γ is upper semicontinuous, with compact convex, nonempty values. We then consider the differential inclusion

$$(\dot x, \dot z) \in \Gamma(x,z). \tag{42}$$

Assuming that $v : \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous supersolution of (15)-(16), we claim that its epigraph

$$\mathrm{epi}\{v\} \doteq \big\{(x,z) \in \mathbb{R}^N \times \mathbb{R}\,;\ z \geq v(x)\big\}$$

is positively invariant w.r.t. the differential inclusion (42). By a basic viability theorem [1], to prove this invariance it suffices to check that, at each point $(x,z) \in \mathrm{epi}\{v\}$, one has

$$\Gamma(x,z) \cap T_{\mathrm{epi}\{v\}}(x,z) \neq \emptyset\,. \tag{43}$$

By $T_S(p)$ we denote here the Bouligand contingent cone to a set S at a point p, namely

$$T_S(p) \doteq \left\{ y \in \mathbb{R}^N\,;\ \liminf_{h\to 0} \frac{d(p+hy;S)}{h} = 0 \right\}.$$

We recall here that the set $D^-v(x)$ of lower differentials to a function v at a point x is

$$D^-v(x) = \left\{ p \in \mathbb{R}^N\,;\ \liminf_{y\to 0} \frac{v(x+y) - v(x) - p\cdot y}{|y|} \geq 0 \right\}. \tag{44}$$

According to Theorem 4.3 in [15], the nonempty intersection property (43) holds at every point $(x,z) \in \mathrm{epi}\{v\}$ if and only if

$$\beta v(x) + \sup_{(y,\eta)\in G(x)} \Big\{ -y\cdot p - \eta \Big\} \geq 0 \tag{45}$$

for every $x \in \mathbb{R}^N$ and $p \in D^-v(x)$. This condition holds if v is an upper solution.

We can thus construct a trajectory $t \mapsto \big(x^*(t),\, z^*(t)\big)$ of the differential inclusion (42), with initial data

$$\big(x^*(0), z^*(0)\big) = (\bar x,\, v(\bar x))\,.$$

Consider the set of times

$$J_i \doteq \Big\{ t \geq 0\,;\ x^*(t) \in \mathcal{M}_i \Big\}.$$

We then have

$$\dot x^*(t) \in T_{\mathcal{M}_i}\big(x^*(t)\big)$$

for a.e. $t \in J_i$. By the property (H2), and using Filippov's measurable selection theorem, we can find measurable control functions $\alpha_i : J_i \mapsto A_i$ such that

$$\dot x^*(t) = f_i\big(x^*(t), \alpha_i(t)\big)\,, \qquad\qquad \ell_i\big(x^*(t), \alpha_i(t)\big) \leq \beta\, z^*(t) - \dot z^*(t)$$

for a.e. $t \in J_i$. Setting $\alpha(t) \doteq \alpha_i(t)$ for $t \in J_i$, the above implies

$$\frac{d}{dt}\left[\int_0^t e^{-\beta s}\ell\big(x^*(s),\, \alpha(s)\big)\, ds + \Big[e^{-\beta s}v\big(x^*(s)\big)\Big]_0^t\right] \leq 0$$

for a.e. $t \geq 0$. Assuming that $v(x) \geq 0 \ \ \forall x \in \mathbb{R}^N$ and letting $t \to \infty$, we conclude

$$v(\bar{x}) \geq \lim_{t\to\infty} \int_0^t e^{-\beta s}\ell\big(x^*(s),\, \alpha(s)\big)\, ds \geq V(\bar{x})\,.$$

as desired. □

4.2. **The lower solution and the value function.**

Theorem 3. *Consider the optimal control problem* (2)*, for the control system* (3) *on a stratified domain. Let the assumptions (H1), (H2) hold. Let V be the value function and let u be a lower solution to the H-J equation* (15)-(16)*. Let the cost functions ℓ_i be Lipschitz continuous w.r.t. x, so that*

$$\big|\ell_i(x,a) - \ell_i(y,a)\big| \leq Lip(\ell_i)\, |x-y|, \quad \forall\ x, y \in \mathcal{M}_i,\ a \in A_i\,, \tag{46}$$

for some Lipschitz constants $Lip(\ell_i)$. If u satisfies the assumptions (H3) and (H4), then

$$u(x) \leq V(x) \qquad\qquad x \in \mathbb{R}^N. \tag{47}$$

Proof. For clarity of exposition, we first give a proof assuming that u, V are both locally Lipschitz. Then we mention the minor changes needed in the more general case where the assumptions (H3) hold.

