Structured Population Models: Measure-Valued Solutions and Difference Approximations

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Outline

- Present distributed rate model: space of measures as an appropriate space for some population models.
- Discuss asymptotic behavior of distributed rate model and survival of the fittest.
- Present a hierarchical continuous-time size-structured model.
- Discuss related literature and our recent work on this model.
- Develop a finite difference scheme to approximate the (singular) solution to this model.
- Present convergence analysis and numerical examples.

Distributed Rate Population Model

- The type of model we consider is based on the simple nonlinear dynamics

$$\frac{d}{dt}X(t) = X(t)\Big[f_1(X(t)) - f_2(X(t))\Big],$$

Which involves a nonlinear growth rate f_1 and a nonlinear mortality rate f_2 for the population, whose size at time t is denoted by $X(t)$.

- We begin with an adjustment to our previous equation, given by

$$\frac{d}{dt}x(t,q) = x(t,q)\Big[q_1 f_1(X(t)) - q_2 f_2(X(t))\Big].$$

Here $q = (q_1, q_2) \in Q$ denotes the subpopulation-specific growth and mortality rate parameters. The function $x \in C^1(\mathbb{R}^+; C(Q))$. The nonlinear terms, moreover, are assumed to depend on the total population level, $X(t) = \int_Q x(t,q)dq$.

Lifting up the Model to Space of Measures

As will be seen the natural space to examine this model is the space of measures. Thus, we want to extend the (density) model considered on the space $C(Q)$

$$\begin{cases} \dot{x}(t) = F(x(t)) \\ x(0) = x_0 \end{cases}$$

to the space $\mathcal{M}$ = finite signed measures on the measurable space $(Q, \mathcal{B}(Q))$ where $\mathcal{B}(Q)$ are the Borel sets on Q.

Indeed, if $f \in C(Q)$, $A \in \mathcal{B}(Q)$, dq is Lebesgue measure and $\mu_f(A) = \int_A f(q)dq$, then $f \to \mu_f$ is an embedding of $C(Q) \hookrightarrow \mathcal{M}$.

Hence we reformulate as follows: if $x \in C^1(\mathbb{R}^+; C(Q))$ is a solution to

$$\begin{cases} \dot{x}(t) = F(x(t)) \\ x(0) = x_0 \end{cases}$$

Then $\frac{d}{dt}\mu(t)(A) = \dot{\mu}_{x(t)}(A) = \int_A \dot{x}(t)dq = \int_A F(x(t))dq$ and $\mu_{x(0)}(A) = \int_A x_0 dq = \mu_0(A)$.

So we define $\hat{F} : \mathcal{M} \to \mathcal{M}$ by $\hat{F}(\mu)(A) = \int_A F(x(t))dq$, and for any $\mu_0 \in \mathcal{M}$ we define the reformulated IVP to be

$$\begin{cases} \dot{\mu} = \hat{F}(\mu) \\ \mu(0) = \mu_0 \end{cases}$$

and the new state space to be $C^1(\mathbb{R}^+; \mathcal{M})$.

- Hence we seek, a measure-valued function $\mu : \mathbb{R}_+ \to \mathcal{M}_+$ (non-negative Borel measures), satisfying the differential equation

$$\frac{d}{dt}\mu(A) = \int_A [q_1 f_1(\mu(Q)) - q_2 f_2(\mu(Q))]\mu(dq) = \mathcal{F}(\mu)(A)$$

for all Borel sets A contained in Q.

- Due to the continuity of f_1 and f_2 and the boundedness of Q, it is clear that $\mathcal{F}$ maps non-negative Borel measures $\mathcal{M}_+$ to signed Borel Measures $\mathcal{M}$. In fact, it is also clear that $\mathcal{F}(\mu)$ is absolutely continuous with respect to μ.

Existence-Uniqueness and Asymptotic Behavior

- We assume Q is a compact subset of $(0,\infty)\times(0,\infty)$.

- If $q=(q_1,q_2)$, q_1 is a scaled per capita reproduction rate and q_2 a scaled per capita mortality rate and the quotient $\frac{q_1}{q_2}$ is a scaled reproductive ratio, i.e. it is a measure of the average amount of offspring an individual of characteristic q produces during its lifetime. The actual reproductive ratio at population density X is given by $\frac{q_1}{q_2}\frac{f_1(X)}{f_2(X)}$.

- We define
$$\mathcal{R}=\max\Big\{\frac{q_1}{q_2};q\in Q\Big\}.$$

Concerning the functions f_1 and f_2, we assume the following.

(A1) f_1 is locally Lipschitz continuous on $\mathbb{R}_+$,nonnegative, and decreasing on $[0, \infty)$.

(A2) f_2 is locally Lipschitz continuous on $_+$, nonnegative and increasing on $[0, \infty)$, and strictly positive on $(0, \infty)$.

(A3) f_1/f_2 is strictly decreasing on $(0, \infty)$.

(A4) $\mathcal{R} \limsup_{X \to \infty} \dfrac{f_1(X)}{f_2(X)} < 1$.

