Zsolt Páles (`pales@math.klte.hu`), University of Debrecen, Debrecen, Hungary and **Vera Zeidan*** (`zeidan@math.msu.edu`), Michgan State University, East Lansing, MI, *Critical Cones and Critical Tangent Cones in Optimization Problems*

The notion of second-order admissible variation defined by Dubovitskii and Milyutin in 1965 turned out to be an essential notion in the theory of second-order necessary conditions for optimum problems. It is introduced in the following definition.

Definition. Let X be a normed space, $\mathbf{Q} \subset X$, $x \in \mathbf{Q}$, and $d \in X$. A vector $v \in X$ is called a *second-order admissible variation of* $\mathbf{Q}$ *at* x *in the direction* d if there exists $\overline{\varepsilon} > 0$ such that

$$x + \varepsilon d + \varepsilon^2(v + u) \in \mathbf{Q} \qquad \text{for all} \quad 0 < \varepsilon < \overline{\varepsilon},\ \|u\| < \overline{\varepsilon},\ u \in X.$$

The set of all such variations is denoted by $V(x, d|\mathbf{Q})$. It follows directly from the definition that $V(x, d|\mathbf{Q})$ is an open set. If $\mathbf{Q}$ is also convex, then $V(x, d|\mathbf{Q})$ is convex as well. In Levitin, Milyutin, and Osmolovskii (1978), a general discussion on second- and higher-order necessary conditions can be found.

In order to derive meaningful second-order optimality conditions for optimum problems, it is required to select directions d for which the set $V(x, d|\mathbf{Q})$ is nonempty. Such directions $d \in X$ are labeled as the *critical directions of* $\mathbf{Q}$ *at* x and form a set called *critical directions cone to* $\mathbf{Q}$ *at* x. Throughout this talk, this cone will be denoted by $C(x|\mathbf{Q})$. It can be easily seen that $C(x|\mathbf{Q})$ is a *convex cone* if $\mathbf{Q}$ is convex.

Define

$$K(x|\mathbf{Q}) := \operatorname{cone}(\mathbf{Q} - x) := \{\lambda(q - x) \mid q \in \mathbf{Q},\ \lambda > 0\},$$

and its closure

$$T(x|\mathbf{Q}) := \overline{\operatorname{cone}}(\mathbf{Q} - x) = \operatorname{cl} K(x|\mathbf{Q}).$$

If $\mathbf{Q}$ is convex, then for the nonemptiness of $V(x, d|\mathbf{Q})$ it is necessary, but not sufficient that the interior of $\mathbf{Q}$ be nonempty and d belong to $T(x|\mathbf{Q})$. However, the nonemptiness of $V(x, d|\mathbf{Q})$ is assured if $\operatorname{intr} \mathbf{Q} \neq \emptyset$ and $d \in K(x|\mathbf{Q})$. Therefore, for convex $\mathbf{Q}$ with nonempty interior, we have

$$K(x|\mathbf{Q}) \subset C(x|\mathbf{Q}) \subset T(x|\mathbf{Q}).$$

In order to demonstrate the use of second-order it admissible variations, consider the following optimization problem:

$$(\mathcal{P}) \qquad \text{Minimize} \quad F(z) \quad \text{subject to} \quad G(z) \in \mathbf{Q},\ H(z) = 0,$$

where $F : \mathcal{D} \to \mathbf{R}$, $G : \mathcal{D} \to X$, $H : \mathcal{D} \to Y$, and X, Y, Z are Banach spaces, $\mathcal{D} \subset Z$ is nonempty and open, and $\mathbf{Q} \subset X$ is a closed convex set with nonempty interior.

The prototype of such problems arise, for instance, in optimal control theory with control and/or state constraints in the inclusion form $x(t) \in \mathcal{Q}(t)$. This pointwise condition would then lead to the set $\mathbf{Q}$ which consists of appropriate selections x. Let us recall the first- and second-order necessary conditions for $(\mathcal{P})$, obtained by the authors in 1994. For, we introduce the following notations and notions.