Fix any point $\bar{x}$ and let $t \mapsto x^*(t)$ be an optimal trajectory, corresponding to the optimal control $t \mapsto \alpha^*(t)$. This will achieve the minimum cost

$$V(\bar{x}) = \int_0^\infty e^{-\beta t}\ell\big(x^*(t), \alpha^*(t)\big)\, dt\,. \tag{48}$$

In order to show that

$$V(\bar{x}) \geq u(\bar{x})\,, \tag{49}$$

for any fixed time interval $[a,b]$ we shall first establish the following basic estimate:

$$\begin{aligned} &e^{-\beta a}u\big(x^*(a)\big) - e^{-\beta b}u\big(x^*(b)\big) \\ &\qquad \leq \int_a^b e^{-\beta t}\ell\big(x^*(t), \alpha^*(t)\big)\, dt \ = \ e^{-\beta a}V\big(x^*(a)\big) - e^{-\beta b}V\big(x^*(b)\big)\,. \end{aligned} \tag{50}$$

To prove (50), we consider various cases.

CASE 1: For all $t \in]a,b[$, the trajectory $x^*(\cdot)$ remains inside one single manifold $\mathcal{M}_j$ of maximal dimension N.

In this case, the estimate (50) follows by standard argument. Assume first that $x^*(t) \in \mathcal{M}_j$ for all $t \in [a,b]$, i.e., including the end-points of the interval. By our assumptions, the O.D.E.

$$\dot{x}(t) = f_j\big(x(t), \alpha^*(t)\big) \tag{51}$$

is Lipschitz continuous w.r.t. x and measurable w.r.t. t. Therefore, for each initial condition

$$x(a) = y \in \mathcal{M}_j\,, \tag{52}$$

the Cauchy problem (51)-(52) admits a unique solution $t \mapsto x(t,y)$. Moreover, for a suitable Lipschitz constant $L = Lip(f_j)$, the solutions corresponding to different initial data $y, \tilde{y}$ satisfy

$$e^{-L(t-a)}|y-\tilde{y}| \leq \big|x(t,y)-x(t,\tilde{y})\big| \leq e^{L(t-a)}|y-\tilde{y}| \qquad\qquad t \geq a\,. \tag{53}$$

Since the function u is differentiable a.e. on the open set $\mathcal{M}_j \subset \mathbb{R}^N$, we can find a sequence of initial points y_n and trajectories $t \mapsto x_n(t) \doteq x(t,y_n)$ such that:

(i) $y_n \to x^*(a)$, and hence $x_n(t) \to x^*(t)$ uniformly for $t \in [a,b]$.

(ii) For each $n \geq 1$, the function u is differentiable at the point $x_n(t)$, for a.e. $t \in [a,b]$.

We now compute

$$\begin{aligned} e^{-\beta a}u(x_n(a)) - e^{-\beta b}u\big(x_n(b)\big) &= -\int_a^b \left[\frac{d}{dt}e^{-\beta t}u\big(x_n(t)\big)\right] dt \\ &= \int_a^b e^{-\beta t}\Big[\beta u\big(x_n(t)\big) - \nabla u\big(x_n(t)\big)\cdot f_j\big(x_n(t),\alpha^*(t)\big)\Big]\, dt \\ &\leq \int_a^b e^{-\beta t}\,\ell_j\big(x_n(t),\,\alpha^*(t)\big)\, dt\,, \end{aligned} \tag{54}$$

because of the definition of lower solution. Letting $n \to \infty$ in (54) we obtain the desired inequality (50).

If now $x^*(t) \in \mathcal{M}_j$ only for $t \in]a,b[$, we can still apply the above result to the smaller closed interval $[a+\varepsilon\,,\ b-\varepsilon]$. This yields

$$e^{-\beta(a+\varepsilon)}u\big(x^*(a+\varepsilon)\big) - e^{-\beta(b-\varepsilon)}u\big(x^*(b-\varepsilon)\big) \leq \int_{a+\varepsilon}^{b-\varepsilon} e^{-\beta t}\ell\big(x^*(t),\alpha^*(t)\big)\, dt\,.$$

Letting $\varepsilon \to 0$ we recover again (50).