- Under these assumptions, we see that for any $q = (q_1, q_2)$ the differential equation

$$\frac{d}{dt}Z(t) = Z(t)\Big[q_1 f_1(Z(t)) - q_2 f_2(Z(t))\Big]$$

has a unique solution.
- If $q = (q_1, q_2) \in Q$ and $\frac{q_1}{q_2} \lim_{X \to 0} \frac{f_1(X)}{f_2(X)} > 1$, the above equation has a unique positive stable equilibrium. We call this equilibrium the *carrying capacity* of populations of characteristic q. This equilibrium is denoted by $K(q)$. If $\frac{q_1}{q_2} \lim_{X \to 0} \frac{f_1(X)}{f_2(X)} \leq 1$, we set $K(q) = 0$. Notice that $K(q)$ only depends on $\frac{q_1}{q_2}$. Therefore we can define

$$K^* = K(q^*), \qquad q^* \in Q^* := \{q \in Q; q_1/q_2 = \mathcal{R}\}.$$

- If $\mathcal{R} \lim_{X \to 0} \frac{f_1(X)}{f_2(X)} > 1$, K^* is the unique positive solution of $\mathcal{R} f_1(K^*) - f_2(K^*) = 0$; otherwise $K^* = 0$. The monotonicity properties of f_1 and f_2 imply that $K^* \geq K(q)$ for all $q \in Q$ which is related to the fact that $\mathcal{R}$ is the highest (scaled) reproductive ratio.

Theorem Let μ_0 be a non-negative Borel measure on Q. Then there exists a unique continuously Differentiable bounded function $\mu : \mathbb{R}_+ \to \mathcal{M}_+$ such that

$$\frac{d}{dt}\mu(A) = \int_A [q_1 f_1(\mu(Q)) - q_2 f_2(\mu(Q))]\mu(dq) = \mathcal{F}(\mu)(A)$$

for all Borel sets A contained in Q. Moreover

$$\mu(t) \leq \max\{\mu_0(Q), K^*\} \qquad \forall\, t \geq 0$$

and

$$\limsup_{t\to\infty} \mu(t)(Q) \leq K^*.$$

Of primary interest here is the asymptotic behavior, as $t \to \infty$, of the solution $\mu(t)(dq)$. The following is an immediate consequence of the previous theorem.

Corollary Let the assumptions (A1) to (A4) be satisfied and $\mathcal{R} \lim_{X \to 0} \frac{f_1(X)}{f_2(X)} \leq 1$. Then $\mu(t)(Q) \to 0$ as $t \to \infty$.

(A5) $\mathcal{R} \lim_{X \to 0} \frac{f_1(X)}{f_2(X)} > 1$.

We will show that $\mu(t)(Q) \to K^*$ as $t \to \infty$ under certain assumptions concerning the initial data.

Proposition Let $Q^* = \{q = (q_1, q_2) \in Q; \frac{q_1}{q_2} = \mathcal{R}$, $\delta_0 > 0$ and U_0 be a relatively open subset of Q which contains Q^*. Let $\mu(t)$ be a solution such that for every $\delta > 0$ there exists some $q^* \in Q^*$ with $\mu(0)(B_\delta(q^*)) > 0$. Then $\mu(t)(Q \setminus U_0) \to 0$ as $t \to \infty$.

Proposition Let (A1) to (A5) be satisfied. Let $\mu(t)$ be a solution such that for every $\delta > 0$ there exists some $q^* \in Q^*$ with $\mu(0)(B_\delta(q^*)) > 0$. Then $\lim_{t\to\infty} \mu(t)(Q) = K^*$, where K^* is the unique solution K of $\mathcal{R}\frac{f_1(K)}{f_2(K)} = 1$.

Finally we add the assumption that

(A6) There exists a unique $q^* \in Q$ such that $\frac{q_1^*}{q_2^*} = \mathcal{R} = \max\{q_1/q_2; q \in Q\}$.

Assumption (A6) states that there is a unique fittest subpopulation. It implies that $\mu(t) \to K^* \delta_{q^*}$ in the weak* sense, i.e., ultimately the population trait characterized by q will be concentrated at q^*.

Theorem Assume that (A1)-(A6) hold. Let $\mu(t)$be a solution such that for every $\delta > 0$ the initial datum satisfies $\mu(0)(B_\delta(q^*)) > 0$. Then for every $f(Q)$,

$$\int_Q f(q)\mu(t)(dq) \to f(q^*)K^*, \quad t \to \infty$$

- Casting the problem in the space of measure of course allows the simultaneous treatment of both discrete and continuous cases. In particular, in the new setting one can choose the initial condition to be a linear combination of Dirac measures centered at a finite collection of points in the parameter space Q.
- These results can be modified to apply to competitive exclusion results for epidemic models similar to those considered by Bremmerman and Thieme (1991), Ackleh and Allen (2003,2005).
- One may wonder why such a general model favors a survival of the fittest asymptotic behavior. One of the reasons for this behavior is that the growth in our model is subpopulation specific (selection).

- If for example, one consider the following generalization of our model (which allows for mutation):

$$\frac{d}{dt}x(t,q) = f_1(X(t)) \int_Q \hat{q}_1 p(q,\hat{q}) x(t,\hat{q}) d\hat{q} \\ -q_2 f_2(X(t)) x(t,q)$$

- Here $p(q,\hat{q})dq$ represents the probability that anindividual of type $\hat{q}$ reproducing an individual of type q. If we let $p(q,\hat{q}) = \delta_q(\hat{q})$ then we obtain the selection (density) model discussed earlier.

Selection-Mutation Model

We obtain the following DE for the selection-mutation model

$$\frac{d}{dt}\mu(t)(A) = f_1(\mu(t)(Q)) \int_Q \hat{q}_1 \Pi(\hat{q})(A)\mu(t)(d\hat{q})$$
$$-f_2(\mu(t)(Q)) \int_A q_2 \mu(t)(dq) = \hat{F}(\mu)(A)$$

where $\Pi(\hat{q})(A) = \int_A P(q,\hat{q})dq$. Note that if $P(q,\hat{q}) = \delta_q(\hat{q})$, then the above model reduces to the selection (measure-valued) model.

