Control Theory, Riccati Equations, and Contraction Semigroups

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- The Thompson metric

Linear Regulator

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and a quadratic "cost" assigned to each control function $u : [0, T] \to \mathbb{R}^m$:

$$C(u) := \int_0^T [x(s)^* Q(s)x(s) + u(s)^* R(s)u(s)]ds + x(T)^* Sx(T),$$

where typically $Q(s)$ is positive semidefinite, $R(s)$ is positive definite, and S is positive semidefinite.

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where typically $Q(s)$ is positive semidefinite, $R(s)$ is positive definite, and S is positive semidefinite.

The integral is the "running cost" and the final term is the "end cost." Note $S = 0$ is possible.

Riccati Equations

To solve the linear regulator optimization problem, one needs the notion of a *Matrix Riccati Differential Equation*, a quadratic differential equation on the space $\mathrm{Sym}(\mathbb{R}^n)$ of $n \times n$ symmetric matrices of the form

$$\dot{K}(t) = R(t)+A(t)K(t)+K(t)A^*(t)-K(t)S(t)K(t), \;\; K(t_0) = K_0, \tag{RDE}$$

where $R(t), S(t), K_0 \in \mathrm{Sym}(\mathbb{R}^n)$ and A is an $n \times n$ real matrix with transpose (=adjoint) A^*.

Existence of Optimal Controls

An *optimal control* for the linear regulator is a control function $\hat{u} : [0, T] \longrightarrow \mathbb{R}^m$ that minimizes the cost $C(u)$ over all possible controls $u(\cdot)$. Its corresponding solution is the *optimal trajectory*.

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Theorem If there exists a solution $P(t)$ on $[0, T]$ of the *matrix Riccati Differential Equation*:

$$\dot{P} = P(BR^{-1}B^*)P - PA - A^*P - Q, \quad P(T) = S, \qquad (RDE)$$

where A, B come from the dynamics, Q, R from the running cost, and S from the end cost, then a unique optimal control and trajectory exist on $[0, T]$.

The Optimal Solution

For $P(t)$ the solution of the matrix Riccati differential equation, the solution of

$$\dot{x}(t) = \left(A(t) - R(t)^{-1} B(t)^* P(t) \right) x(t), \quad x(0) = x_0$$

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The minimal cost is given by $V(x_0) = x_0'P(0)x_0$.

Importance of Riccati Equations

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Our approach will be somewhat indirect through the study of the symplectic group and symplectic semigroup.

The Sympletic Form

Let E be $\mathbb{R}^n$ equipped with the usual inner product and V_E be $\mathbb{R}^n \bigoplus \mathbb{R}^n$ (with column notation) equipped with the sympletic form

$$Q\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) := x_1 \cdot y_2 - y_1 \cdot x_2.$$

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The sympletic form $\mathbb{Q}$ is a skew-symmetric, nondegenerate bilinear form. Any such form on $V_E = \mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{R}^{2n}$ is form-preserving isomorphic to the one just given.

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The constructions we consider generalize to arbitrary Hilbert spaces E.

Block Matrix Representation

We denote the set of linear operators on E resp. $V_E = E \oplus E$ by $\mathcal{L}(E)$ resp. $\mathcal{L}(V_E)$. Any linear operator $A\colon V_E \to V_E$ has a *block matrix representation* of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ where } A_{ij} := \pi_i \circ A \circ \iota_j \colon E \to E,\ i, j = 1, 2.$$

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Note that the adjoint A^* is given by

$$A^* = \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix}.$$

We denote the symmetric members of $\mathcal{L}(E)$ and $\mathcal{L}(V_E)$ by $\mathrm{Sym}(E)$ and $\mathrm{Sym}(V_E)$ resp.

The Symplectic Group

The *sympletic group* is the subgroup of of the general linear group $\mathrm{GL}(V_E)$ preserving Q:

$$\text{Sp}(V_E) := \{M \in \mathrm{GL}(V_E)\colon \forall x, y \in V_E,\ Q(Mx, My) = Q(x, y)\}.$$

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Important examples of members of $\mathrm{Sp}(V_E)$ are the block diagonal and strictly triangular matrices:

$$\begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix}, \quad \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} I & 0 \\ C & I \end{bmatrix},$$

where $A \in \mathrm{GL}(E)$ and $B, C \in \mathrm{Sym}(E)$. The set of such elements generates $\mathrm{Sp}(V_E)$.

Regular Elements

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M is regular $\Leftrightarrow$ M has an UDL-*triple decomposition*:

$$M = \begin{bmatrix} I & U \\ 0 & I \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & (D^*)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ L & I \end{bmatrix}$$

$\Leftrightarrow$ M has an LDU-triple decomposition. Triple decompositions are unique.

The Sympletic Lie Algebra

The *symplectic Lie algebra* $\mathfrak{sp}(V_E)$ consists of all $X \in \mathrm{End}(V_E)$ such that $\exp(tX) \in \mathrm{Sp}(V_E)$ for all $t \in \mathbb{R}$, or alternatively the set of tangent vectors to $\mathrm{Sp}(V_E)$ at the identity. Members of $\mathfrak{sp}(V_E)$ are often called *Hamiltonian operators*.