CASE 2: We assume now that $x^*(a), x^*(b) \in \mathcal{M}_j$, the dimension of $\mathcal{M}_j$ is $d_j = N-1$, and moreover the trajectory $t \mapsto x^*(t)$ remains either inside $\mathcal{M}_j$ or inside other manifolds of dimension N, for all $t \in [a,b]$.

Using a local chart, we can assume that

$$\mathcal{M}_j = \big\{(x_1,\ldots,x_N) \in \mathbb{R}^N\,;\ x_N = 0\ \big\}. \tag{55}$$

By continuity, $x^*(\cdot)$ leaves $\mathcal{M}_j$ and enters some other N-dimensional manifold $\mathcal{M}_k$ on an open set of times, say

$$\big\{t \in [a,b]\,;\ x^*(t) \notin \mathcal{M}_j\big\} = \bigcup_{i\in I}]a_i,\ b_i[\,.$$

Here I is a finite or countable set of indices.

For every $i \in I$, by the analysis in Case 1 we already know that

$$e^{-\beta a_i}u\big(x^*(a_i)\big) - e^{-\beta b_i}u\big(x^*(b_i)\big) \leq e^{-\beta a_i}V\big(x^*(a_i)\big) - e^{-\beta b_i}V\big(x^*(b_i)\big) \tag{56}$$

A further estimate will be needed. For each $i \in I$, by the assumption (H1) of Lipschitz continuity of the functions f_j, f_k, and by the assumption (H2) of upper semicontinuity of the velocity sets, we have

$$\overline{\mathrm{co}}\left(\bigcup_{t\in[a_i,b_i]} F_k\big(x^*(t)\big)\right)\cap T_{\mathcal{M}_j} \subseteq B\Big(F\big(x^*(a_i)\big)\,,\ L(b_i-a_i)\Big)\,, \tag{57}$$

for some Lipschitz constant L. Here $B(S,r)$ denotes the closed neighborhood of radius r around the set S.

By (57) we can choose a constant control $\alpha_{j,i}\in A_j$ such that

$$\left| f_j\big(x^*(a_i),\,\alpha_{j,i}\big) - \frac{1}{b_i-a_i}\int_{a_i}^{b_i} f_k\big(x^*(t),\alpha^*(t)\big)\,dt\right| \le L(b_i-a_i) \tag{58}$$

observing that

$$f_j\big(x^*(a_i),\,\alpha_{j,i}\big)\in F(x^*(a_i))\,,$$

$$\frac{x^*(b_i)-x^*(a_i)}{b_i-a_i} \;=\; \frac{1}{b_i-a_i}\int_{a_i}^{b_i} f_k\big(x^*(t),\alpha^*(t)\big)\,dt \;\in\; \overline{\mathrm{co}}\left(\bigcup_{t\in[a_i,b_i]} F_k\big(x^*(t)\big)\right)\cap T_{\mathcal{M}_j}\,.$$

Moreover, by the Jensen's inequality and the Lipschitz continuity of the cost function ℓ_k, we can also achieve

$$\ell_j\big(x^*(a_i),\,\alpha_{j,i}\big) \le \frac{1}{b_i-a_i}\int_{a_i}^{b_i} \ell_k\big(x^*(t),\alpha^*(t)\big)\,dt + C(b_i-a_i)\,. \tag{59}$$

Using the constant control $\alpha_{j,i}$ on the whole interval $[a_i,b_i]$, the solution of

$$\dot x(t) = f_j(x(t),\alpha_{j,i}) \qquad\qquad x(a_i)=x^*(a_i)\,,$$

satisfies

$$y_i \doteq x(b_i) = x^*(a_i) + f_j(x^*(a_i),\alpha_{j,i})(b_i-a_i) + \mathcal{O}(1)(b_i-a_i)^2,$$

and $y_i\in\mathcal{M}_j$. Hence, setting

$$\mathbf{v}_i \doteq x^*(b_i) - y_i$$

we have

$$|\mathbf{v}_i| \le \kappa(b_i-a_i)^2\,. \tag{60}$$

for some constant κ, uniformly valid for all $i\in I$.

