Tracking and Parameter Identification for Model Reference Adaptive Control*

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Abstract

We provide barrier Lyapunov functions for model reference adaptive control algorithms, allowing us to prove robustness in the input-to-state stability framework, and to compute rates of exponential convergence of the tracking and parameter identification errors to zero. Our results ensure identification of all entries of the unknown weight and control effectiveness matrices. We provide easily checked sufficient conditions for our relaxed persistency of excitation conditions to hold. Our illustrative numerical example demonstrates the performance of the control methods.

1 Introduction

Adaptive control is effective in applications where one needs to simultaneously ensure that a tracking control objective is realized and to cope with uncertain parameters [2, 14, 15, 26]. While usually dependent on some structural properties of the system, adaptive control is valuable because of its ability to solve tracking problems for a wide range of possible values of the unknown model parameters including higher-order systems. One area where adaptive control is important is in aerospace problems [3, 4, 8, 24, 27]; see also applications in [5, 11, 12, 13, 28, 33, 37]. In basic adaptive control for some classes of control systems (such as linear systems), one can often use nonstrict (or weak) Lyapunov functions to ensure that tracking objectives are realized, and then achieve parameter identification (i.e., convergence of the parameter estimates to the true parameter values) provided that a persistency of excitation (or PE) condition is also satisfied.

In its basic form, the PE condition is the requirement that the reference trajectory is such that the regressor satisfies a PE inequality when evaluated along the given reference trajectory [7]; see Section 4.1 for our relaxed PE condition. For 2D curve tracking for gyroscopic models, our works [17, 18] proved globally asymptotically stable tracking and parameter identification results using a novel barrier Lyapunov function approach that ensured robustness with respect to actuator uncertainties under polygonal state constraints; see also [19] for 3D analogs. The adaptive control work [29] provides time-varying gains to ensure exponential convergence for some nonlinear systems and to ensure convergence of the parameter estimates to a constant vector (which might not be the true value of the unknown parameter vector).

Here we combine the approaches of, and are inspired by, the works [23, 29, 36], but we provide substantial benefits that were not present in earlier works. For instance, [23] used barrier Lyapunov functions to identify unknown parameters for many nonlinear systems (by cancelling the effects of undesirable terms in the dynamics), and [23] also provided integral input-to-state stability (or integral ISS) for a DC motor model. By contrast, the present paper provides parameter estimators that can identify unknown weight and control effectiveness matrices, using existing controls for the original plant. Also, the robustness results we provide here apply to a large class of model reference adaptive control systems, including the frequency limited architecture from [36] that can reduce oscillations. Moreover, while [29] only ensured exponential convergence of the parameter estimate to a constant vector, here we achieve tracking and parameter convergence to the true parameter values, including formulas for rates of exponential convergence. For time-varying systems, our use of an integrator state leads to a problem of identifying unknown constant parameters, so our work

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can be viewed as a disturbance observation system. This sets our work apart from the valuable work [32, pp.27-28] which is confined to a time invariant version of the dynamics that we study here but whose time-varying analog would produce a (more difficult to identify) time-varying unknown aggregated weight matrix. These benefits are made possible by our new global strict barrier Lyapunov functions that were not present in [23], and that contain a new coupling of state components and unknown parameters. These new Lyapunov function constructions (rather than trajectory tracking mechanisms for the systems in this work, which were reported in [36] in the special case of time invariant systems) are the focus of this work. Unlike in [23], the unknown parameters in our dynamics enter nonlinearly (through products of entries of unknown weight and control effectiveness matrices) which also puts our work outside the scope of works such as [23]. Also, we do not restrict the dimensions of the systems, so our work can cover higher-order systems; see Section 5.