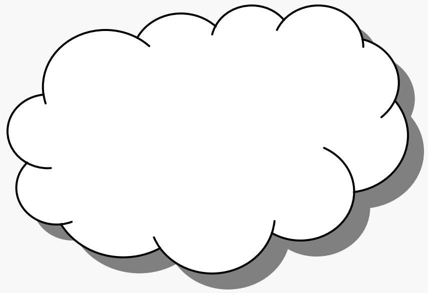
The Hierarchical Structured Model

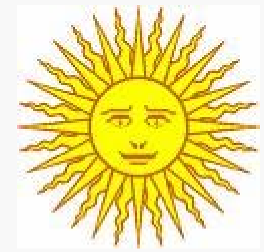
- We consider the following hierarchical size-structured population model:

$$u_t + (g(x, Q(x,t))u)_x + m(x, Q(x,t))u = 0$$
$$g(x_0, Q(x_0,t))u(x_0,t) = C(t) + \int_{x_0}^{x_1} \beta(x, Q(x,t))u(x,t)dx$$
$$u(x,0) = u^0(x).$$

Here $u(x,t)$ is the density of individuals having size x at time t, and for $0 \leq \alpha < 1$,

$$Q(x,t) = \alpha \int_{x_0}^{x} w(\xi)u(\xi,t)d\xi + \int_{x}^{x_1} w(\xi)u(\xi,t)d\xi,$$





When $\alpha = 0$ such models have been used to describe the dynamics of tree populations when light is limiting

Review of Related Literature

- **The case** $\alpha = 1$

$$Q(x,t) = Q(t) = \int_{x_0}^{x_1} w(\xi)u(\xi,t)d\xi,$$

- This problem has been studied by several authors.
  - Calsina and Saldana (1995) and Ackleh and Ito (1997) studied this problem.
  - Diekmann, Gyllenberg, Metz, Thieme, and their co-workers (1998, 2001).

- **The case** $0 \leq \alpha < 1, \text{ and } g_Q \leq 0$
  - Cushing (1994) and Cushing and Henson (1996) Ackleh and Deng (2005) studied this problem for the age structured case (i.e., $g(x, Q) = 1$).
  - Calsina and Saldana (1997), Kraev (2001), Blayneh (2002) and Ackleh et al. (2004) studied the case $g_Q(x, Q) \leq 0$.

What happens if $g_Q \le 0$ is not satisfied?

- To formally explain this, assume for simplicity that $\alpha = 0$, $w = 1$, $g = g(Q)$, $\beta = \beta(Q)$, and $m = m(Q)$.

- Consider the IBVP

$$u_t + g u_x = -(\frac{dg}{dx} + m)u$$

$$= -(g_Q Q_x + m)u = -mu + g_Q u^2$$

- Integrating the equation over (x, x_1) we get:

$$Q_t + g(Q)Q_x + M(Q) = 0,$$
$$\frac{d}{dt}Q(x_0,t) = C(t) + B(Q(x_0,t)) - M(Q(x_0,t)),$$
$$Q(x,0) = Q^0(x).$$

- where $M(Q) = \int_0^Q m(s)ds$ and $B(Q) = \int_0^Q \beta(s)ds$
- This is a local quasilinear hyperbolic equation in Q and for $a < b$ we have

$$Q^0(a) = \int_a^{x_1} u^0(x)dx \geq \int_b^{x_1} u^0(x)dx = Q^0(b)$$

- Hence, if the condition $g_Q \leq 0$ is not imposed then the characteristic curves may intersect causing a discontinuity in Q which corresponds to a Dirac delta measure in u since $Q_x = -u$.

Vanishing Viscosity Method

- For the sake of simplicity we first assume:

$$m(x,Q) = m(Q),\ \beta(x,Q) = \beta(Q),$$
$$w(x) = 1,\ \text{and } \alpha = 0.$$

- In particular, for $\epsilon > 0$ we consider the following viscous problem

$$\begin{cases} u_t + (g(x,Q)\,u)_x + m(Q)u = \epsilon\, u_{xx} \\ g(x_0, Q(x_0,t))\,u(x_0,t) - \epsilon\, u_x(x_0,t) \\ \qquad = C(t) + \int_{x_0}^{x_1} \beta(Q(y,t))u(y,t)dy \\ u_x(x_1,t) = 0, \end{cases} \tag{1}$$

- where

$$Q(x,t) = \int_x^{x_1} u(x,t)\,dx.$$

- Integrating the previous PDE on (x, x_1), we obtain

$$Q_t + g(x,Q)Q_x + M(Q) = \epsilon Q_{xx}$$
$$\frac{d}{dt}Q(x_0,t) = C(t) + B(Q(x_0,t)) - M(Q(x_0,t))$$
$$Q(x_1,t) = 0,$$

- where

$$M(Q) = \int_0^Q m(s)\,ds, \quad B(Q) = \int_0^Q \beta(s)\,ds.$$

- This problem is a parabolic equation with a non-homogeneous but local boundary value at $x = x_0$. Clearly the first boundary equation has a unique solution $Q(x_0,t)$ which satisfies

$$|Q(x_0,t)| \leq e^{\omega t}|Q(x_0,0)| + \int_0^t e^{\omega(t-s)}|C(s)|\,dx \leq K_1, \quad t \in [0,T],$$

for some positive constant K_1.