We are now ready to define a family of perturbed trajectories. Define the control function

$$\alpha^\dagger(t) \doteq \begin{cases}\alpha^*(t) & \text{if } t\notin\bigcup_i\,]a_i,b_i[\,,\\ \alpha_{j,i} & \text{if } t\in]a_i,b_i[\quad \text{for some } i\in I.\end{cases} \tag{61}$$

For each initial point $y\in\mathcal{M}_j$ close to $x^*(a)$, let $t\mapsto x(t,y)$ be the solution of the impulsive Cauchy problem

$$\dot x(t) = f_j\big(x(t),\alpha^\dagger(t)\big)$$

$$x(a)=y\,,\qquad\qquad x(b_i)=x(b_i-)+\mathbf{v}_i\,.$$

The figure 2 illustrates the solution. Notice that this trajectory is unique, because it corresponds to the unique fixed point of the integral transformation $x(\cdot)\mapsto\mathcal{T}x(\cdot)$, defined as

$$\mathcal{T}x(t) = y + \int_0^t f_j\big(x(t),\alpha^\dagger(t)\big)\,dt + \sum_{b_i\le t}\mathbf{v}_i\,.$$

As in Case 1, we can select a sequence of initial points y_n, with corresponding trajectories $x(\cdot,y_n)$, such that

(i) As $n \to \infty$, one has $y_n \to x^*(a)$, and hence $x_n(t) \to x^*(t)$ uniformly for $t \in [a, b]$.

(ii) For each $n \geq 1$, the restriction of u to $\mathcal{M}_j$ is differentiable at the point $x_n(t)$, for a.e. $t \in [a, b]$.

Using (59) and (60) we now compute

$$\begin{aligned}
&e^{-\beta a} u\big(x_n(a)\big) - e^{-\beta b} u\big(x_n(b)\big) \\
&\quad = -\int_a^b \left[\frac{d}{dt} e^{-\beta t} u\big(x_n(t)\big)\right] dt - \sum_{i \in I} e^{-\beta b_i} \big[u\big(x_n(b_i)\big) - u\big(x_n(b_i -)\big)\big] \\
&\quad \leq \int_a^b e^{-\beta t} \Big[\beta u\big(x_n(t)\big) - \nabla u\big(x_n(t)\big) \cdot f_j\big(x_n(t), \alpha^\dagger(t)\big)\Big]\, dt \\
&\qquad\qquad + \sum_{i \in I} e^{-\beta b_i} L_u \kappa (b_i - a_i)^2 \\
&\quad \leq \int_a^b e^{-\beta t}\, \ell_j\big(x_n(t),\, \alpha^\dagger(t)\big)\, dt + \sum_{i \in I} e^{-\beta b_i} L_u \kappa (b_i - a_i)^2 \\
&\quad \leq \int_a^b e^{-\beta t}\, \ell\big(x_n(t),\, \alpha^*(t)\big)\, dt + \sum_{i \in I} e^{-\beta a_i} C (b_i - a_i)^2 \\
&\qquad\qquad + \sum_{i \in I} e^{-\beta b_i} L_u \kappa (b_i - a_i)^2\,,
\end{aligned} \tag{62}$$

because u is a lower solution. The Lipschitz constant of u is denoted by L_u.

Let $\varepsilon > 0$ be given. Choose a finite subset of indices $I' \subset I$ such that

$$\sum_{i \in I \setminus I'} e^{-\beta a_i} C (b_i - a_i)^2 + \sum_{i \in I \setminus I'} e^{-\beta b_i} L_u \kappa (b_i - a_i)^2 < \varepsilon\,.$$

To fix the ideas, let $I' = \{1, \ldots, \nu\}$, with

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_\nu < b_\nu \leq b\,.$$

We can now use the estimate (56) on each of the subintervals $[a_k, b_k]$, $k = 1, \ldots, \nu$, and an estimate of the form (62) on the remaining finitely many intervals

$$[a, a_1]\,,\ [b_1, a_2]\,,\ \ldots\ ,\ [b_\nu, b]\,.$$

Setting for convenience $b_0 = a$, $a_{\nu+1} = b$, we thus obtain

$$
\begin{aligned}
&e^{-\beta a}u\big(x_n(a)\big) - e^{-\beta b}u\big(x_n(b)\big) \\
&\quad = \sum_{k=1}^{\nu+1}\Big(e^{-\beta b_{k-1}}u\big(x_n(b_{k-1})\big) - e^{-\beta a_k}u\big(x_n(a_k)\big)\Big) \\
&\qquad\qquad\qquad + \sum_{k=1}^{\nu}\Big(e^{-\beta a_k}u\big(x_n(a_k)\big) - e^{-\beta b_k}u\big(x_n(b_k)\big)\Big) \\
&\quad \le \int_a^b e^{-\beta t}\,\ell\big(x_n(t),\,\alpha^*(t)\big)\,dt + \sum_{i\in I\setminus I'} e^{-\beta a_i}C(b_i-a_i)^2 \\
&\qquad\qquad\qquad + \sum_{i\in I\setminus I'} e^{-\beta b_i}L_u\kappa(b_i-a_i)^2\,.
\end{aligned}
\tag{63}
$$