In addition to the preceding benefits, the systems in this work are motivated by frequency-limited model reference adaptive control methods from [36], which improved on basic adaptive control by ensuring better transient performance. However, while [36] used nonstrict Lyapunov functions and so did not ensure parameter convergence, here we combine barrier Lyapunov functions with a different class of update laws from those in [36], to overcome the obstacles that prevented [23] from being applicable to model reference adaptive control with unknown weight matrices. We use penalty terms in our update laws, and known intervals containing the unknown parameter component values. This differs from the projection update laws in [36]. Our main motivations for our different class of update laws from [36] are that they enable parameter identification, ISS, and rate of convergence computations that were not possible in [23, 36].After providing our notation, definitions, and theory in the next three sections, we illustrate our work in numerical simulations in Section 5, and we close in Section 6 by summarizing our findings and ideas for future research.

2 Notation and Definitions

The dimensions of our Euclidean spaces are arbitrary unless otherwise noted, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $|\cdot|$ is the usual Euclidean norm and corresponding matrix norm, $|\cdot|_\infty$ is the corresponding essential sup norm, and we sometimes write our controls as functions of time, but later they will be feedback controls and so will depend on time through their dependence on states of the original system or on the state of a dynamic controller. Let $\mathbb{D}_+^{m\times m}$ denote the set of all $m\times m$ diagonal matrices whose main diagonal entries are all positive real constants. We use $\mathrm{diag}\{a_1,\dots,a_m\}$ to denote a diagonal matrix having $a_i\in\mathbb{R}$ as its ith diagonal entry for each i. We also let $I_{n\times n}$ and $0_{n\times m}$ denote the $n\times n$ identity and $n\times m$ zero matrices for any dimensions n and m, respectively. We assume that the initial times of our solutions of our time-varying systems are 0, unless otherwise indicated. A continuous function $\gamma:[0,\infty)\to[0,\infty)$ belongs to class $\mathcal{K}_\infty$ (written $\gamma\in\mathcal{K}_\infty$) provided it is strictly increasing and unbounded and $\gamma(0)=0$ [9]. A continuous function $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ is of class $\mathcal{KL}$ (written $\beta\in\mathcal{KL}$) provided for each $s\geq 0$, the function $\beta(\cdot,s)$ belongs to class $\mathcal{K}_\infty$, and for each $r\geq 0$, the function $\beta(r,\cdot)$ is non-increasing and $\beta(r,s)\to 0$ as $s\to\infty$ [9]. A continuous function $\mathcal{L}:\mathcal{S}\to[0,\infty)$ is called a modulus with respect to a set $\mathcal{S}\subseteq\mathbb{R}^n$ containing 0 provided $\mathcal{L}(0)=0$, $\mathcal{L}(q)>0$ for all nonzero values of $q\in\mathcal{S}$, and $\mathcal{L}(q)\to\infty$ as $q\to\mathrm{boundary}(\mathcal{S})$ or as $|q|\to\infty$ with q staying in $\mathcal{S}$. Also, $|f|_\mathcal{C}$ denotes the essential supremum over any set $\mathcal{C}$. Let $\mathrm{row}_i K$ (resp., $\mathrm{col}_i K$) mean the ith row (resp., column) of any matrix K for each row (resp., column) index i of K. We then use K_{ij} (or $K_{i,j}$) to denote the entry in the ith row and jth column of K for all i and j. Also, $\lambda_{\min}(Q)$ denotes the smallest eigenvalue of any positive definite matrix Q. We use the standard definition and properties of fundamental solutions (also called state transition matrices in [25, Section 5.7]) for time-varying systems, which are well known analogs of the matrix exponential; see [6, Chapter 5] or [30, Appendix C.4]. We use $M_1\geq M_2$ for any $p\times p$ matrices M_i to mean that M_1-M_2 is nonnegative definite, and we use almost all in the Lebesgue measure sense. By C^1 of a matrix valued function, we mean that all of its entries are C^1.

