Impulsive Dynamical Systems

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Outline

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I. The Chemostat

A standard model for a Sequential Batch Reactor (SBR) is

\[
\begin{align*}
\dot{x}_i(t) &= \left( \mu_i(s) - \frac{u(t)}{v(t)} \right) x_i(t) \\
\dot{s}(t) &= -\sum_{j=1}^{n} \mu_j(s) x_j(t) + \frac{u(t)}{v(t)} (s_{in} - s) \\
\dot{v}(t) &= u(t),
\end{align*}
\]

plus initial conditions. Here, \(x_i, s, \) and \(v\) are the concentrations of the \(i^{th}\) species, substrate, and total volume, respectively. The variable \(u\) is the input flow rate and is the control variable.

The problem becomes nonstandard if the control variable \(u\) is allowed to be unbounded.
II. Differential inclusions

A standard (nonlinear) Control Dynamical (CD) system has the form

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \quad \text{a.e. } t \in [0, T] \\
u(t) &\in \mathcal{U} \quad \text{a.e. } t \in [0, T] \\
x(0) &= x_0,
\end{align*}
\]

where \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \mathcal{U} \subseteq \mathbb{R}^m, \) and \( x_0 \in \mathbb{R}^n. \)

A Differential Inclusion has the form

\[
\begin{align*}
\dot{x}(t) &\in F(x(t)) \quad \text{a.e. } t \in [0, T] \\
x(0) &= x_0,
\end{align*}
\]

where \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is a multifunction (i.e. a set-valued map).

Remark: The trajectories \( x(\cdot) \) of (CD) and (DI) coincide if

\[ F(x) = \left\{ f(x, u) : u \in \mathcal{U} \right\}. \]
Basic Theory of (DI):

Hypotheses

A well-developed theory is established under Standard Hypotheses (SH) on $F$:

- Each set-value $F(x)$ is nonempty, compact, and convex.

- The graph $\{(x, v) : v \in F(x)\}$ is closed ($F(\cdot)$ is upper or outer semicontinuous.)

- $F(\cdot)$ has linear growth: $\exists c > 0$ so that $v \in F(x) \Rightarrow \|v\| \leq c(1 + \|x\|)$.

An additional Lipschitz hypothesis (SH)$_+$ is often invoked to obtain additional results:

- $F(x) \subseteq F(y) + k\|x - y\|B$. 
Basic Theory of (DI):

Results

Under assumptions (SH):

• Existence of absolutely continuous solutions.
• Compactness of trajectories: a sequence of “approximate” solutions have a cluster point that is a solution.
• Characterization of weak invariance on a closed set $C$: if $x_0 \in C$, then there exists a solution $x(\cdot)$ to (DI) that remains in $C$ for all time.

The (normal-type) characterization is that

$$-H(x, -\zeta) \leq 0 \quad \forall x \in C, \zeta \in N^P_C(x),$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the Hamiltonian given by

$$H(x, \zeta) := \max_{v \in F(x)} \langle v, \zeta \rangle.$$
Notice that

$$-H(x, -\xi) = \min_{v \in F(x)} \langle v, \xi \rangle,$$

and recall the proximal normal cone $N^P_C(x)$ is defined by

$$N^P_C(x) := \left\{ \zeta : \exists \sigma > 0 \text{ s.t. } \langle y - x, \zeta \rangle \leq \sigma \|y - x\|^2 \forall y \in C \right\}.$$

Thus the condition

$$-H(x, -\zeta) \leq 0 \quad \forall x \in C, \zeta \in N^P_C(x),$$

says that at each $x \in C$, there exists a velocity vector $v \in F(x)$ that “points into” $C$. 
Under assumptions \((\text{SH})_+\):

- The reachable set
  \[ R^T(x_0) := \{ x(T) : x(\cdot) \text{ solves } (\text{DI}) \} \]
  can be characterized in several ways.

- Every solution is a limit of Euler approximate trajectories (Sampling).

- Relaxation.

