Impulsive Dynamical Systems

Peter R. Wolenski
Louisiana State University

Joint work with **Stanislav Žabić**

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Outline

(I) The Chemostat

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I. The Chemostat

A standard model for a Sequential Batch Reactor (SBR) is

$$\begin{aligned} \dot{x}_i(t) &= \left(\mu_i(s) - \frac{u(t)}{v(t)}\right) x_i(t) \\ \dot{s}(t) &= -\sum_{j=1}^{n} \mu_j(s) x_j(t) + \frac{u(t)}{v(t)}\left(s_{\text{in}} - s\right) \\ \dot{v}(t) &= u(t), \end{aligned}$$

plus initial conditions. Here, x_i, s, and v are the concentrations of the i^{th} species, substrate, and total volume, respectively. The variable u is the input flow rate and is the control variable.

The problem becomes nonstandard if the control variable u is allowed to be

unbounded

II. Differential inclusions

A standard (nonlinear) **C**ontrol **D**ynamical **(*CD*)** system has the form

$$(CD) \qquad \begin{cases} \dot{x}(t) &= f\big(x(t), u(t)\big) \quad \text{a.e. } t \in [0, T] \\ u(t) &\in \mathcal{U} \quad \text{a.e. } t \in [0, T] \\ x(0) &= x_0, \end{cases}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^m$, and $x_0 \in \mathbb{R}^n$.

A **D***ifferential* **I***nclusion* has the form

$$(DI) \qquad \begin{cases} \dot{x}(t) &\in F\big(x(t)\big) \quad \text{a.e. } t \in [0, T] \\ x(0) &= x_0, \end{cases}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a *multifunction* (i.e. a set-valued map).

Remark: The trajectories $x(\cdot)$ of **(*CD*)** and **(*DI*)** coincide if

$$F(x) = \Big\{ f(x, u) : u \in \mathcal{U} \Big\}.$$

Basic Theory of **(DI)**:

Hypotheses

A well-developed theory is established under Standard Hypotheses **(SH)** on F:

- Each set-value $F(x)$ is nonempty, **compact**, and convex.

- The graph $\{(x,v) : v \in F(x)\}$ is closed ($F(\cdot)$ is *upper* or *outer* semicontinuous.)

- $F(\cdot)$ has linear growth: $\exists\, c > 0$ so that

$$v \in F(x) \Rightarrow \|v\| \le c(1 + \|x\|).$$

An *additional* Lipschitz hypothesis **(SH)**$_+$ is often invoked to obtain additional results:

- $F(x) \subseteq F(y) + k\|x - y\|\overline{B}$.

Basic Theory of (DI):

Results

Under assumptions **(SH)**:

- Existence of absolutely continuous solutions.
- Compactness of trajectories: a sequence of "approximate" solutions have a cluster point that is a solution.
- Characterization of weak invariance on a closed set C: if $x_0 \in C$, then there exists a solution $x(\cdot)$ to (DI) that remains in C for all time.

The (normal-type) characterization is that

$$-H(x,-\zeta) \leq 0 \quad \forall x \in C, \zeta \in N_C^P(x),$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the **Hamiltonian** given by

$$H(x,\zeta) := \max_{v \in F(x)} \langle v, \zeta \rangle.$$

Notice that

$$-H(x,-\xi) = \min_{v \in F(x)} \langle v, \xi \rangle,$$

and recall the proximal normal cone $N_C^P(x)$ is defined by $N_C^P(x) :=$

$$\left\{ \zeta : \; \exists\, \sigma > 0 \text{ s.t. } \langle y - x, \zeta \rangle \leq \sigma \|y - x\|^2 \, \forall y \in C \right\}.$$

Thus the condition

$$-H(x,-\zeta) \leq 0 \quad \forall x \in C, \zeta \in N_C^P(x),$$

says that at each $x \in C$, there exists a velocity vector $v \in F(x)$ that "points into" C.

Under assumptions **(SH)$_+$**:

- The reachable set

$$R^T(x_0) := \Big\{ x(T) : \ x(\cdot) \text{ solves (DI)} \Big\}$$

can be characterized in several ways.

- *Every* solution is a limit of *Euler* approximate trajectories (Sampling).

- Relaxation.

- Characterization of strong invariance on a closed set C:

$$H(x,\zeta) \leq 0 \quad \forall x \in C, \zeta \in N_C^P(x).$$

III. Impulsive Systems

Impulsive systems arise naturally by asking

What if
$F(x)$ is unbounded?

The theory loses many desirable properties, especially compactness of trajectories. In particular, trajectories can tend to limit to arcs that have **jumps**.