Define

$$\tilde{Q}(x,t) = Q(x,t) - \frac{x_1 - x}{x_1 - x_0} Q(x_0,t).$$

Then we have

$$\begin{cases} \tilde{Q}_t + g(x,Q)Q_x + M(Q) = \epsilon\,\tilde{Q}_{xx} - \frac{x_1-x}{x_1-x_0}\frac{d}{dt}Q(x_0,t) \\ \tilde{Q}(x_1,t) = \tilde{Q}(x_0,t) = 0. \end{cases}$$

Let $H = L^2(x_0,x_1)$ and $V = H_0^1(x_0,x_1)$.
Applying standard techniques for parabolic systems which are based on the Gelfand triple $V \subset H = H^* \subset V^*$ we see that for $\epsilon > 0$ the PDE has a unique solution $\tilde{Q}_\epsilon \in C(0,T;V) \cap L^2(0,T;H^2(x_0,x_1)) \cap H^1(0,T;H)$.
Hence, PDE with nonhomogeneous boundary has a unique solution $Q_\epsilon(0,T;H^1) \cap L^2(0,T;H^2(x_0,x_1)) \cap H^1(0,T;H)$.

Apriori Estimates and Existence

- We will prove that $\lim_{\epsilon\to 0^+} Q_\epsilon$ and $\lim_{\epsilon\to 0^+} u_\epsilon$ exist, and thatlim$_{\epsilon\to 0^+} u_\epsilon$ defines a measure-valued solution to (1).
- Multiplying by the test function $\phi = \max(0, Q_\epsilon(x,t) - K_2)$ we obtain

$$|Q_\epsilon(x,t)| \leq \max(|Q_\epsilon(x,0)|_\infty, |Q_\epsilon(x_0,t)|) \leq K_2,$$

where K_2 is a positive constant independent of ϵ.

- Using the fact that $u_\epsilon(x,t) \geq 0$ and

$$\int_{x_0}^{x_1} |u_\epsilon(x,t)|\, dx \leq e^{\omega t} \int_{x_0}^{x_1} |u_0(x)|\, dx + \int_0^t e^{\omega(t-s)} |C(s)|\, ds \leq K_3.$$

Multiplying by $\tilde{Q}$, integrating over (x_0, x_1), and using the above estimates we get

$$\frac{d}{dt}\int_{x_0}^{x_1} \frac{|\tilde{Q}_\epsilon(x,t)|^2}{2}\,dx + \epsilon\int_{x_0}^{x_1} |(\tilde{Q}_\epsilon)_x|^2\,dx \leq K_4,$$

for some positive constant K_4 (independent of $\epsilon > 0$).

Upon integrating the above inequality over $(0, T)$ we see that

$$\epsilon\int_0^T\int_{x_0}^{x_1} |(\tilde{Q}_\epsilon)_x|^2\,dxdt$$

is uniformly bounded in $\epsilon > 0$.

Existence of Vanishing Viscosity Solution

Therefore Q_ϵ is bounded in $L^\infty((0,T)\times(x_0,x_1))\cap L^\infty((0,T);BV(x_0,x_1))$. Furthermore, it follows that

$$\frac{d}{dt}\tilde{Q}_\epsilon \text{ is bounded in } L^2(0,T;V^*).$$

Now, using Aubin's lemma to pass to the limit $\epsilon \to 0^+$, we get Q_ϵ has a strong convergent subsequence in $L^2((0,T)\times(x_0,x_1))$ where the limit $Q \in L^\infty((0,T)\times(x_0,x_1)) \cap L^\infty(0,T;BV(x_0,x_1))$ satisfies

$$\begin{aligned}
&\int_0^t \int_{x_0}^{x_1} (-Q\phi_s - G(x,Q)\phi_x + (-G_x(x,Q) + M(Q))\,\phi)\,dx\,ds \\
&\qquad + \int_{x_0}^{x_1} (Q(x,t)\phi(x,t) - Q(x,0)\phi(x,0))\,dx = 0, \\
&\tfrac{d}{dt}Q(x_0,t) = C(t) + B(Q(x_0,t)) - M(Q(x_0,t)),
\end{aligned}$$

for all $\phi \in C^1((0,T)\times(x_0,x_1))$ satisfying $\phi(x_0,t) = \phi(x_1,t) = 0$.

Moreover $u(\cdot,t) = -Q_x(\cdot,t)$ is a measure-valued function and satisfies

$$\begin{aligned}&\int_0^t \int_{x_0}^{x_1} (-u\psi_s - g(x,Q)u\psi_x + m(Q)u\psi)\,dx\,ds \\ &\quad + \int_{x_0}^{x_1} (u(x,t)\psi(x,t) - u_0(x)\psi(x,0))\,dx \\ &+ \int_0^t (C(s) + B(Q(x_0,s))\psi(x_0,s)\,ds = 0,\end{aligned}$$

for all $\psi \in C^1((0,T)\times(x_0,x_1))$,
where

$$\begin{aligned}-g(x,Q)u &= (G(x,Q))_x - G_x(x,Q), \\ m(Q)u &= -(M(Q))_x \quad \in L^\infty(0,T;C^*).\end{aligned}$$

The Finite-Difference Method

We let $x_0 = 0$ and $x_1 = 1$. We consider a semi-implicit method as follows. Let $\Delta x = \frac{1}{N}$ and $\Delta t = \frac{T}{M}$. For $x_j = j\Delta x, j = 0, 1, 2, \cdots, N$ and $t_k = k\Delta t$, $t = 0, 1, 2, \cdots, M$, we will denote by u_j^k and Q_j^k the difference approximation of $u(t_k, x_j)$ and $Q(t_k, x_j)$ and we define

$$g_j^k = g(x_j, Q_j^k), \quad \beta_j^k = \beta(x_j, Q_j^k),$$

$$m_j^k = m(x_j, Q_j^k) \text{ and } C^k = C(t_k)$$

We assume that $g(x, Q) > 0$ and $g(1, Q) = 0$. The semi-implicit scheme is given by

$$\begin{cases} \frac{u_j^{k+1} - u_j^k}{\Delta t} + \frac{g_j^k u_j^{k+1} - g_{j-1}^k u_{j-1}^{k+1}}{\Delta x} + m_j^k u_j^{k+1} = 0, \quad 1 \le j \le N \\ g_0^k u_0^{k+1} = C^k + \sum_{i=1}^N \beta_i^k u_i^{k+1} \, \Delta x \end{cases}$$

with

$$Q_j^k = \alpha \sum_{i=1}^{j} w_i u_i^k \, \Delta x + \sum_{i=j+1}^{N} w_i u_i^k \, \Delta x$$