3 Model Reference Adaptive Control

3.1 Basic Case

To help make this paper more self-contained, this subsection provides a concise overview of model reference adaptive control for the main class of models from [36], generalized to allow time-varying coefficients and more general integrator states; see Remark 2 and Section 3.2 for other adaptive controls that are covered by our construction including the approach from [32, pp.27-28]. Consider the system

$$\dot{x}_p(t) = A_p(t)x_p(t) + B_p(t)\Lambda u(t) + B_p(t)\delta_p\big(t, x_p(t)\big), \tag{1}$$

where the accessible state $x_p(t)$ is valued in $\mathbb{R}^{n_p}$, the control $u(t)$ is valued in $\mathbb{R}^m$, $\delta_p : \mathbb{R}^{n_p+1} \to \mathbb{R}^m$ is an *uncertainty*, A_p and B_p will be specified below, and $\Lambda \in \mathbb{D}_+^{m\times m}$ is an *unknown* control effectiveness matrix (but we can allow some entries of Λ to be negative, by multiplying the corresponding entries of u and the update laws and the integral terms in our Lyapunov functions below by -1). The structure (1) is motivated by many applications, since a considerable set of aerospace and mechanical systems such as fixed-wing airplanes and robots are accurately enough represented by (1) when only the most significant features are modeled, and because time-varying systems naturally arise when linearizing a tracking dynamics around a reference trajectory. As in [36], we also assume that the uncertainty is parameterized as

$$\delta_p\big(t, x_p\big) = W_p^\top \sigma_p\big(t, x_p\big), \tag{2}$$

where $W_p \in \mathbb{R}^{s\times m}$ is an *unknown* constant weight matrix and $\sigma_p = \begin{bmatrix}\sigma_{\mathrm{p}_1} & \sigma_{\mathrm{p}_2} & \dots & \sigma_{\mathrm{p}_\mathrm{s}}\end{bmatrix}^\top : \mathbb{R}^{n_p+1} \to \mathbb{R}^s$ is a known C^1 function (but see Section 4.5 for time-varying weight matrices). We also assume the following time-varying analog of the controllability of constant pairs (A_p, B_p), where the C^1 property of B_p will be used in Section 4 to ensure C^1 of the regressor G in our adaptive controller:

Assumption 1. *The known functions $A_p : \mathbb{R} \to \mathbb{R}^{n_p\times n_p}$ and $B_p : \mathbb{R} \to \mathbb{R}^{n_p\times m}$ are bounded, A_p is continuous, all entries of B_p are C^1 with bounded first derivatives, and there is a bounded C^1 function $K_p : \mathbb{R} \to \mathbb{R}^{m\times n_p}$ such that*

$$\dot{z}(t) = \big(A_p(t) + B_p(t)K_p(t)\big)z(t) \tag{3}$$

is uniformly globally exponentially stable to 0, and $\dot{K}_p$ is bounded. □

See Remark 1 for ways to satisfy Assumption 1. Set $A_c = A_p + B_pK_p$. To specify our allowable reference trajectories, we first choose an integer $n_c > 0$, a bounded C^1 function $K_r : \mathbb{R} \to \mathbb{R}^{m\times n_c}$ such that $\dot{K}_r$ is bounded, and continuous functions $E_p : \mathbb{R} \to \mathbb{R}^{n_c\times n_p}$ and $E_r : \mathbb{R} \to \mathbb{R}^{n_c\times n_c}$ such that with the choice

$$A_r(t) = \begin{bmatrix} A_c(t) & B_p(t)K_r(t) \\ E_p(t) & E_r(t) \end{bmatrix}, \tag{4}$$

the system $\dot{Z}(t) = A_r(t)Z(t)$ is uniformly globally exponentially stable to 0, and we set $n = n_p + n_c$, so A_r is valued in $\mathbb{R}^{n\times n}$. For instance, we can choose any integer $n_c > 0$, $E_p = 0_{n_c\times n_p}$, and $E_r(t) = -I_{n_c\times n_c}$. We can also allow cases where $E_r = 0_{n_c\times n_c}$, which is the E_r choice made in [36]; see Remark 1 below. Choose any piecewise C^1 function $x_{pr} : \mathbb{R} \to \mathbb{R}^{n_p}$ and any bounded piecewise C^1 function $r : \mathbb{R} \to \mathbb{R}^{n_c}$ such that