We consider the impulsive dynamical systems having the following *differential* form:

$$(\star) \qquad \begin{cases} dx & \in \; F\big(x(t)\big)\,dt + G\big(x(t)\big)\,d\mu(dt) \\ \mu & \in \; \mathcal{B}_K([0,T]) \\ x(0-) & = \; x_0, \end{cases}$$

where $x(\cdot)$ is an arc of bounded variation.

Given Data

- The multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has linear growth with nonempty, compact, and convex values;

- The multifunction $G : \mathbb{R}^n \rightrightarrows \mathcal{M}_{n\times m}$ also with linear growth with nonempty, compact, and convex values, where $\mathcal{M}_{n\times m}$ denotes the $n \times m$ dimensional matrices with real entries;

- The measure μ belongs to $\mathcal{B}_K[0, T]$, the set of vector-valued Borel measures taking values in the closed convex cone $K \subseteq \mathbb{R}^m$.

If F and G are as above, we say **(SH)** is satisfied, and if they are in addition locally Lipschitz, then **(SH)**$_+$ is satisfied.

$$(\star)\begin{cases} dx \in F\Big(x(t)\Big)\,dt + G\Big(x(t)\Big)\,d\mu(dt) \\ \mu \in \mathcal{B}_K([0,T]) \\ x(0-) = x_0 \end{cases}$$

We are interested in the following

Questions:

(Q1) What is meant by a solution to $(\star)$?

(Q2) What are the natural invariance notions?

(Q3) How can the invariance properties be infinitesimally characterized?

(Q4) Can solutions be generated by time discretization?

IV. Solution Concepts

Suppose $\mu \in \mathcal{B}_K[0,T]$ is fixed. Recall

$$(\star) \qquad dx \in F\Big(x(t)\Big)\,dt + G\Big(x(t)\Big)\,d\mu(dt)$$

(Q1) What is meant by a solution to $(\star)$?

The case $m = 1$ or, more generally, when the columns of $G(x)$ commute, can be handled in a natural way:

Approximate μ by *continuous* (w.r.t. Lebesgue) measures $d\mu_i = \dot{u}_i(t)dt$, and take limits of the solutions $\{x_i(\cdot)\}$ to the differential inclusions

$$\dot{x}_i(t) \in F\Big(x_i(t)\Big) + G\Big(x_i(t)\Big)\dot{u}_i(t).$$

This is not a well-defined concept for more general $G(\cdot)$! (Bressan-Rampazzo)

Graph completions

The *distribution* of μ and the *time reparametrization* are defined respectively by

$$u(t) = \mu\Big([0,t]\Big),$$
$$\eta(t) = t + \Big|\mu\Big|\Big([0,t]\Big)$$

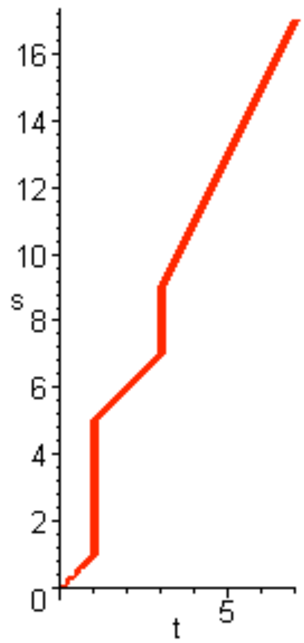
Let $S = \eta(T)$. A *graph completion* of $u(\cdot)$ is a Lipschitz map

$$(\phi_0, \phi) : [0,S] \to [0,T] \times \mathbb{R}^n$$

with $\phi_0(\cdot)$ nondecreasing and mapping onto $[0,T]$, and such that for every $t \in [0,T]$, there exists an $s \in [0,S]$ with

$$\Big(\phi_0(s), \phi(s)\Big) = \Big(\eta(t), u(t)\Big).$$

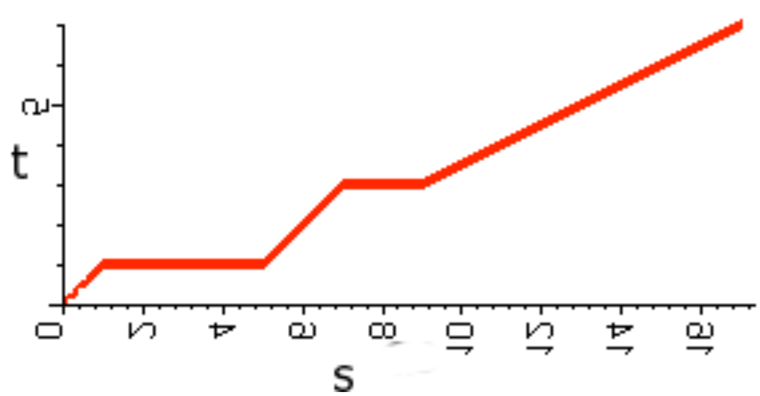
Simple Example



In this example, μ has 2 point masses, and so $\eta(\cdot)$ has 2 jumps.