Convergence Analysis

If we define

$$c_j^k = 1 + \frac{\Delta t}{\Delta x} g_j^k + \Delta t\, m_j^k, \quad 1 \le j \le N$$

then (2.1) is equivalently written as the following system oflinear equation for $\vec{u}^{k+1} = [u_0^{k+1}, u_1^{k+1}, \ldots, u_N^{k+1}]^T \in R^{N+1}$

$$A^k \vec{u}^{k+1} = \vec{f}^k \tag{3.1}$$

where

$$\vec{f}^k = [C^k, u_1^k, u_2^k, \ldots, u_N^k]^T$$

and

$$A^k = \begin{pmatrix} g_0^k & -\Delta x \beta_1^k & -\Delta x \beta_2^k & \cdots & -\Delta x \beta_N^k \\ -\frac{\Delta t}{\Delta x} g_0^k & c_1^k & 0 & \cdots & 0 \\ 0 & -\frac{\Delta t}{\Delta x} g_1^k & c_2^k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & -\frac{\Delta t}{\Delta x} g_{N-1}^k & c_N^k \end{pmatrix}.$$

If $\frac{\Delta t}{\Delta x} \leq \hat{C}$ then (3.1) has a unique solution.

Lemma 1. $u_j^k \geq 0$ provided that $u_j^0 \geq 0$.

Lemma 2. *Assume* $m(x,Q) - \beta(x,Q) \leq \omega_1$ *and* $\omega_1 \Delta t < 1$. *We have the estimate*

$$\|u^k\|_1 \leq Q_{max} = \left(\tfrac{1}{1-\omega_1 \Delta t}\right)^k \|u_0\|_1 + \textstyle\sum_{i=1}^{k} \left(\tfrac{1}{1-\omega_1 \Delta t}\right)^{k+1-i} |C^i| \, \Delta t.$$

and

$$0 \leq Q_j^k \leq Q_{max}.$$

Lemma 3. *There exists an $L > 0$, independent of $\Delta x, \Delta t$, such that for any $m > p$*

$$\sum_{j=1}^{N} \left|\frac{Q_j^m - Q_j^p}{\Delta t}\right| \Delta x \leq L(m-p)$$

Define a family of functions $\{Q_{\Delta x, \Delta t}\}$ by

$$Q_{\Delta x, \Delta t}(x,t) = Q_j^k \quad \text{for } \; x \in [x_{j-1}, x_j), \;\; t \in [t_{k-1}, t_k)$$

Then, the set of functions $\{Q_{\Delta x, \Delta t}\}$ is compact in the topology of $L^1((0,1) \times (0,T))$ and we have the following theorem.

Theorem 4. *There exists a subsequence* $\{Q_{\Delta x_i,\Delta t_i}\} \subset \{Q_{\Delta x,\Delta t}\}$ *which converges to a* $BV([0,1]\times[0,T])$ *function* $Q(x,t)$ *in the sense that for all* $t > 0$