$$\dot{x}_{pr}(t) = A_c(t)x_{pr}(t) + B_p(t)K_r(t)r(t) \tag{5}$$

for almost all $t \geq 0$, where $A_c = A_p + B_pK_p$ is as before and K_p is from Assumption 1, and we refer to x_{pr} as a reference trajectory and r as a reference input. We next use the augmented state vector $x(t) = [x_p^\top(t)\ x_c^\top(t)]^\top$ (having dimension $n = n_p + n_c$) and $c(t) = E_p(t)x_{pr}(t) + E_r(t)r(t) - \dot{r}(t)$ (which is defined for almost all t, because r is piecewise C^1), where x_c satisfies

$$\dot{x}_c(t) = E_p(t)x_p(t) + E_r(t)x_c(t) - c(t), \tag{6}$$

which agrees with the integrator state dynamics from [36] when $E_p(t)$ is a nonzero constant matrix and $E_r = 0_{n_c\times n_c}$; see [35] and Remark 2 below for motivation for the integrator state. Then (1) and (6) yield

$$\dot{x}(t) \ = \ A(t)x(t) + B(t)\Lambda u(t) + B(t)W_p^\top\sigma_p\big(t, x_p(t)\big) + B_rc(t), \tag{7}$$

where the known matrix valued functions A, B, and B_r are as follows:

$$A(t) = \begin{bmatrix} A_p(t) & 0_{n_p\times n_c} \\ E_p(t) & E_r(t) \end{bmatrix},\ \ B(t) = \begin{bmatrix} B_p(t) \\ 0_{n_c\times m} \end{bmatrix},\ \text{and}\ B_r = \begin{bmatrix} 0_{n_p\times n_c} \\ -I_{n_c\times n_c} \end{bmatrix} \in \mathbb{R}^{n\times n_c}. \tag{8}$$

Next, we design the control u in (1) and (7). Consider the $\mathbb{R}^m$-valued feedback $u(t) = K(t)x(t) + u_a(t)$, where $u_{\rm n}(t) = K(t)x(t)$ and $u_a(t)$ are called the nominal and adaptive control laws, respectively, $K(t) = [K_p(t)\ K_r(t)]$ with K_p and K_r satisfying the conditions above, and u_a will be specified below. Using this feedback in (7) yields

$$\dot{x}(t) \;=\; A_r(t)x(t) + B_r c(t) + B(t)\Lambda\big(u_a(t) + W^\top\sigma\big(t, x(t)\big)\big), \tag{9}$$

where $W^\top = \big[\Lambda^{-1}W_p^\top \quad \Lambda^{-1}{-}I_{m\times m}\big]\in \mathbb{R}^{m\times(s+m)}$ is an *unknown* (aggregated) weight matrix, $A_r = A + BK$ was defined in (4), and the $\mathbb{R}^{s+m}$-valued known (aggregated) basis function σ is defined by

$$\sigma^\top(t, x(t)) = [\sigma_p^\top(t, x_p(t)) \;\; -x^\top(t)K^\top(t)].$$

The unknown matrices Λ and W enter (9) nonlinearly, through the product $B_p(t)\Lambda W^\top$ where B_p will usually not be invertible (e.g., for dynamics of aircraft lateral dynamics with four states, and with the aileron and rudder as the controls, which gives a constant matrix $B_p \in \mathbb{R}^{4\times 2}$), which puts (9) outside the scope of parameter identification results requiring the unknown parameters to enter linearly. [1]