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For $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(V_e)$

$X \in \mathfrak{sp}(V_E) \Leftrightarrow B$ and C are symmetric, and $D = -A^*$.

The Fundamental Equation

We consider on $\mathrm{Sp}(V_E)$ the *fundamental equation*

$$\dot{X}(t) = \tilde{M}(t)X(t),\ X(t_0) = I_{V_E},\ \tilde{M}(t) \in \mathfrak{sp}(V_E) \tag{1}$$

with *fundamental solution* $\Phi(t, t_0)$. From the homogeneity of $\mathrm{Sp}(V_E)$, this equation has a global solution for quite general control functions $\tilde{M}(\cdot)$.

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The general solution of the fundamental equation $\dot{X}(t) = \tilde{M}(t)X(t)$ for $X(t_0) = X_0 \in \mathrm{Sp}(E)$ is given by $X(t) = \Phi(t, t_0)X_0$, and is called a *trajectory*.

A Second Column Solution

We consider the solution of $\dot{X}(t) = \tilde{M}(t)X(t)$ over some interval I where $X_{22}(t)$ is invertible (i.e. X is regular). Then $K(t) := X_{12}(t)X_{22}^{-1}(t)$ is symmetric (since $X(t) \in \mathrm{Sp}(E)$) and satisfies

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$$\begin{aligned}
\dot{K}(t) &= \frac{d}{dt}(X_{12}(t)X_{22}^{-1}(t)) \\
&= \dot{X}_{12}(t)X_{22}^{-1}(t) - X_{12}(t)X_{22}^{-1}(t)\dot{X}_{22}(t)X_{22}^{-1}(t) \\
&= M_{12}(t) + M_{11}(t)K(t) + K(t)M_{11}^{*}(t) \\
&\quad -K(t)M_{21}(t)K(t).
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Note that M_{12} and M_{21} are symmetric since $\tilde{M} \in \mathfrak{sp}(V_E)$. Thus $K(t)$ satisfies the Riccati matrix differential equation on I (Radon, 1927; J. Levin, 1959).

The Lifted Riccati Equation

Given a Riccati differential equation

$$\dot{K}(t) = R(t)+A(t)K(t)+K(t)A^*(t)-K(t)S(t)K(t),\ K(t_0) = K_0,$$

where $R(t), S(t), K_0 \in \mathrm{Sym}(E)$, we consider the *sympletic second column method* for a solution.

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where $R(t), S(t), K_0 \in \mathrm{Sym}(E)$, we consider the *sympletic second column method* for a solution.
We first solve the fundamental equation

$$\dot{X}(t) = \begin{bmatrix} A(t) & R(t) \\ S(t) & -A^*(t) \end{bmatrix} X(t),\ X(t_0) = \begin{bmatrix} I & K_0 \\ 0 & I \end{bmatrix}$$

over any interval for which X_{22}^{-1} exists and then obtain $K(t)$ as

$$K(t) = X_{12}(t)X_{22}^{-1}(t).$$

The sympletic second column method allows one to “lift” control problems involving Riccati differential equations to the sympletic group. Note that even when the Riccati equation “blows up,” one still has a “virtual” solution at the sympletic group level.

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These ideas are not original, but a reworking of earlier insights of R. Hermann and M. A. Shayman.

Positive Semidefinite Case

A important special case of the Riccati equation

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is the case that $R, S \geq 0$, that is, are positive semidefinite.

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In this case the corresponding sympletic fundamental equation is given by

$$\dot{X}(t) = \begin{bmatrix} A(t) & R(t) \\ S(t) & -A^*(t) \end{bmatrix} X(t), \; X(0) = I_{V_E},$$

where $R(t), S(t) \geq 0$.

The Symplectic Semigroup

We set

$$\mathcal{W} := \{\tilde{M} = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} : B, C \geq 0\} \subseteq \mathfrak{sp}(E).$$

Then we can consider the reachable set for

$$\dot{X}(t) = \tilde{M}(t)X(t),\ X(0) = I$$

as $t \mapsto \tilde{M}(t)$ varies over all admissible steering functions from $[0, \infty)$ to $\mathcal{W}$. This set is a subsemigroup of $\mathrm{Sp}(E)$, called the *sympletic semigroup* and denoted $\mathcal{S}$.

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Note that $\mathcal{S}$ consists of the points belonging to lifts beginning at I of all Riccati equations with $R(t), S(t) \geq 0$. All points of $\mathcal{S}$ are reachable from I via piecewise constant controls.

Positive Symplectic Operators

If $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(E)$ satisfies $BA^*, A^*C \geq 0$, A invertible, then M is called a *positive symplectic operator*.

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(1) $M \in \mathcal{S}$, the symplectic semigroup;

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Positive Sympletic Operators

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The following are equivalent for $M \in \mathrm{Sp}(E)$:

(1) $M \in \mathcal{S}$, the symplectic semigroup;

(2) M is a positive symplectic operator.