$$\int_0^1 |Q_{\Delta x_i,\Delta t_i}(x,t) - Q(x,t)|\, dx \to 0,$$

and

$$\int_0^T \int_0^1 |Q_{\Delta x_i,\Delta t_i}(x,t) - Q(x,t)|\, dx\, dt \to 0,$$

as $i \to \infty$. *Furthermore the limit function satisfies*

$$\|Q\|_{BV([0,1]\times[0,T])} \leq c(|u_0|_1, |C|_\infty).$$

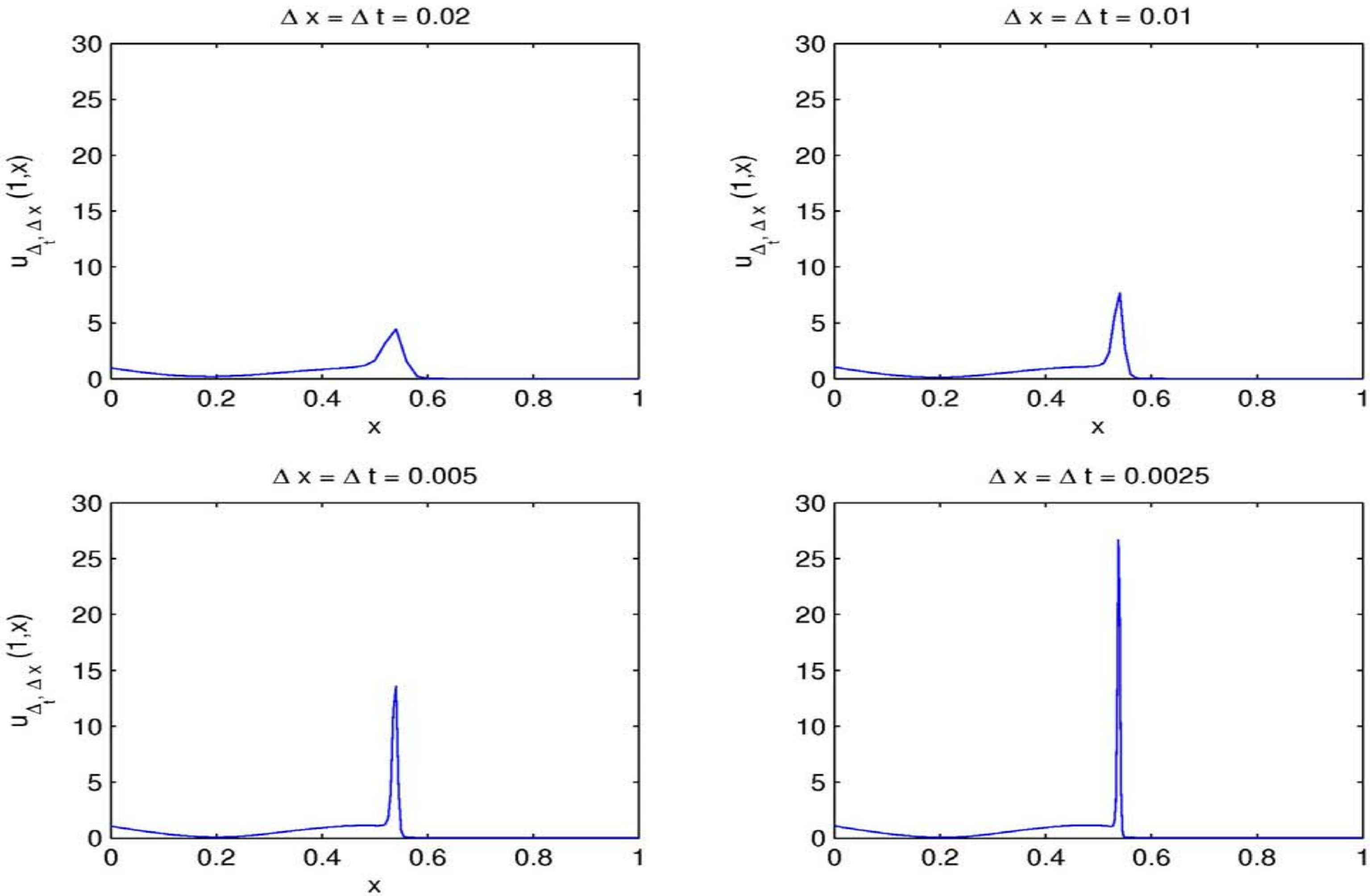
Theorem 5. The sequence $Q_{\Delta x, \Delta t}(x, t)$ constructed via our difference scheme converges in $L^\infty((0, T); L^1(0, 1))$ to the unique entropy solution $Q(x, t)$.

Corollary 6. Let $u_{\Delta x, \Delta t}(x, t) = u_j^k$ for $x \in [x_{j-1}, x_j)$, $t \in [t_{k-1}, t_k)$. Then, $u_{\Delta x, \Delta t} \to u$ in weak star topology of C^* for every $t \in [0, T]$.

