A New Tuning Function-Based Robust Adaptive Controller for Parametric Strict-Feedback Systems

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The Adaptive Control Problem

Consider the system

$$\dot{x} = f(x, \theta, d) + g(x)u$$

affine in the uncertain parameter θ, where $d(t)$ is an unknown exogenous disturbance. Find a dynamic feedback

$$\begin{aligned} u &= u(x, \hat{\theta}) \\ \dot{\hat{\theta}} &= \tau(x, \hat{\theta}), \end{aligned}$$

where $\hat{\theta}$ is the estimate of θ, that drives the output to zero while keeping all closed-loop signals bounded.

Background

- In presence of bounded disturbances, performance of adaptive controllers can deteriorate and even lead to instability.

- Common solutions:
  - Robustifying (leakage) term: σ-modification (Ioannou, 1983) and e_1-modification (Narendra, 1989).
  - Projection operators.

- Leakage modifications don't recover disturbance-free stability performance of unmodified adaptation law if disturbance disappears.

- Projection operators preserve ideal properties of adaptive controller if disturbance disappears, but require parameter bounds to be known a priori.

Background

- Robust adaptive controllers
  - Pomet and Praly (1992)
  - Zhang and Ioannou (1996)
  - Polycarpou and Ioannou (1996)
  - Freeman, Krstic, and Kokotovic (1998)
  - Ikhouane and Krstic (1998)
  - Marino and Tomei (1998)
  - Pan and Başar (1998)
  - Jiang and Hill (1999)
  - Ge and Wang (2003)
  - Cai, de Queiroz, and Dawson (2006)

Background

- Use of adaptive control was broaden with advent of the integrator backstepping design (KKK, 1995).

- Allows one to adaptively stabilize nonlinear systems in parametric strict-feedback (PSF) form.

$$
\begin{aligned}
\dot{x}_1 &= \varphi_1^{\top}(x_1)\,\theta + x_2 \\
\dot{x}_2 &= \varphi_2^{\top}(x_1, x_2)\,\theta + x_3 \\
&\vdots \\
\dot{x}_i &= \varphi_i^{\top}(x_1, \ldots, x_i)\,\theta + x_{i+1} \\
&\vdots \\
\dot{x}_n &= \varphi_n^{\top}(x_1, \ldots, x_n)\,\theta + u
\end{aligned}
$$

Background

- Standard backstepping
  - Recursive procedure generates at each step a new adaptation law $\Rightarrow$ Over-parametrization.
  - nth-order PSF system ($n \geq 3$) requires the $(n-2)$th derivative of 1st adaptation law, the $(n-3)$th derivative of 2nd adaptation law, and so on.
  - Most projection operators are, at best, Lipschitz continuous (e.g., Pomet and Praly, 1992).
  - Sufficiently-smooth version of Pomet/Praly projection (Cai, de Queiroz, and Dawson, 2006).

Background

- When $d = 0$, tuning functions method (KKK, 1995) avoids differentiation of adaptation laws and overparametrization.

- When $d \neq 0$ and $\dot{\hat{\theta}} = \mathrm{Proj}(x, \hat{\theta})$, tuning functions don't work.
  - Tuning functions in time derivative of Lyapunov-like function aren't cancelled by adaptation law nor can be proven nonpositive.

- Alternative: inject extra nonlinear damping terms to handle projection-related terms in time derivative of Lyapunov-like function (Marino and Tomei, 1998).
  - Avoids overparametrization.
  - High control effort.

Background

Common assumption: Disturbance is output of autonomous exosystem,

$$\dot{d} = s(d),$$

with partially known structure; e.g.,

$$s(d) = Sd$$

where S is an unknown stable matrix.

- Restricts class of disturbances.
- Facilitates control design.
  - Internal model principle to compensate for disturbance.
  - Observer to estimate disturbance.
  - No overparametrization.