- Characterization of strong invariance on a closed set \(C\):
  \[ H(x, \zeta) \leq 0 \quad \forall x \in C, \zeta \in N^P_C(x). \]
III. Impulsive Systems

Impulsive systems arise naturally by asking

What if $F(x)$ is unbounded?

The theory loses many desirable properties, especially compactness of trajectories. In particular, trajectories can tend to limit to arcs that have jumps.

We consider the impulsive dynamical systems having the following differential form:

\[
\begin{cases}
\begin{aligned}
  dx & \in F(x(t)) \, dt + G(x(t)) \, d\mu(dt) \\
  \mu & \in \mathcal{B}_K([0,T]) \\
  x(0-) & = x_0,
\end{aligned}
\end{cases}
\]

where $x(\cdot)$ is an arc of bounded variation.
**Given Data**

- The multifunction \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) has linear growth with nonempty, compact, and convex values;

- The multifunction \( G : \mathbb{R}^n \rightrightarrows \mathcal{M}_{n \times m} \) also with linear growth with nonempty, compact, and convex values, where \( \mathcal{M}_{n \times m} \) denotes the \( n \times m \) dimensional matrices with real entries;

- The measure \( \mu \) belongs to \( \mathcal{B}_K[0, T] \), the set of vector-valued Borel measures taking values in the closed convex cone \( K \subseteq \mathbb{R}^m \).

If \( F \) and \( G \) are as above, we say \((SH)\) is satisfied, and if they are in addition locally Lipschitz, then \((SH)_+\) is satisfied.
We are interested in the following

**Questions:**

(Q1) What is meant by a solution to (⋆)?

(Q2) What are the natural invariance notions?

(Q3) How can the invariance properties be infinitesimally characterized?

(Q4) Can solutions be generated by time discretization?
IV. Solution Concepts

Suppose \( \mu \in B_K[0, T] \) is fixed. Recall

\[
(\star) \quad dx \in F(x(t)) \, dt + G(x(t)) \, d\mu(dt)
\]

(Q1) What is meant by a solution to (\( \star \))? 

The case \( m = 1 \) or, more generally, when the columns of \( G(x) \) commute, can be handled in a natural way:

Approximate \( \mu \) by continuous (w.r.t. Lebesgue) measures \( d\mu_i = \dot{u}_i(t) \, dt \), and take limits of the solutions \( \{x_i(\cdot)\} \) to the differential inclusions

\[
\dot{x}_i(t) \in F(x_i(t)) + G(x_i(t)) \dot{u}_i(t).
\]

This is not a well-defined concept for more general \( G(\cdot)! \) (Bressan-Rampazzo)
Graph completions

The distribution of $\mu$ and the time reparametrization are defined respectively by

\[
\begin{align*}
u(t) &= \mu([0, t]), \\
\eta(t) &= t + |\mu([0, t])|
\end{align*}
\]

Let $S = \eta(T)$. A graph completion of $u(\cdot)$ is a Lipschitz map

\[
(\phi_0, \phi) : [0, S] \to [0, T] \times \mathbb{R}^n
\]

with $\phi_0(\cdot)$ nondecreasing and mapping onto $[0, T]$, and such that for every $t \in [0, T]$, there exists an $s \in [0, S]$ with

\[
(\phi_0(s), \phi(s)) = (\eta(t), u(t)).
\]
Simple Example

In this example, $\mu$ has 2 point masses, and so $\eta(\cdot)$ has 2 jumps.

We shall take $\phi_0(\cdot)$ as the graph inverse of $\eta$ and refer to a graph completion as just $\phi(\cdot)$.

The graph of $\phi_0(\cdot)$ looks like
Solution data

Given $\mu \in B_K([0, T])$ with atoms $\{t_i\}_{i \in I}$, consider

$$X_\mu = \left( \begin{array}{c} x(\cdot) \\text{arc of bounded variation} \\ \phi(\cdot) \\text{graph completion} \\ \{y_i(\cdot)\}_{i \in I} \\text{arcs defined on the jump intervals} \end{array} \right)$$

The jump intervals have the form

$$I_i = [\eta(t_i-), \eta(t_i+)] \subseteq [0, S], \quad i \in I$$

and have length $|\mu(t_i)|$, which is the magnitude of the atom.