To specify the adaptive part u_a of the control, we first fix a bounded C^1 matrix valued function $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ and positive constants c_a and c_b such that the time derivative of the function $V_r(t, Z) = Z^\top P(t)Z$ along all solutions of our system $\dot{Z}(t) = A_r(t)Z(t)$ for all $t \geq 0$ satisfies $\dot{V} \leq -c_a|Z(t)|^2$ and such that $V(t, Z) \geq c_b|Z|^2$ for all $t \geq 0$ and $Z \in \mathbb{R}^n$, where A_r was defined in (4); the existence of the required function P and constants c_a and c_b follows from [9, Theorem 4.14].[2] Let the adaptive control in (9) be

$$u_a(t) = -\hat{W}^\top(t)\sigma\big(t, x(t)\big), \tag{10}$$

where $\hat{W}(t)$ is the $\mathbb{R}^{(s+m)\times m}$-valued estimate of W satisfying the update law

$$\dot{\hat{W}}(t) \;=\; \gamma\sigma\big(t, x(t)\big)e^\top(t)P(t)B(t), \tag{11}$$

from [36] where $\gamma > 0$ is a known constant learning rate, and $e(t) = x(t) - x_r(t)$ is the system error with x_r being an $\mathbb{R}^n$-valued reference state vector satisfying the reference system

$$\dot{x}_r(t) \;=\; A_r(t)x_r(t) + B_r c(t) \tag{12}$$

where c was specified above, namely, $x_r = [x_{pr}^\top \;\, r^\top]^\top$ where x_{pr} is the reference trajectory for (5) and the reference input r. Then the system error dynamics is given by using (9), (10), and (12), and has the form

$$\dot{e}(t) \;=\; A_r(t)e(t) - B(t)\Lambda\tilde{W}^\top(t)\sigma\big(t, x(t)\big), \tag{13}$$

where $\tilde{W}(t) = \hat{W}(t) - W$ is the weight error and is valued in $\mathbb{R}^{(s+m)\times m}$. See Fig. 1 for a flow chart of this adaptive control method.

When A_p and B_p are time invariant, [36] shows that the preceding model reference adaptive controller guarantees that the dynamics for the system error $e(t)$ and the weight error are Lyapunov stable, and $\lim_{t\to\infty} e(t) = 0$ from all initial states. In particular, while $\lim_{t\to\infty} e(t) = 0$ holds, the state vector $x(t)$ can be far different from $x_r(t)$ during the transient time (which is the learning phase), unless a high learning rate γ is used in the update law (11). However, update laws with high learning rates in the face of large system uncertainties and abrupt changes may result in signals with high-frequency oscillations, which can violate actuator rate saturation constraints and/or excite unmodeled system dynamics, resulting in system instability for practical applications. Moreover, the update law (11) will not guarantee convergence of the parameter estimates to the true parameter values without a PE condition, which is one of our motivations for replacing (11) by a new update law with penalty terms under a relaxed PE condition. Our new update law also allows us to construct barrier type strict Lyapunov functions for the augmented tracking and parameter identification dynamics for $(e, \tilde{W})$, which is key to our ISS robustness and rate of convergence analysis below.

[1] The preceding W and σ are different from [36], even when σ_p only depends on the state x_p. The work [36] used $\sigma^\top(x(t)) = [\sigma_p^\top(x_p(t)) \;\; x^\top(t)]$ and then incorporated K into the formula for W. Our new W and σ are motivated by the fact that identifying our W is equivalent to identifying Λ and W_p. Note that W is constant, even when the system is time-varying, since W_p is constant, which illustrates the technical merit of our integrator state x_c; see Remark 2 below for a comparison with [32].

[2] The function P can be expressed in terms of the fundamental matrix Φ_{A_r} for the system $\dot{Z}(t) = A_r(t)Z(t)$, and Φ_{A_r} can be computed using the dynamic extension method of computing fundamental solutions that we discuss in Section 4.3 below.

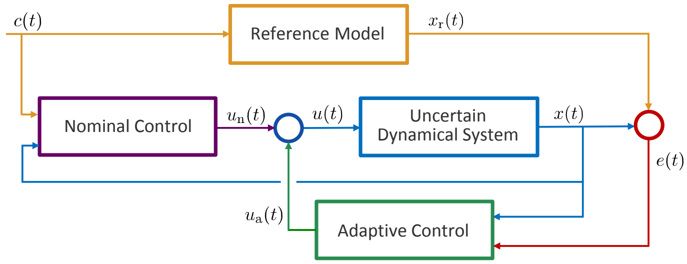

Figure 1: Flow Chart for Model Reference Adaptive Control.