Problem Statement

$$
\begin{aligned}
\dot{x}_1 &= \varphi_1^\top(x_1)\,\theta + x_2 + d_1 \\
&\vdots \\
\dot{x}_i &= \varphi_i^\top(x_1, \ldots, x_i)\,\theta + x_{i+1} + d_i \\
&\vdots \\
\dot{x}_n &= \varphi_n^\top(x_1, \ldots, x_n)\,\theta + d_n + u \\
y &= x_1
\end{aligned}
$$

$x_i \in \mathbb{R}^{m_i}$, $i = 1, \ldots, n$

$\varphi_i \in \mathbb{R}^{p \times m_i}$, $i = 1, \ldots, n$ are known nonlinearities

$\theta \in \mathbb{R}^p$ is an uncertain constant parameter vector

$d_i(t) \in \mathbb{R}^{m_i}$, $i = 1, \ldots, n$ are unknown additive disturbances

Problem Statement

Assumptions:

- $\varphi_i \in \mathcal{C}^{n-i}$, $i = 1, ..., n$.
- $d_i \in \mathcal{C}^0$, $i = 1, ..., n$ and $\|d_i(t)\|_{\mathcal{L}_\infty} \leq \overline{d}_i$ where $\overline{d}_i$ is unknown positive constant.
- Elements of θ satisfy $\underline{\theta}_i < \theta_i < \overline{\theta}_i$, $i = 1, ..., p$ where $\underline{\theta}_i, \overline{\theta}_i$ are known bounds.
- Reference trajectory $y_r \in \mathcal{C}^n$ and $y_r^{(i)}(t) \in \mathcal{L}_\infty$, $i = 0, ..., n+1$.

Goal: Design a feedback $u(x_1, \ldots, x_n, t)$ that ensures:

- $e(t) := y(t) - y_r(t) \to 0$ as $t \to \infty$, or $\|e(t)\| \to \varepsilon$ as $t \to \infty$ where ε can be made arbitrarily small.
- Boundedness of all closed-loop signals.

Preliminaries

Parameter transformation: Let

$$\overline{\theta} = \left(\overline{\theta}_1, \ldots, \overline{\theta}_p\right)^\top, \quad \underline{\theta} = \left(\underline{\theta}_1, \ldots, \underline{\theta}_p\right)^\top, \quad \Delta = \left\|\overline{\theta} - \underline{\theta}\right\|, \quad \tau = \frac{\overline{\theta} - \theta}{\Delta}$$

then

$$\|\underline{\theta}\| < \|\theta\| < \left\|\overline{\theta}\right\| \quad \text{and} \quad \|\tau\| < 1$$

where $\|\cdot\|$ denotes the 2-norm.

Lemma: Consider the function

$$\begin{aligned} W \;&=\; \tau^\top \Gamma^{-1} \ln(\tau) + (1_p - \tau)^\top \Gamma^{-1} \ln(1_p - \tau) \\ &\quad -\tau^\top \Gamma^{-1} \ln(\hat{\tau}) - (1_p - \tau)^\top \Gamma^{-1} \ln(1_p - \hat{\tau}) \end{aligned} \tag{1}$$

where $\hat{\tau} \in \mathbb{R}^p$ denotes the estimate of the transformed parameter τ, $\ln(\xi) := \left(\ln \xi_i, \ldots, \ln \xi_p\right)^\top$, $1_p := (1, \ldots, 1)_{p \times 1}$, and Γ is a positive-definite diagonal matrix. Then, (1) is nonnegative and radially unbounded in

$$S := \{\hat{\tau} \in \mathbb{R}^p \mid 0 < \hat{\tau}_i < 1 \; \forall i\}.$$

Robust Adaptive Control Design

Design follows the tuning-function-based backstepping procedure.