Bressan-Rampazzo concept

The idea: Recast the dynamics in the reparameterize time interval, and reduce the system to a classical control problem.
Given

\[ X_\mu = \left( x(\cdot), \phi(\cdot), \{ y_i(\cdot) \}_{i \in \mathcal{I}} \right), \]

define \( y(\cdot) : [0, S] \rightarrow \mathbb{R}^n \) by

\[
y(s) = \begin{cases} 
  x(\phi_0(s)) & \text{if } s \notin \bigcup_{i \in \mathcal{I}} I_i \\
  y_i(s) & \text{if } i \in I_i
\end{cases}
\]

Then \( X_\mu \) is a **Bressan-Rampazzo** solution to

\[
(\star) \quad \left\{ \begin{array}{l}
  dx \in F(x(t)) \, dt + G(x(t)) \, d\mu(dt) \\
  x(0^-) = x_0
\end{array} \right.
\]

provided \( y(\cdot) \) is Lipschitz and satisfies \( y(0) = x_0 \) and

\[
\dot{y}(s) \in F(y(s))\dot{\phi}_0(s) + G(y(s))\dot{\phi}(s) \text{ a.e. } s \in [0, S].
\]
Another definition

Recall

\[ d\mu = \dot{u}(t) \, dt + d\mu_\sigma + d\mu_D. \]

where \( d\mu_\sigma \) is continuous (Lebesgue) singular and \( d\mu_D \) is discrete. Similarly, one has

\[ dx = \dot{x}(t) \, dt + dx_\sigma + dx_D. \]

\[ \star \left\{ \begin{array}{l}
  dx \in F(x(t)) \, dt + G(x(t)) \, d\mu(dt) \\
  x(0-) = x_0.
\end{array} \right. \]

We introduce a new solution concept by requiring the “parts” of the decomposition to “match up”.
We say

\[ X_\mu = \left( x(\cdot), \phi(\cdot), \{ y_i(\cdot) \}_{i \in \mathcal{I}} \right), \]

is a solution to \((\ast)\) if \(x(0-) = x_0\) and

- \(\dot{x}(t) \in F(x(t)) + G(x(t)) \dot{u}(t)\) a.e. \(t \in [0, T]\),

- \(dx_\sigma = \gamma(t) \, d\mu_\sigma\) for some \(\mu_\sigma\)-measurable selection \(\gamma(\cdot)\) of \(G(x(\cdot))\).

- the atoms of \(dx\) and \(\mu\) coincide, and for each \(i \in \mathcal{I}\), the arc \(y_i(\cdot)\) satisfies
  
  \[
  \dot{y}_i(s) \in G(y_i(s)) \phi(s) \quad \text{a.e. } s \in I_i \\
  y_i(t_i-) = x(t_i-) \\
  y_i(t_i+) = x(t_i+) 
  \]
Theorem 1

*Under the assumption \((SH)\), the two notions of solution coincide.*
V. invariance

We next consider

(Q2) What are the invariance notions?

The graph $\text{gr} \ X_\mu$ of the 3-tuple $X_\mu$ is defined by

$$
\text{gr} \ X_\mu := \{(t, x(t)) : t \in [0, T]\} \bigcup \{(t_i, y_i(s)) : s \in I_i, i \in I\}
$$

(⋆) is weakly invariant on $C \subseteq \mathbb{R}^n$:

For all $S > 0$ and $x_0 \in C$, there exist $T \in [0, S]$, $\mu \in \mathcal{B}_K([0, T])$, a graph completion $\phi(\cdot)$, and a solution $X_\mu$ so that the projection of the graph of $X_\mu$ into the second component is contained in $C$.

Strong invariance is defined similarly, where every trajectory satisfies the projection property.
Let \( K_1 = \{ k : k \in K, \|k\| = 1 \} \).