We shall take $\phi_0(\cdot)$ as the *graph inverse* of η and refer to a graph completion as just $\phi(\cdot)$.

The graph of $\phi_0(\cdot)$ looks like



Solution data

Given $\mu \in \mathcal{B}_K([0,T])$ with atoms $\{t_i\}_{i\in\mathcal{I}}$, consider

$$X_\mu = \Big(\underbrace{x(\cdot)}_{\text{arc of bounded variation}}, \underbrace{\phi(\cdot)}_{\text{graph completion}}, \underbrace{\{y_i(\cdot)\}_{i\in\mathcal{I}}}_{\text{arcs defined on the jump intervals}} \Big)$$

The **jump intervals** have the form

$$I_i = [\eta(t_i-), \eta(t_i+)] \subseteq [0,S], \quad i \in \mathcal{I}$$

and have length $|\mu(t_i)|$, which is the magnitude of the atom.

Bressan-Rampazzo concept

The idea: Recast the dynamics in the reparameterize time interval, and reduce the system to a classical control problem

Given

$$X_\mu = \Big(x(\cdot), \phi(\cdot), \{y_i(\cdot)\}_{i\in\mathcal{I}} \Big),$$

define $y(\cdot) : [0, S] \to I\!R^n$ by

$$y(s) = \begin{cases} x\Big(\phi_0(s)\Big) & \text{if } s \notin \cup_{i\in\mathcal{I}} I_i \\ y_i(s) & \text{if } i \in I_i \end{cases}$$

Then X_μ is a **Bressan-Rampazzo** solution to

$$\Big(\star\Big) \qquad \begin{cases} dx & \in & F\Big(x(t)\Big)\,dt + G\Big(x(t)\Big)\,d\mu(dt) \\ x(0-) & = & x_0. \end{cases}$$

provided $y(\cdot)$ is Lipschitz and satisfies $y(0) = x_0$ and

$\dot{y}(s) \in F\Big(y(s)\Big)\dot{\phi}_0(s) + G\Big(y(s)\Big)\dot{\phi}(s)$ a.e. $s \in [0, S]$.

Another definition

Recall

$$d\mu = \dot{u}(t)\,dt + d\mu_\sigma + d\mu_{\mathsf{D}}.$$

where $d\mu_\sigma$ is continuous (Lebesgue) singular and $d\mu_{\mathsf{D}}$ is discrete. Similarly, one has

$$dx = \dot{x}(t)\,dt + dx_\sigma + dx_{\mathsf{D}}.$$

$$\big(\star\big) \qquad \begin{cases} dx & \in & F\Big(x(t)\Big)\,dt + G\Big(x(t)\Big)\,d\mu(dt) \\ x(0-) & = & x_0. \end{cases}$$

We introduce a new solution concept by requiring the "parts" of the decomposition to "match up".

We say

$$X_\mu = \Big(x(\cdot), \phi(\cdot), \{y_i(\cdot)\}_{i\in\mathcal{I}}\Big),$$

is a **solution** to $(\star)$ if $x(0-) = x_0$ and

- $\dot{x}(t) \in F\big(x(t)\big) + G\big(x(t)\big)\dot{u}(t) \quad$ a.e. $t \in [0, T]$,

- $dx_\sigma = \gamma(t)\, d\mu_\sigma$ for some μ_σ-measurable selection $\gamma(\cdot)$ of $G\big(x(\cdot)\big)$.

- the atoms of dx and μ coincide, and for each $i \in \mathcal{I}$, the arc $y_i(\cdot)$ satisfies

$$\begin{array}{rcl} \dot{y}_i(s) & \in & G\big(y_i(s)\big)\dot{\phi}(s) \quad \text{a.e. } s \in I_i \\ y_i(t_i-) & = & x(t_i-) \\ y_i(t_i+) & = & x(t_i+) \end{array}$$

Theorem 1

Under the assumption **(SH)**, *the two notions of solution coincide.*

V. invariance

We next consider

(Q2) What are the invariance notions?

The graph gr X_μ of the 3-tuple X_μ is defined by

$$\text{gr } X_\mu := \Big\{(t, x(t)) : t \in [0, T]\Big\} \bigcup \Big\{(t_i, y_i(s)) : s \in I_i,\ i \in \mathcal{I}\Big\}$$

$(\star)$ is **weakly invariant** on $C \subseteq \mathbb{R}^n$:

For all $S > 0$ and $x_0 \in C$, there exist $T \in [0, S]$, $\mu \in \mathcal{B}_K([0, T])$, a graph completion $\phi(\cdot)$, and a solution X_μ so that the projection of the graph of X_μ into the second component is contained in C.