Step 1

- Let $z_1 = e$, then

$$\dot{z}_1 = \varphi_1^{\mathsf{T}} \left(\overline{\theta} - \tau \boldsymbol{\Delta} \right) - \dot{y}_r + \alpha_1 + z_2$$

where $z_2 = x_2 - \alpha_1$.

- Design stabilizing function α_1 as

$$\alpha_1 = -c_1 z_1 - \varphi_1^{\mathsf{T}} \left(\overline{\theta} - \hat{\tau} \boldsymbol{\Delta} \right) + \dot{y}_r$$

where $c_1 > 0$.

Robust Adaptive Control Design

- Define function

$$V_1 = \frac{1}{2} z_1^2 + W(\hat{\tau}),$$

whose derivative along closed-loop z_1-subsystem is

$$\dot{V}_1 = -c_1 z_1^2 - \tilde{\tau}^\top \left(\varsigma_1 + \Gamma^{-1} N^{-1} \dot{\hat{\tau}} \right) + z_1 z_2$$

where

$$\tilde{\tau} = \tau - \hat{\tau}, \quad \varsigma_1 = \varphi_1 z_1 \Delta, \quad N = \mathrm{diag}\left[\hat{\tau}_i \left(1 - \hat{\tau}_i\right)\right]_{p \times p}.$$

Robust Adaptive Control Design

Step i $(2 \le i \le n-1)$

- Let $z_i = x_i - \alpha_{i-1}$, then

$$\begin{aligned}\dot{z}_i &= \varphi_i^\mathsf{T}\left(\overline{\theta} - \tau\Delta\right) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\left[\varphi_j^\mathsf{T}\left(\overline{\theta} - \tau\Delta\right) + x_{j+1}\right] \\ &\quad -\frac{\partial \alpha_{i-1}}{\partial \hat{\tau}}\dot{\hat{\tau}} - \sum_{j=1}^{i} \frac{\partial \alpha_{i-1}}{\partial y_r^{(j-1)}} y_r^{(j)} + \alpha_i + z_{i+1}.\end{aligned}$$

- Design stabilizing function α_i as

$$\begin{aligned}\alpha_i &= -c_i z_i - \varphi_i^\mathsf{T}\left(\overline{\theta} - \hat{\tau}\Delta\right) + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\left[\varphi_j^\mathsf{T}\left(\overline{\theta} - \hat{\tau}\Delta\right) + x_{j+1}\right] \\ &\quad + \sum_{j=1}^{i} \frac{\partial \alpha_{i-1}}{\partial y_r^{(j-1)}} y_r^{(j)} - z_{i-1} + \phi_i\end{aligned}$$

where $c_i > 0$ and ϕ_i is the tuning function.

Robust Adaptive Control Design

- Define function

$$V_i = V_{i-1} + \frac{1}{2} z_i^2,$$

whose derivative along closed-loop z_i-subsystem is

$$\begin{aligned} \dot{V}_i &= -\sum_{j=1}^{i} c_j z_j^2 - \tilde{\tau}^\top \left[\varsigma_i + \Gamma^{-1} N^{-1} \dot{\hat{\tau}} \right] \\ & \quad - \sum_{j=1}^{i-2} \left(z_{j+1} \frac{\partial \alpha_j}{\partial \hat{\tau}} \right) N \Gamma \left[\varsigma_{i-1} + \Gamma^{-1} N^{-1} \dot{\hat{\tau}} \right] \\ & \quad + z_i \left[\phi_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\tau}} \dot{\hat{\tau}} \right] + z_i z_{i+1} \end{aligned}$$

where

$$\varsigma_i = \varsigma_{i-1} + \left(\varphi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j \right) z_i \Delta.$$

Robust Adaptive Control Design

- Set tuning function to

$$\phi_i = -\frac{\partial \alpha_{i-1}}{\partial \hat{\tau}} N \Gamma \varsigma_i + \bar{\phi}_i,$$

where $\bar{\phi}_i$ is an auxiliary tuning term.