Letting $n\to\infty$, from (63) it follows

$$
\begin{aligned}
&e^{-\beta a}u\big(x^*(a)\big) - e^{-\beta b}u\big(x^*(b)\big) \\
&\quad \le \int_a^b \ell\big(x^*(t),\,\alpha^*(t)\big)\,dt + \sum_{i\in I\setminus I'} e^{-\beta a_i}C(b_i-a_i)^2 \\
&\qquad\qquad\qquad + \sum_{i\in I\setminus I'} e^{-\beta b_i}L_u\kappa(b_i-a_i)^2 \\
&\quad \le e^{-\beta a}V\big(x^*(a)\big) - e^{-\beta b}V\big(x^*(b)\big) + \varepsilon\,.
\end{aligned}
\tag{64}
$$

Since $\varepsilon>0$ was arbitrary, once again we obtain the basic inequality (50).

CASE 3: During the interval $[a,b]$ the optimal trajectory $x^*(\cdot)$ remains inside manifolds of dimension N or $N-1$.

This is a slight generalization of the previous case. The validity of (50) is clear, observing that we can find finitely many times $a=t_0<t_1<\cdots<t_n=b$ such that the restriction of x^* to each subinterval $[t_{i-1},\,t_i]$ satisfies the conditions in Case 2.

CASE 4: We now assume that the estimate (50) holds whenever the optimal trajectory $x^*(\cdot)$ remains on manifolds of dimension $\ge m+1$, and prove that it still holds when $x^*(\cdot)$ stays on manifolds of dimension $\ge m$. By induction, this will establish (50) in the general case.

The proof of this inductive step relies on the same ideas used in Case 2. We thus only sketch the main lines.

Assume that $x^*(\cdot)$ remains inside a manifold $\mathcal{M}_j$ of dimension $d_j=m$, or other manifolds of strictly higher dimension. Using a local chart, we can assume that

$$
\mathcal{M}_j = \big\{(x_1,\ldots,x_N)\in\mathbb{R}^N\,;\ x_i=0,\ i=m+1,\ldots,N\big\}\,.
$$

By continuity, we again have

$$
\big\{t\in[a,b]\,;\ x^*(t)\notin\mathcal{M}_j\big\} = \bigcup_{i\in I}\,]a_i,\ b_i[\,,
$$

where I is a finite or countable set of indices. For every $i\in I$, by the inductive assumption we still have (56). Furthermore, for each subinterval $[a_i,b_i]$ we can find a control $\alpha_{j,i}$ such that (58) and (59) hold. We thus define the control $\alpha^\dagger$ as in (61), choose a sequence of trajectories $x_n = x(y_n, \alpha^\dagger)$ and retrace all steps (62)–(64). This concludes the proof of (50).

We now conclude the proof of (49). For any given initial condition $\bar{x}$, let $\alpha^*(\cdot)$ and $x^*(\cdot)$ be a corresponding optimal control and optimal trajectory. For every $T > 0$, using (50) on the interval $[0, T]$ we find

$$u(\bar{x}) \leq V(\bar{x}) + e^{-\beta T} u\big(x^*(T)\big) \,. \tag{65}$$

Letting $T \to \infty$, by (H4) we have

$$e^{-\beta T} u\big(x^*(T)\big) \to 0.$$

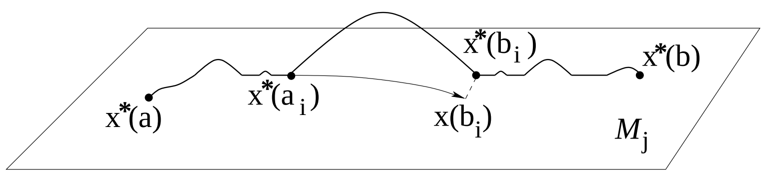

FIGURE 2. The solution of the impulsive Cauchy problem

Finally, we observe that the above result remains valid if the assumption of Lipschitz continuity of the lower solution u is replaced by the assumption (H3). The proof would go through as before, except that the last term in (62) would be replaced by

$$\sum_{i \in I} e^{-\beta b_i} L_u \kappa (b_i - a_i) \,.$$

Now L_u denotes the Hölder constant of u. This estimate suffices to complete the remainder of the proof. □