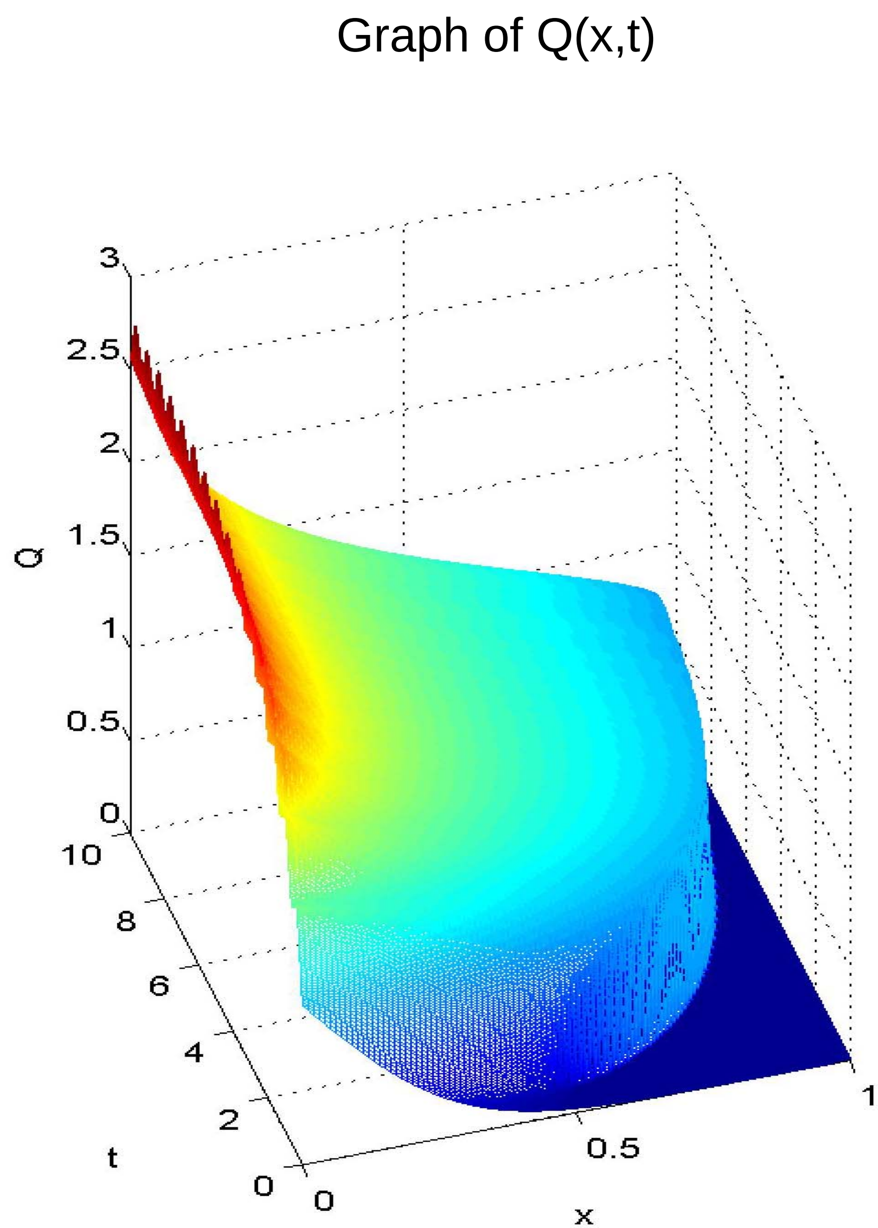
Numerical Results

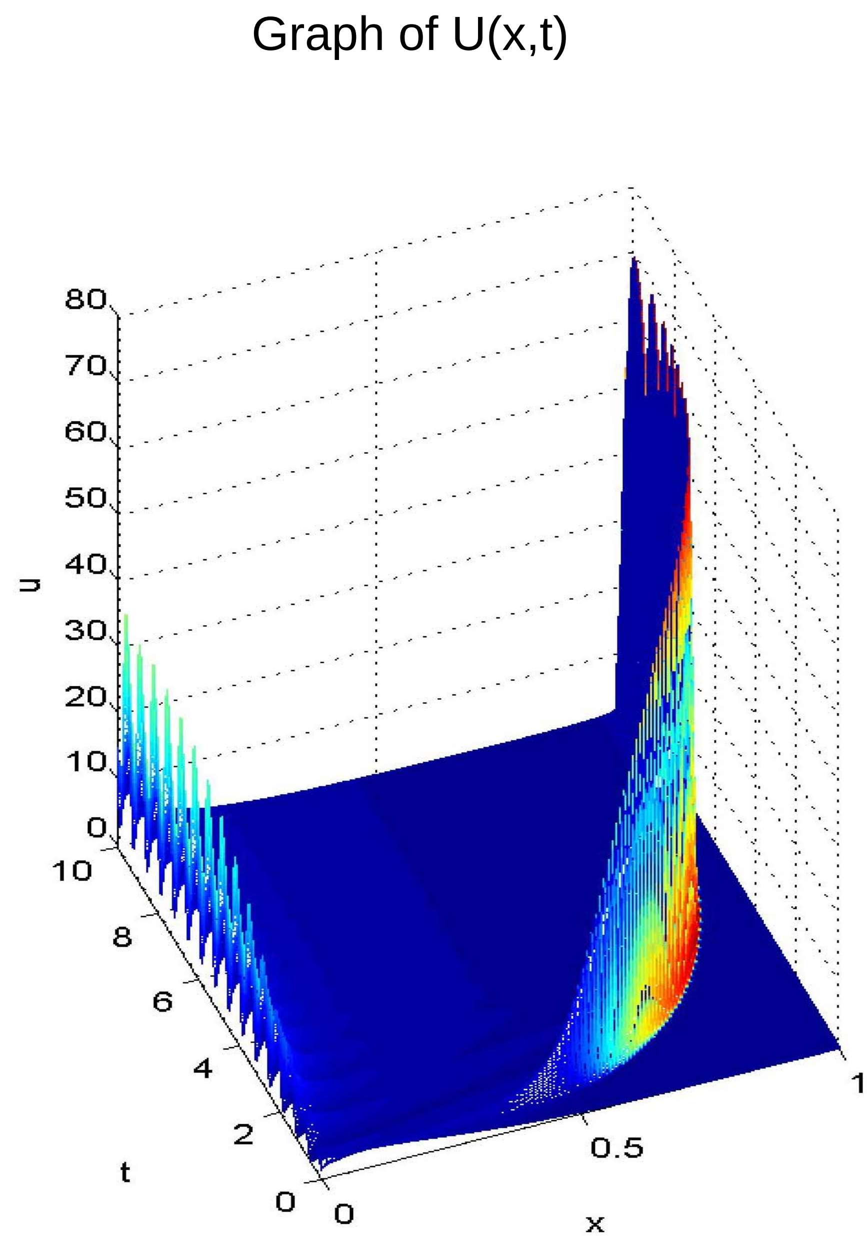
Example 1. Let $\alpha = 0$ and choose the parameters g, m and β as follows:

$$
\begin{aligned}
&g(x,Q) = 5(1-x)(Q+0.01)\exp(-2Q),\\
&m(x,Q) = (Q+1)\exp(x)\\
&\beta(x,Q) = 0.2x\exp(-0.2Q)\\
&C(t) = 1+\sin(2\pi t).
\end{aligned}
$$



Δ x = Δ t = 0.02
Δ x = Δ t = 0.01
Δ x = Δ t = 0.005
Δ x = Δ t = 0.0025
$u_{\Delta_t, \Delta x}(1,x)$
x


Graph of Q(x,t)
Q
3
2.5
2
1.5
1
0.5
0
t
10
8
6
4
2
0
x
0
0.5
1


Graph of U(x,t)
u
80
70
60
50
40
30
20
10
0
t
10
8
6
4
2
0
x
0
0.5
1

Indefinite Growth Rate

If $g(x, Q)$ is indefinite but $g(0, Q) > 0$ and $g(1, Q) = 0$, we use the following approximation;

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} + \frac{g_j^k u_j^{k+1} - 0}{\Delta x} + m_j^k u_j^{k+1} = 0, \quad 1 \leq j \leq N$$

if $g_j^k > 0$ and $g_{j-1}^k < 0$.