$$\begin{aligned}\dot{V}_i &= -\sum_{j=1}^{i} c_j z_j^2 - \tilde{\tau}^\top \left[\varsigma_i + \Gamma^{-1} N^{-1} \dot{\hat{\tau}}\right] \\ &\quad - \sum_{j=1}^{i-2} \left(z_{j+1} \frac{\partial \alpha_j}{\partial \hat{\tau}} \right) N \Gamma \left[\varsigma_{i-1} + \Gamma^{-1} N^{-1} \dot{\hat{\tau}}\right] \\ &\quad - z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\tau}} N \Gamma \left[\varsigma_i + \Gamma^{-1} N^{-1} \dot{\hat{\tau}}\right] + z_i \bar{\phi}_i + z_i z_{i+1}.\end{aligned}$$

Robust Adaptive Control Design

- Set auxiliary tuning function to

$$\bar{\phi}_i = -\sum_{j=1}^{i-2}\left(z_{j+1}\frac{\partial\alpha_j}{\partial\hat{\tau}}\right)N\Gamma\left(\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right)\Delta$$

so that

$$\begin{aligned}\dot{V}_i &= -\sum_{j=1}^{i}c_j z_j^2 - \tilde{\tau}^\top\left[\varsigma_i + \Gamma^{-1}N^{-1}\dot{\hat{\tau}}\right] \\ &\quad -\sum_{j=1}^{i-1}\left(z_{j+1}\frac{\partial\alpha_j}{\partial\hat{\tau}}\right)N\Gamma\left[\varsigma_i + \Gamma^{-1}N^{-1}\dot{\hat{\tau}}\right] + z_i z_{i+1}.\end{aligned}$$

Robust Adaptive Control Design

Step n

- Let $z_n = x_n - \alpha_{n-1}$, then

$$\begin{aligned} \dot{z}_n &= \varphi_n^{\mathsf{T}} \left(\overline{\theta} - \tau \boldsymbol{\Delta} \right) - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \left[\varphi_j^{\mathsf{T}} \left(\overline{\theta} - \tau \boldsymbol{\Delta} \right) + x_{j+1} \right] \\ & \quad - \frac{\partial \alpha_{n-1}}{\partial \hat{\tau}} \dot{\hat{\tau}} - \sum_{j=1}^{n} \frac{\partial \alpha_{n-1}}{\partial y_r^{(j-1)}} y_r^{(j)} + d_n + u. \end{aligned}$$

- Design control input

$$\begin{aligned} u &= - \left(c_n + k_n \right) z_n - \varphi_n^{\mathsf{T}} \left(\overline{\theta} - \hat{\tau} \boldsymbol{\Delta} \right) \\ & \quad + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \left[\varphi_j^{\mathsf{T}} \left(\overline{\theta} - \hat{\tau} \boldsymbol{\Delta} \right) + x_{j+1} \right] \\ & \quad + \sum_{j=1}^{n} \frac{\partial \alpha_{n-1}}{\partial y_r^{(j-1)}} y_r^{(j)} - z_{n-1} + \phi_n \end{aligned}$$

where $c_n, k_n > 0$ and ϕ_n is final tuning function.

Robust Adaptive Control Design

- Define function

$$V_n = V_{n-1} + \frac{1}{2} z_n^2,$$

whose derivative along closed-loop z_n-subsystem is

$$\begin{aligned}
\dot{V}_n &= -\sum_{j=1}^{n} c_j z_j^2 - \tilde{\tau}^\top \left[\varsigma_n + \Gamma^{-1} N^{-1} \dot{\hat{\tau}} \right] \\
&\quad - \sum_{j=1}^{n-2} \left(z_{j+1} \frac{\partial \alpha_j}{\partial \hat{\tau}} \right) N \Gamma \left[\varsigma_{n-1} + \Gamma^{-1} N^{-1} \dot{\hat{\tau}} \right] \\
&\quad + z_n \left[\phi_n - \frac{\partial \alpha_{i-1}}{\partial \hat{\tau}} \dot{\hat{\tau}} \right] + z_n d_n - k_n z_n^2
\end{aligned}$$

where

$$\varsigma_n = \varsigma_{n-1} + \left(\varphi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \varphi_j \right) z_n \Delta.$$