- A point $\hat{z} \in \mathcal{D}$ is called an *admissible point* for $(\mathcal{P})$ if $G(\hat{z}) \in \mathbf{Q}$ and $H(\hat{z}) = 0$ hold. A point $\hat{z} \in \mathcal{D}$ is a *solution (local minimum)* of the problem if it is admissible and there exists a neighborhood U of $\hat{z}$ such that $F(z) \geq F(\hat{z})$ for all admissible points $z \in U$.

- A point $\hat{z} \in \mathcal{D}$ is called a *regular point* for $(calP)$ if F, G, and H are strictly Fréchet differentiable at $\hat{z}$ and the range of the linear operator $H'(\hat{z})$ is a closed subspace of Y.

Let $\hat{z}$ be an admissible regular point for $(\mathcal{P})$ and $d \in Z$.

- A vector $\delta z \in Z$ is called a *critical direction* at $\hat{z}$ for $(\mathcal{P})$ if

$$F'(\hat{z})\delta z \leq 0, \qquad G'(\hat{z})\delta z \in C(G(\hat{z})|\mathbf{Q}) \qquad E'(\hat{z})\delta z = 0.$$

- A vector $\delta z \in Z$ is called a *regular direction* at $\hat{z}$ for $(\mathcal{P})$ if the second-order directional derivative of $L := (F, G, H)$

$$L''(\hat{z}, \delta z) := \lim_{\varepsilon \to 0+} 2\frac{L(\hat{z} + \varepsilon\delta z) - L(\hat{z}) - \varepsilon L'(\hat{z})\delta z}{\varepsilon^2}$$

exists.

Clearly, the *zero* vector is always a regular critical direction at $\hat{z}$ for $(\mathcal{P})$. Now we are ready to state a particular case of the result developed in 1994.

Theorem 0.1 *(P/Z) Let $\hat{z}$ be a regular local solution of the above problem $(\mathcal{P})$. Then, for all regular critical directions δz, there correspond Lagrange multipliers $\lambda \geq 0$, $\xi \in X^*$, and $\eta \in Y^*$ (which depend on δz), such that at least one of them is different from zero and the following relations hold*

$$\xi \in N(G(\hat{z})|\mathbf{Q}), \tag{1}$$

$$\lambda F'(\hat{z})z + < \xi, G'(\hat{z})z > + < \eta, E'(\hat{z})z > \, = \, 0 \qquad \text{for} \quad z \in Z, \tag{2}$$

and

$$\lambda F''(\hat{z}, \delta z) + < \xi, G''(\hat{z}, \delta z) > + < \eta, E''(\hat{z}, \delta z) > \quad \geq 2\delta^*(\xi | V(G(\hat{z}), G'(\hat{z})\delta z|\mathbf{Q})). \tag{3}$$

(Here δ^* stands for the support function and $N(x|\mathbf{Q})$ denotes the adjoint cone of $T(x|\mathbf{Q})$, that is the cone of outward normals to the set $\mathbf{Q}$ at the point x.)

The result presented in the above theorem was derived under the Mangasarian-Fromovitz condition, by Kawasaki in 1988 for the case when $\mathbf{Q}$ is a cone, and by Cominetti 1990 for the case when $\mathbf{Q}$ is convex. The possibility of removing the Mangasarian-Fromovitz condition, as is obtained in the above theorem, was noted by Ioffe 1989.

If $d \in K(x|\mathbf{Q})$, then $V(x, d|\mathbf{Q})$ is nonempty and $V(x, d|\mathbf{Q}) = \text{cone}(\text{cone}(\text{intr}\, \mathbf{Q} - x) - d)$, that is, V is a cone. Thus, if this is the case for $d := G'(\hat{z})\delta z$ and $x := G(\hat{z})$, then the right-hand side in the second-order condition inequality (3) vanishes. This is the case of the result obtained for instance by Ben-Tal and Zowe in 1982. However examples are provided by Kawasaki in 1988 that show that the necessary conditions with *nonzero extra term*, which occurs only if $d \in T(x|\mathbf{Q}) \setminus K(x|\mathbf{Q})$, handle situations that cannot be handled with previous results, that is, when d is taken from $K(x|\mathbf{Q})$. Thus, one has to also consider directions $d \in T(x|\mathbf{Q}) \setminus K(x|\mathbf{Q})$. In this important case two questions naturally arise from the theorem:

(i) How can we check the nonemptiness of $V(x, d|\mathbf{Q})$, that is, how can the critical cone $C(x|\mathbf{Q})$ be characterized in terms of $\mathbf{Q}$.