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} + \frac{g_{j+1}^k u_{j+1}^{k+1} - g_j^k u_j^{k+1}}{\Delta x} + m_j^k u_j^{k+1} = 0, \quad 1 \leq j \leq N$$

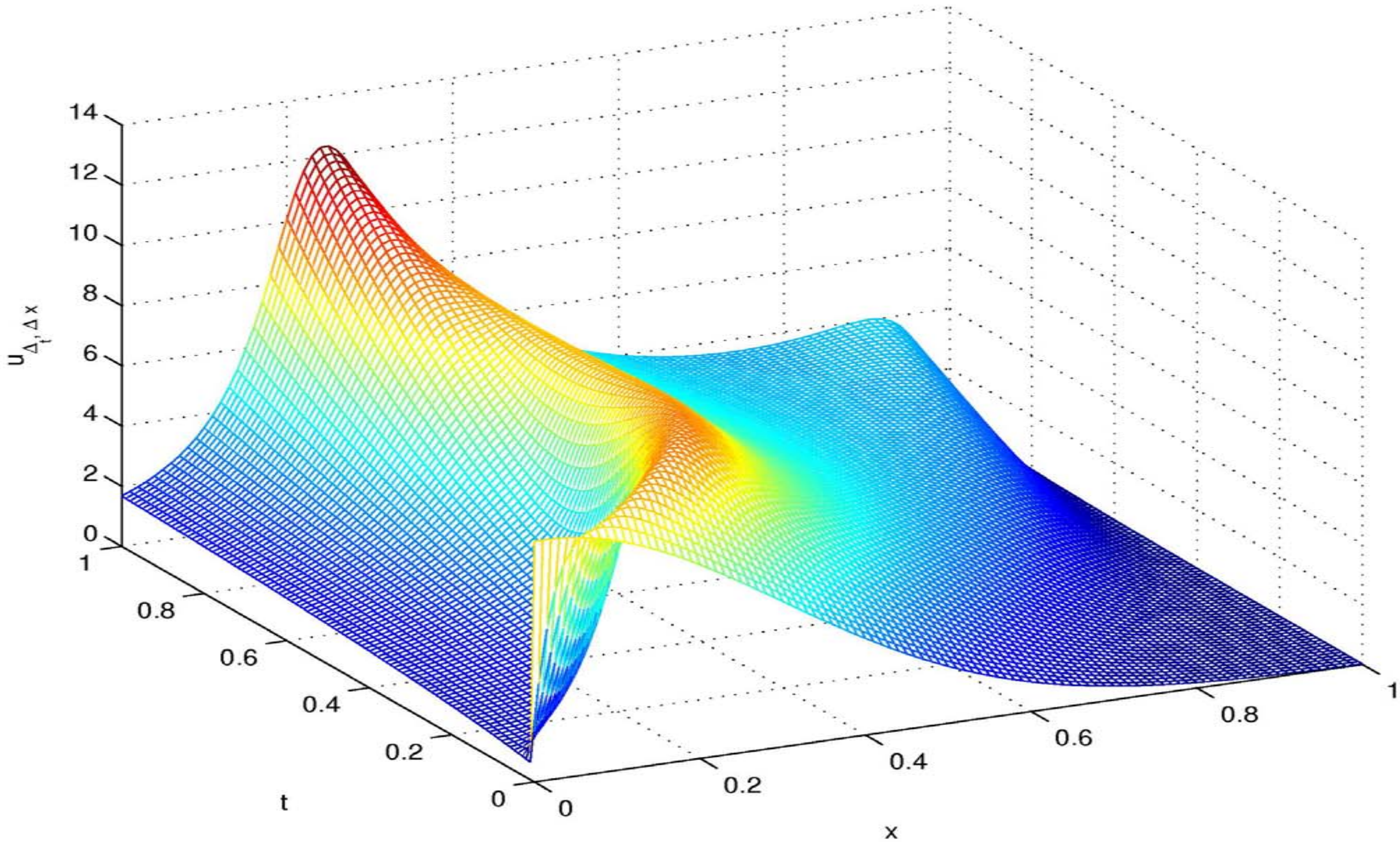
if $g_j^k \leq 0$ and $g_{j+1}^k \leq 0$.

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} + \frac{0 - g_j^k u_j^{k+1}}{\Delta x} + m_j^k u_j^{k+1} = 0, \quad 1 \leq j \leq N$$

if $g_j^k \leq 0$ and $g_{j+1}^k > 0$.

Example 2. Let $\alpha = 0$ and choose the parameters g, m andβ as follows:

$$g(x, Q) = 2(1 - x)(1 - xQ),$$
$$m(x, Q) = 0.01(Q + 1)\exp(x)$$
$$\beta(x, Q) = 0.2x\exp(-0.2Q)$$
$$C(t) = 0.$$



14
12
10
8
6
4
2
0
1
0.8
0.6
0.4
0.2
0
t
0
0.2
0.4
0.6
0.8
1
x

Concluding Remarks

- We have developed a finite difference approximation to the singular solutions of the model. We have established convergence of the approximating environments $Q_{\Delta x,\Delta t}$ strongly in $L^\infty((0,T);L^1(0,1))$ to the unique solution Q and showed that the approximating measures $u_{\Delta x,\Delta t}$ converge in the weak* topology to u the measure-valued solution of the model.
- While uniqueness of the entropy solution Q has been established. We have not established uniqueness of u.
- Investigate the case $Q(x,t) = \int_0^1 k(x,y)u(y,t)dy$.