Robust Adaptive Control Design

- Set tuning function to

$$
\begin{aligned}
\phi_n &= -\frac{\partial \alpha_{n-1}}{\partial \hat{\tau}} N \Gamma \varsigma_n \\
&\quad - \sum_{j=1}^{n-2} \left(z_{j+1} \frac{\partial \alpha_j}{\partial \hat{\tau}} \right) N \Gamma \left(\varphi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \varphi_j \right) \Delta
\end{aligned}
$$

so that

$$
\begin{aligned}
\dot{V}_n &= -\sum_{j=1}^{n} c_j z_j^2 - \tilde{\tau}^\top \left[\varsigma_n + \Gamma^{-1} N^{-1} \dot{\hat{\tau}} \right] \\
&\quad - \sum_{j=1}^{n-1} \left(z_{j+1} \frac{\partial \alpha_j}{\partial \hat{\tau}} \right) N \Gamma \left[\varsigma_n + \Gamma^{-1} N^{-1} \dot{\hat{\tau}} \right] + z_n d_n - k_n z_n^2 .
\end{aligned}
$$

Robust Adaptive Control Design

- Design adaptation law as

$$\begin{aligned} \dot{\hat{\tau}} &= -N\Gamma\varsigma_n, \quad \hat{\tau}(0) \in \mathcal{S} \\ N &= \mathrm{diag}\left[\hat{\tau}_i\left(1-\hat{\tau}_i\right)\right]_{p\times p}, \end{aligned}$$

  which ensures $\hat{\tau}(t) \in \mathcal{S}$ for all time.

$$\dot{V}_n \leq -\sum_{j=1}^{n} c_j z_j^2 + \frac{d_n^2}{4k_n} \leq -\min_j \left(c_j\right) \|z\|^2 + \frac{\overline{d}_n^2}{4k_n}$$

where $z := (z_1, \ldots, z_n)^\top$.

- V_n is nonnegative in $\mathbb{R}^n \times \mathcal{S}$ and $\dot{V}_n$ is negative in the set

$$\left\{ z \in \mathbb{R}^n \mid \|z\| > \frac{\overline{d}_n}{2\sqrt{k_n \min_j \left(c_j\right)}} \right\}.$$

Robust Adaptive Control Design

- z is ultimately bounded, and bound can be made arbitrarily small.
- From Barbalat's lemma, if

$$d_n(t) \to 0 \quad \text{for} \quad t \geq T > 0,$$

then

$$z(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Simulation Results

- System model

$$\varphi_1 = \begin{bmatrix} -x_1^2 \\ \frac{1}{4}x_1^3 \end{bmatrix} \quad \varphi_2 = \begin{bmatrix} -x_1^2 \\ \frac{1}{4}x_2^3 \end{bmatrix} \quad \varphi_3 = \begin{bmatrix} -x_2 \\ \frac{1}{4}x_1^2 x_3 \end{bmatrix}$$

$$\theta = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \underline{\theta} = \begin{bmatrix} 1.5 \\ 0.677 \end{bmatrix} \quad \overline{\theta} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$d_3(t) = \begin{cases} \sin 3t & 0 \le t \le 15 \\ 0 & \text{otherwise.} \end{cases}$$