**Theorem 2**

(a) **Weak invariance** is equivalent to: For each \( x \in C \) and \( \zeta \in N_P^C(x) \) (\( = \) the proximal normal cone to \( C \) at \( x \)), there exists \( \lambda \in [0, 1] \) and \( v \in [\lambda F(x) + (1 - \lambda)G(x)K_1] \) so that
\[
\langle v, \zeta \rangle \leq 0.
\]

(b) **Strong invariance** is equivalent to: For each \( x \in C, \zeta \in N_P^C(x), \lambda \in [0, 1], \) and
\[
v \in [\lambda F(x) + (1 - \lambda)G(x)K_1],
\]
one has
\[
\langle v, \zeta \rangle \leq 0.
\]
VI. Sampling Methods

Given the measure $\mu$, a natural sampling method of

$$\dot{y}(s) \in F(y(s))\phi_0(s) + G(y(s))\phi(s)$$

is the following. Let $S > 0$, and partition $[0, S]$: $N \in \mathbb{N}$, $h = S/N$, $s_j = jh$, $t_j = \phi_0(s_j)$, $\lambda_j = t_{j+1} - t_j$.

$$x_0 = x_0; \quad f_0 \in F(x_0); \quad g_0 \in G(x_0);$$

$$x_1 = x_0 + \lambda_1 f_0 + (g_0)(\phi(s_1) - \phi(s_0));$$

$$f_1 \in F(x_1); \quad g_1 \in G(x_1);$$

$$\vdots$$

$$x_{j+1} = x_j + \lambda_j f_j + (g_j)(\phi(s_j) - \phi(s_{j-1}))$$

$$f_{j+1} \in F(x_{j+1}); \quad g_{j+1} \in G(x_{j+1});$$

$$\vdots$$
But the issue here is that we must also produce the measure and graph completion. In particular, the $\lambda_j$ must also be selected. Let $S > 0$, and partition $[0, S]$ with $N \in \mathbb{N}$ and $h = S/N$.

$$x_0 := x_0, \quad \lambda_0 \in [0, 1], \quad f_0 \in F(x_0), \quad k_0 \in K_1, \quad g_0 \in G(x_0);$$

$$x_1 := x_0 + \lambda_0 hf_0 + (1 - \lambda_0) hg_0 k_0, \quad \lambda_1 \in [0, 1], \quad f_1 \in F(x_1), \quad k_1 \in K_1, \quad g_1 \in G(x_1);$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_{j+1} := x_j + \lambda_j hf_j + (1 - \lambda_j) hg_j k_j, \quad \lambda_{j+1} \in [0, 1], \quad f_{j+1} \in F(x_{j+1}), \quad k_{j+1} \in K_1, \quad g_{j+1} \in G(x_{j+1});$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_N := x_{N-1} + \lambda_{N-1} hf_{N-1} + (1 - \lambda_{N-1}) hg_{N-1} k_{N-1}.$$

Let $t_j = \sum_{i=1}^{j} \lambda_i$, and $\chi^N = \{(t_j, x_j) : j = 1, \ldots, N\}$ be the sampled graph.
Our answer to

(Q4) **How can solutions be generated by time discretization?**

is the following theorem.

**Theorem 3 (a)** Suppose assumption (SH) holds, and let $\{\chi^N\}$ be a sequence of sampled graphs. Then there exists a measure $\mu \in B_K[0,T]$ and a solution $X_\mu$ for which

$$\liminf_{N \to \infty} \text{dist}_H(\chi^N, \text{gr } X_\mu) = 0,$$

where $\text{gr } X_\mu$ is the graph of $X_\mu$; i.e. equals

$$\{(t, x(t)) : t \in [0, T]\} \cup \{(t_i, y_i(s)) : i \in I, s \in I_i\}.$$ 

**(b)** Suppose assumption (SH)$_+$ holds. Then the graph of every solution is the limit of a sampled sequence.
Conclusions

• A solution concept for impulsive equations is defined through “matching” the decomposition of the measures.

• Weak and strong invariance is infinitesimally characterized through Hamilton-Jacobi inequalities.

• The proof technique of the invariance results rely on an Euler-type sampling method in the reparameterized time space and they generate solutions.