William M. McEneaney (`wmceneaney@ucsd.edu`), Dept. of Mathematics and Dept. of Mechanical and Aerospace Eng., University of California, San Diego, *A Max–Plus Method for Bellman Equations via Summation of Dual-Space Operators*

The solution of nonlinear optimal control problems and many nonlinear H_∞/L_2-gain control problems can be obtained via solution of the corresponding Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDE's). These HJB PDE's are fully nonlinear and first-order. Interestingly, the semi-groups associated with with these HJB PDE's are linear over the either the max-plus or min-plus algebra (a point which the author believes was first pointed out by Maslov in 1987). For simplicity, we discuss only the max-plus case. This led to development of a class of numerical methods which are essentially max-plus spectral methods (documented in various papers not included here due to space limitations). This class of methods is distinct from the well-known finite element and characteristic-based methods.

Here we take a different approach. To put the following discussion in perspective, note that in the linear/quadratic case, the solutions of the HJB PDE's are given simply by the solution of (finite-dimensional) Riccati equations. We will use the solutions of these simple problems to build approximate solutions of more complex ones, via max-plus summation of some associated operators over the semi-convex dual space.

In this abstract, we consider only the subclass consisting of steady-state HJB PDE's over all of $\mathbf{R}^n$. Let S_τ^k be the semi-group (for time propagation τ) associated with the k^{th} HJB PDE in a set of K such (steady-state) PDE's. The solution of the PDE is also the fixed point of $W = S_\tau^k[W]$ (for any $\tau > 0$), or equivalently by noting the max-plus linearity of S_τ^k, the max-plus eigenvector of the max-plus linear operator corresponding to max-plus eigenvalue zero.

The solutions lie in the space of semiconvex functions which we will view as a space over the max-plus algebra as opposed to the standard field. Let this space be spanned by basis functions $\{\psi_i\}$. Truncating this expansion at M terms, an approximate solution is given by $W^{k,\tau} = \bigoplus_{i=1}^M e_i^k \otimes \psi_i$. Further, the vector of e_i^k's satisfies the finite-dimensional eigenvector problem $\vec{e}^{\,k} = B^{k,\tau} \otimes \vec{e}^{\,k}$ where $B^{k,\tau}$ is an $M \times M$ matrix associated with S_τ^k. Note that in the linear/quadratic case, $B^{k,\tau}$ is relatively easy to compute.

Define $\bar{S}_\tau[\phi] = \max_{k \in \{1,2,\dots K\}} S_\tau^k[\phi]$. Let $\overline{B}$ be the $M \times M$ matrix associated with $\bar{S}_\tau$. Then $\overline{B} = \bigoplus_{k=1}^K B^{k,\tau}$. Thus the solution of $W = \bar{S}_\tau[W]$ is approximately given by the vector of coefficients satisfying $\overline{\vec{e}} = \overline{B} \otimes \overline{\vec{e}} = (\bigoplus_{k=1}^K B^{k,\tau}) \otimes \overline{\vec{e}}$.

Returning to the PDE's, let the k^{th} PDE be $0 = H^k(x, \nabla W)$. Then consider the PDE $0 = \overline{H}(x, \nabla W) \doteq \max_{k \le K} H^k(x, \nabla W)$. Taking τ small and M large, we prove that the solution $\overline{W}$ of $0 = \overline{H}(x, \nabla W)$ can be arbitrarily closely approximated on compact sets by the function $\bigoplus_{i=1}^M \overline{\vec{e}}_i \otimes \psi_i$. In the case where each of the H^k correspond to linear/quadratic problems, $\overline{B}$ is relatively easy to compute, and this leads to an approach for solution of HJB PDE's which are given (or approximated by) maxima of linear/quadratic HJB PDE's.

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