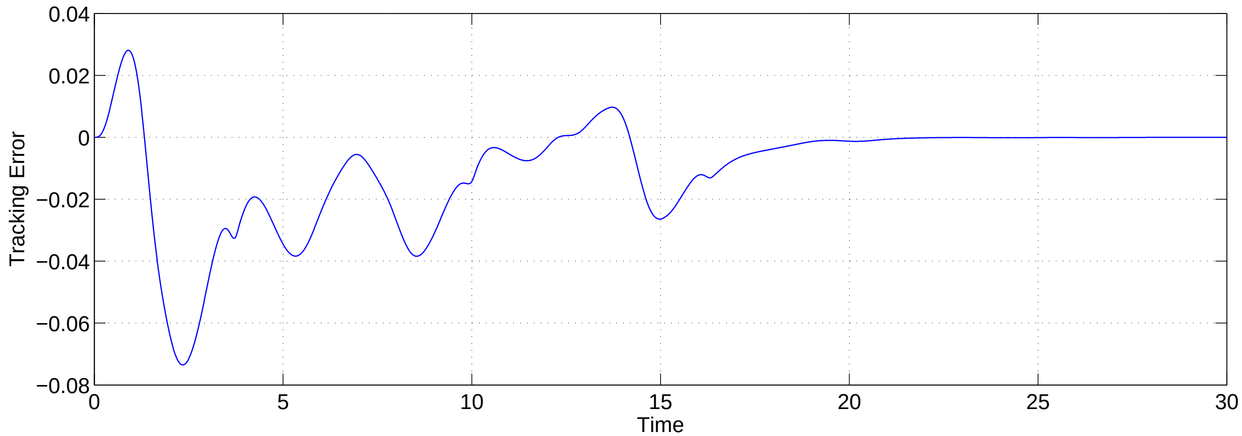
- Reference trajectory: $y_r = \sin t \left(1 - \exp\left(-\frac{1}{3}t^3\right)\right)$

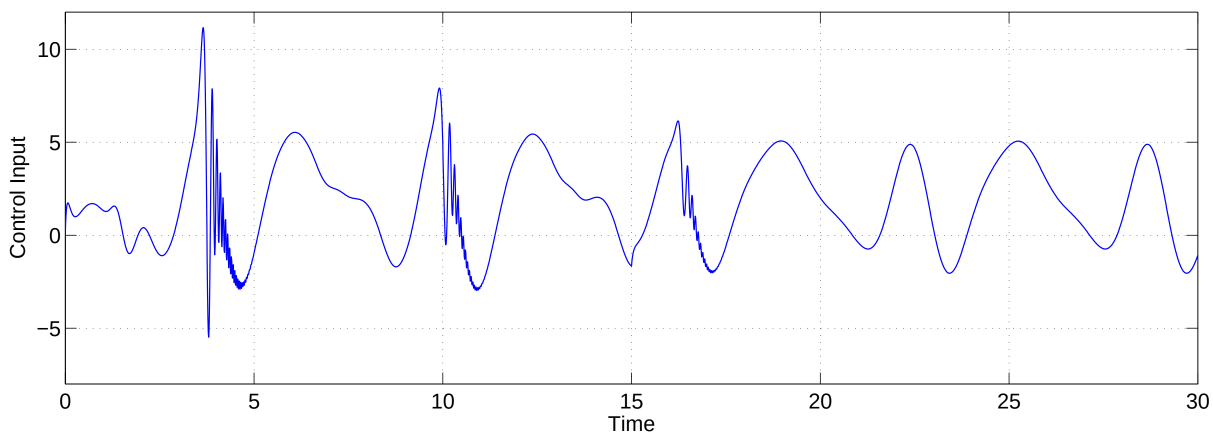
- Initial conditions: $x_1(0) = x_2(0) = x_3(0) = 0, \quad \hat{\tau}(0) = (0.6, 0.4)^\top$

- Control parameters: $c_1 = 1, \quad c_2 = 10, \quad c_3 = 10, \quad k_3 = 10, \quad \Gamma = I_2$