Strong invariance is defined similarly, where **every** trajectory satisfies the projection property.

(Q3) **How are the invariance properties characterized infinitesimally?**

Let $K_1 = \{k : k \in K,\ \|k\| = 1\}$.

Theorem 2

(a) *Weak invariance is equivalent to: For each $x \in C$ and $\zeta \in N_C^P(x)$ (= the proximal normal cone to C at x), there exists $\lambda \in [0,1]$ and $v \in \Big[\lambda F(x) + (1-\lambda)G(x)K_1\Big]$ so that*

$$\langle v, \zeta \rangle \leq 0.$$

(b) *Strong invariance is equivalent to: For each*

$$x \in C,\ \zeta \in N_C^P(x),\ \lambda \in [0,1], \text{ and } \\ v \in \Big[\lambda F(x) + (1-\lambda)G(x)K_1\Big],$$

one has

$$\langle v, \zeta \rangle \leq 0.$$

VI. Sampling Methods

Given the measure μ, a natural sampling method of

$$\dot{y}(s) \in F\Big(y(s)\Big)\dot{\phi}_0(s) + G\Big(y(s)\Big)\dot{\phi}(s)$$

is the following. Let $S > 0$, and partition $[0, S]$: $N \in \mathbb{N}$, $h = S/N$, $s_j = jh$, $t_j = \phi_0(s_j)$, $\lambda_j = t_{j+1} - t_j$.

$$x_0 = x_0; \quad f_0 \in F(x_0); \quad g_0 \in G(x_0);$$

$$x_1 = x_0 + \lambda_1 f_0 + \Big(g_0\Big)\Big(\phi(s_1) - \phi(s_0)\Big);$$
$$f_1 \in F(x_1); \quad g_1 \in G(x_1);$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{j+1} = x_j + \lambda_j f_j + \Big(g_j\Big)\Big(\phi(s_j) - \phi(s_{j-1})\Big)$$
$$f_{j+1} \in F(x_{j+1}); \quad g_{j+1} \in G(x_{j+1});$$

$$\vdots \qquad \vdots \qquad \vdots$$

But the issue here is that we must also produce the measure and graph completion. In particular, the λ_j must also be selected. Let $S > 0$, and partition $[0, S]$ with $N \in \mathbb{N}$ and $h = S/N$.

$$x_0 := x_0, \quad \lambda_0 \in [0,1], \quad f_0 \in F(x_0), \quad k_0 \in K_1, \quad g_0 \in G(x_0);$$

$$x_1 := x_0 + \lambda_0 h f_0 + (1-\lambda_0) h g_0 k_0, \quad \lambda_1 \in [0,1], \quad f_1 \in F(x_1), \quad k_1 \in K_1, \quad g_1 \in G(x_1);$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{j+1} := x_j + \lambda_j h f_j + (1-\lambda_j) h g_j k_j, \quad \lambda_{j+1} \in [0,1], \quad f_{j+1} \in F(x_{j+1}), \quad k_{j+1} \in K_1, \quad g_{j+1} \in G(x_{j+1});$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_N := x_{N-1} + \lambda_{N-1} h f_{N-1} + (1-\lambda_{N-1}) h g_{N-1} k_{N-1}.$$

Let $t_j = \sum_{i=1}^{j} \lambda_i$, and $\chi^N = \{(t_j, x_j) : j = 1, \ldots, N\}$ be the sampled graph.

Our answer to

(Q4)	**How can solutions be generated by time discretization?**

is the following theorem.

Theorem 3 **(a)** *Suppose assumption* **(SH)** *holds, and let* $\{\chi^N\}$ *be a sequence of sampled graphs. Then there exists a measure* $\mu \in \mathcal{B}_K[0,T]$ *and a solution* X_μ *for which*

$$\liminf_{N\to\infty} dist_H\Big(\chi^N, gr\ X_\mu\Big) = 0,$$

where $gr\ X_\mu$ *is the graph of* X_μ*; i.e. equals*

$$\Big\{(t, x(t)) : t \in [0,T]\Big\} \cup \Big\{(t_i, y_i(s)) : i \in \mathcal{I},\ s \in I_i\Big\}.$$

(b) *Suppose assumption* $\mathbf{(SH)}_+$ *holds. Then the graph of every solution is the limit of a sampled sequence.*

Conclusions

- A solution concept for impulsive equations is defined through "matching" the decomposition of the measures.

- Weak and strong invariance is infinitesimally characterized through Hamilton-Jacobi inequalities.

- The proof technique of the invariance results rely on an Euler-type sampling method in the reparameterized time space and they generate solutions.