(ii) How to evaluate the support function of $V(x, d|\mathbf{Q})$ in terms of that of $\mathbf{Q}$ itself.

A significant setting is the case when $\mathbf{Q}$ is a subset of the space of continuous functions, $\mathcal{C}(T, \mathbf{R}^\kappa)$, and is defined by

$$\mathbf{Q} = \mathrm{sel}_C(\mathcal{Q}) := \{x \in \mathcal{C}(T, \mathbf{R}^\kappa) \,|\, x(t) \in \mathcal{Q}(t) \text{ for all } t \in T\}, \tag{4}$$

where $\mathcal{Q}$ is a lower semicontinuous set-valued map whose images are closed, convex sets with nonempty interior, and T is a compact Hausdorff space. This type of constraints represents the state constraints in control problems.

Another case of interest is when $\mathbf{Q}$ is a subset of the space of essentially bounded functions, $\mathcal{L}^\infty(\Omega, \mathbf{R}^\gamma)$, and is defined by

$$\mathbf{Q} = \mathrm{sel}_\infty(\mathcal{Q}) := \{x \in \mathcal{L}^\infty(\Omega, \mathbf{R}^\gamma) \,|\, x(t) \in \mathcal{Q}(t) \text{ for a.e. } t \in \Omega\}, \tag{5}$$

where $\mathcal{Q}$ is a measurable set-valued map whose images are closed and have nonempty interior, and $(\Omega, \mathcal{A}, \nu)$ is a complete finite measure space. This type of constraints is typical for control constraints in control problems.

However, there are optimization problems for which the constraint set $\mathbf{Q}$ in neither a subset of $\mathcal{C}(T, \mathbf{R}^\kappa)$ nor a subset of $\mathcal{L}^\infty(\Omega, \mathbf{R}^\gamma)$, but a subset of a general normed space X.

The main goal of this talk is to answer the above questions (i)-(ii) when the underlying space X is *any* normed space. For the special case when $\mathbf{Q}$ is given by (4) or (5), we wish the answer to (i)-(ii) be phrased in terms of the original data, that is, the set-valued map $\mathcal{Q}$.

To reach the first goal, a characterization of the critical cone $C(x|\mathbf{Q})$ is derived in terms of the set $\mathbf{Q}$. This leads to the introduction of $CT(x|\mathbf{Q})$, the *critical tangent cone* of a set $\mathbf{Q} \subset X$ at x. Properties and examples of this cone will be presented. For the second goal, we managed to develop in terms of the support function of $\mathbf{Q}$, a *formula* for the extra term in inequality (3), that is, for the support function $V(x, d|\mathbf{Q})$ at $\xi \in X^*$. When $\mathbf{Q}$ is given by either (4) or (5), the characterization of $C(x|\mathbf{Q})$ simplifies drastically. Furthermore, in these cases, the extra term can then be expressed in terms of the support function of the images of the set-valued map $\mathcal{Q}$ as long as in the case of equation (5) the extra term in (3) is to be evaluated at $\xi \in \mathcal{L}^1(\Omega, \mathbf{R}^\gamma)$.

The proofs of these results are stimulated by a string of papers that the authors wrote in the last decade on the special cases considered in equations (4) and (5). However, in those cases they were able to reduce the infinite dimension feature of questions (i)-(ii) to one of finite dimension through the images $\mathcal{Q}(t)$ of the set-valued map $\mathcal{Q}$. In the general setting, which is the issue in this talk, no such structure is given for $\mathbf{Q}$, but nevertheless answers to (i)-(ii) can still be provided.