Simulation Results

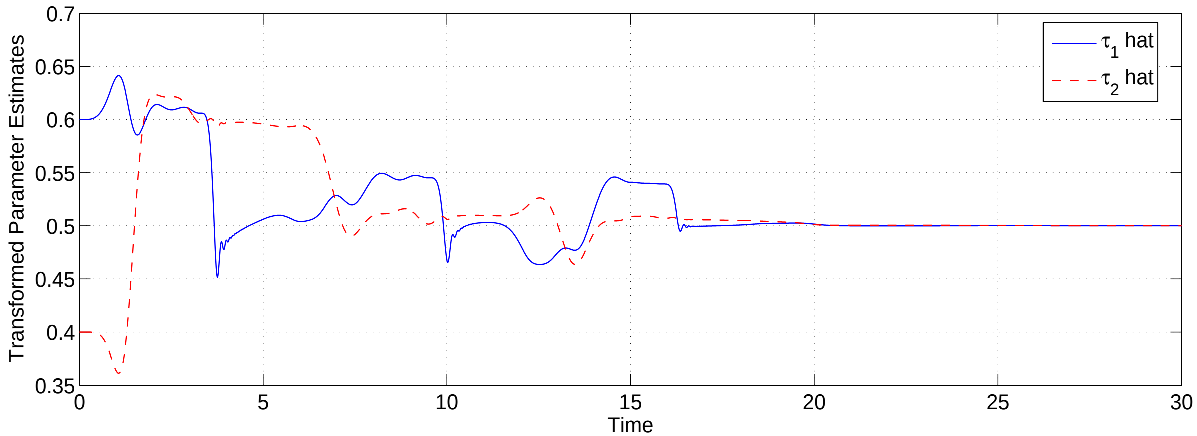
- Tracking error $e(t)$ and control input $u(t)$

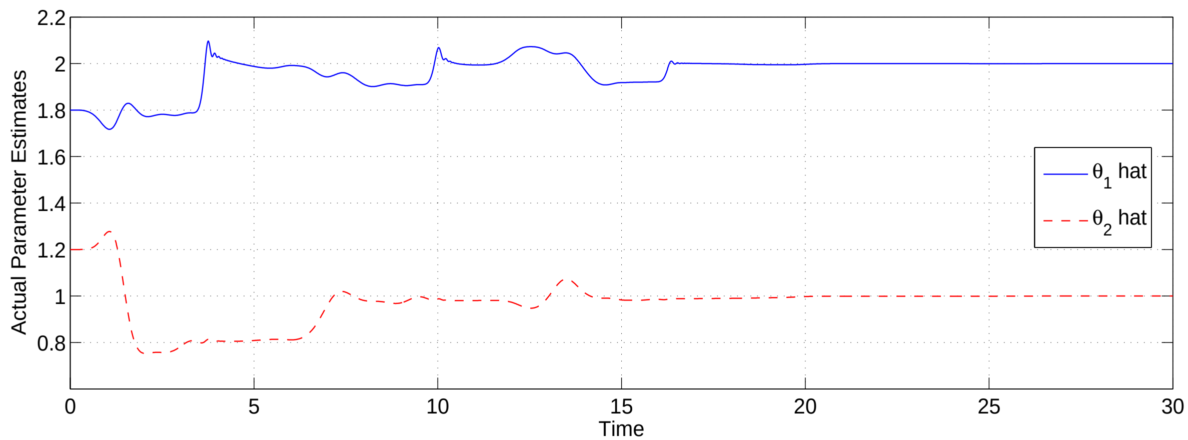




Simulation Results

- Parameter estimates $\hat{\tau}(t)$ and $\hat{\theta}(t)$





Concluding Remarks

- New robust adaptive controller for PSF systems.
- Additive disturbance is not exosystem generated.
- Only assumed disturbance is continuous and bounded.
- New projection-like operator.
  - Enabled use of tuning function method.
  - No overparametrization.
  - Avoided need for extra nonlinear damping terms.
- Practical tracking in presence of disturbance; asymptotic tracking when disturbance disappears.