From the above comparison theorems one immediately obtains a uniqueness result:

Corollary 1. *Consider the optimal control problem* (2)*, for the control system* (3) *on a stratified domain* (1)*. Let the assumptions (H1), (H2), (H4) and* (46) *hold. Let's assume that the value function V satisfies the regularity assumptions (H3). Then V is the unique non-negative solution to the H-J equation* (15)-(16) *with such regularity properties.*

Remark 1. All the results in this paper remain valid in the more general case where we allow the control set A_i to be empty, i.e. $A_i = \emptyset$, on some manifold $\mathcal{M}_i$ of dimension $d_i < N$.

Notice that, in this case, there is no control which keeps the system inside $\mathcal{M}_i$. The assumption (H2) now implies that

$$\Big\{(y, \eta) \in G(x) \,;\; y \in T_{\mathcal{M}_{i(x)}}(x)\Big\} = \emptyset \,,$$

for all $x \in \mathcal{M}_i$. In particular, this means that all trajectories cross the manifold $\mathcal{M}_i$ transversally, spending a zero amount of time inside $\mathcal{M}_i$.

Acknowledgements. This material is based upon work supported by the National Science Foundation under Grant No. 0505430.

REFERENCES

[1] J. P. Aubin and A. Cellina, "Differential Inclusions," Springer-Verlag, New York 1984.
[2] M. Bardi and I. Capuzzo-Dolcetta, "Optimal Control and Viscosity solutions of Hamilton-Jacobi-Bellman Equations," Birkhäuser, Boston, 1997.
[3] V. G. Boltyanskii, *Sufficient conditions for optimality and the justification of the dynamic programming principle*, SIAM J. Control Optim., **4** (1966), 326–361.
[4] P. Brunovský, *Every normal linear system has a regular time-optimal synthesis*, Math. Slovaca, **28** (1978), 81–100.
[5] F. Camilli and A. Siconolfi, *Hamilton-Jacobi Equations with measurable dependence on the state vairable*, Adv. Differential Equations, **8** (2003), no. 6, 733–768.
[6] A. F. Filippov, *On certain questions in the theory of optimal control*, SIAM on Journal Control, **1** (1962), 76–84.
[7] W. H. Fleming and R. W. Rishel, "Deterministic and Stochastic Optimal Control," Springer-Verlag, New York, 1975.
[8] P. L. Lions, "Generalized Solutions of Hamilton-Jacobi Equations," Pitman publishing inc, Marshfield, Massachusetts, 1982.
[9] P. L. Lions and A. S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math., **37** (1984), 511–537.
[10] R. T. Newcomb and J. Su, *Eikonal equations with discontinuities*, Differential Integral Equations, **8** (1995), no. 8, 1947–1960.
[11] B. Piccoli, *A maximum principle for hybrid optimal control problems*, Proceedings of the 38th IEEE Conference on Decision and Control, **1** (1999), 410 – 415.
[12] B. Piccoli and H. J. Sussmann, *Regular synthesis and sufficiency conditions for optimality*, SIAM J. Control Optim., **39** (2000), no. 2, 359–410.
[13] O. Serea, *On reflecting boundary problem for optimal control*, SIAM J. Control Optim., **42** (2003), no. 2, 559–575.
[14] P. Soravia, *Boundary value problems for Hamilton-Jacobi equations with discontinuous Lagrangian*, Indiana Univ. Math. J., **51** (2002), no. 2, 451–477.
[15] A.I. Subbotin, "Generalized Solutions of First-order PDEs," Birkhäuser, Boston, 1995.
[16] H.J. Sussmann, *A maximum principle for hybrid optimal control problems*, Proceedings of the 38th IEEE Conference on Decision and Control, **1** (1999), 425 – 430.

Received December 2006; revised February 2007.

E-mail address: `bressan@math.psu.edu`
E-mail address: `yyh101@psu.edu`