Members of $\mathcal{S}$ are all regular, and hence each has a triple decomposition. The factors of the decomposition remain in $\mathcal{S}$. This observation sometimes allows arguments concerning members of $\mathcal{S}$ to be reduce to the strictly block triangular and block diagonal case.

Lie Semgroup Theory

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\emptyset \neq \Omega \subseteq \mathfrak{g}$. We consider the fundamental differential equation

$$\dot{X}(t) = U(t)X(t), \quad X(0) = e, \text{ where } U : [0, \infty) \to \Omega.$$

All points lying on solution trajectories make up the reachable set from e, which is a subsemigroup $S(\Omega)$.

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The *Lie wedge* of $S(\Omega)$ is the largest possible $\mathcal{W} \subseteq \mathfrak{g}$ such that $S(\mathcal{W}) \subseteq \overline{S(\Omega)}$, and is always a closed convex cone.

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The fundamental equation for $U(t) \in \Omega$ and $X(0) \in S(\Omega)$ has a global solution with positive time values contained in $S(\Omega)$.

Existence of Solutions

It thus follows from the general considerations of Lie semigroup theory and the regularity of members of $\mathcal{S}$ that the Riccati equation

$$\dot{K} = R + AK + KA^{*} - KSK, \quad K(0) = K_0$$

for $R, S, K_0 \geq 0$ has a positive semidefinite solution that exists for all $t \geq 0$.

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The Riccati equation arising in the linear regulator optimization problem is a time reversed version of the preceding one, hence has a solution over the interval in question.

SYM(E)

The vector space $\mathrm{Sym}(E)$ embeds in the coset space $\mathrm{Sp}(V_E)/\mathcal{P}$ as a dense open subset, where $\mathcal{P}$ is the subgroup of lower triangular block matrices:

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The action of $\mathcal{S}$ on $\mathrm{Sp}(V_E)/\mathcal{P}$ carries the embedded image of the set $\mathrm{Sym}^+(E)$ of positive definite operators into itself, and the action is given by fractional transformation:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} .P = (AP+B)(CP+D)^{-1} \in \mathrm{Sym}^+(E).$$

Thompson Metric

There is a naturally defined metric on $\mathrm{Sym}^+(E)$, the cone of positive definite operators, called the *Thompson metric* (or *part metric*), and defined from the Loewner order: $A \leq B$ if $B - A \geq 0$.

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For A, B positive definite define $M(A/B) = \inf\{t : A \leq tB\}$ and define the Thompson metric by

$$d(A,B) = \log(\max\{M(A,B), M(B,A)\}).$$

There is a naturally defined Finsler metric on $\mathrm{Sym}^+(E)$ such that the Thompson metric is the distance metric for the Finsler metric.

Contractions

As we have seen earlier the sympletic semigroup acts on the space of positive definite operators by fractional transformations:

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Theorem (Y. Lim, J.L.) With respect to the Thompson metric ρ, the action of $\mathcal{S}$ on $\mathrm{Sym}^+(E)$ by fractional transformations is by strict contractions.

Birkhoff Formula

The coefficent of contraction is given by the Birkhoff formula:

$$\tanh\left(\frac{\Delta(g)}{4}\right) = \sup\left\{\frac{\rho(g(A), g(B))}{\rho(A, B)} : 0 < A, B,\ A \neq B\right\},\ \ g \in \mathcal{S},$$

where $\Delta(g)$ denotes the diameter of $g(\mathrm{Sym}^+(E))$. In particular, it is always a contraction and a strict contraction for $g \in$ int$(\mathcal{S})$.

Applications

The contractive property of the operators from $\mathcal{S}$ allows one to study convergence properties and even estimate convergence rates for the classical Riccati differential equation and a number of other types of Riccati equations.

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For example, it can be shown that any two solutions $K_1(t), K_2(t)$ of the Riccati matrix differential equation

$$\dot{K}(t) = R(t) + A(t)K(t) + K(t)A^*(t) - K(t)S(t)K(t)$$

with initial points $K_1, K_2 > 0$ resp. converge exponentially toward each other as long as $S(t)^{1/2}R(t)S(t)^{1/2} > 0$ remains appropriately bounded away from 0.

Applications (cont.)

Consider the discrete algebraic Riccati equation

$$X = A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA + H$$

on a Hilbert space that arises in the context of minimizing a quadratic cost on discrete-time autonomous systems.

Applications (cont.)

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on a Hilbert space that arises in the context of minimizing a quadratic cost on discrete-time autonomous systems.

If $R, H, BR^{-1}B^* > 0$ and A is invertible, then iterating the right-hand side beginning with any positive definite operator converges to the unique fixed point X_∞ with

$$p(X_\infty, X_n) \le \frac{L^n}{1-L} p(X_1, X_0)$$

where $L = \tanh((1/4)\log\|I + \Lambda\Lambda^*\|)$,
$\Lambda = H^{-1/2}A^*(BR^{-1}B^*)^{-1/2}$.

Future Work

The authors intend as a future research project to carry these investigations further, and look for additional applications of the Birkhoff contraction theorem to Riccati equations and problems in control theory.