Remark 1. *Assumption 1 is satisfied if (A_p, B_p) is a constant controllable pair (by using a constant matrix K_p that can be found using the Pole-Shifting Theorem [30]). It also covers cases of the form $(A_p(t), B_p(t)) = (A_0 + A_v(t), B_0 + B_v(t))$ for a constant controllable pair (A_0, B_0) when the sup norms of the continuous matrix valued functions A_v and $B_v \in C^1$ are small enough and $\dot{B}_v$ is bounded; this is shown using the Lyapunov function $V_p(z) = z^\top P_c z$ where the positive definite matrix P_c satisfies $P_c A_a + A_a^\top P_c = -I_{n_p \times n_p}$ and $A_a = A_0 + B_0 K_0$ for a constant K_0 such that A_a is Hurwitz (where K_0 can be found using the Pole-Shifting Theorem) and choosing $K_p = K_0$.*

In the special case where A_p, B_p, K_p, E_p, and K_r are constant and $E_r = 0_{n_c \times n_c}$ and (A_p, B_p) is a controllable pair, our matrix A_r in (4) agrees with the matrix A_r in the more basic case from [16] and [36] where there is no frequency limiting control, and in that case the error dynamics have the form (13) and so are covered by the analysis in this work. In that case, if (A_p, B_p) is a controllable pair, and if

$$\mathit{rank} \begin{bmatrix} A_p & B_p \\ E_p & 0_{n_c \times m} \end{bmatrix} = n_p + n_c, \tag{14}$$

then with the choices $n = n_p + n_c$ and

$$A = \begin{bmatrix} A_p & 0_{n_p \times n_c} \\ E_p & 0_{n_c \times n_c} \end{bmatrix} \in \mathbb{R}^{n \times n} \ \ \mathit{and} \ \ B = \begin{bmatrix} B_p^\top & 0_{n_c \times m}^\top \end{bmatrix}^\top \in \mathbb{R}^{n \times m}, \tag{15}$$

the pair (A, B) will be controllable, and then we can choose any K_p and K_r such that with the choice $K = [K_p \ \ K_r]$, the matrix $A + BK$ is Hurwitz; this follows from the Popov-Belevitch-Hautus criterion rank $[A - \lambda I, B] = n$ for controllability of the pair (A, B) and by separately considering zero and nonzero values of $\lambda \in \mathbb{C}$. However, we will not require A_p, B_p, K, E_p, or E_r to be constant in the theory to follow. □

Remark 2. *An alternative approach for (1) for cases where A_p and B_p are constant is provided by [32, pp.27-28], which does not use the integrator state x_c that we used above. However, this earlier work also produces error dynamics of the form (13) that are covered by our work. Instead of an integrator state, [32, pp.27-28] uses the unknown aggregated weight matrix $W^\top = \Lambda^{-1}[K_p \ \ K_r \ \ -W_p^\top]$ and the aggregated basis function $\sigma^\top(t, x, r) = -[x^\top \ r^\top \ \sigma_p^\top(x)]$, where K_p and K_r are constant matrices that satisfy the requirements we gave above. Also, in terms of the parametrization (2) above, the control in [32, pp.27-28] is*

$$u = \hat{W}_1 x_p + \hat{W}_2 r + \hat{W}_3 \sigma_p, \tag{16}$$

where $\hat{W}_1$, $\hat{W}_2$, and $\hat{W}_3$ are estimators for $\Lambda^{-1}K_p$, $\Lambda^{-1}K_r$, and $-\Lambda^{-1}W_p^\top$, respectively. However, whereas our aggregated weight matrix W is constant even if A_p, B_p, K_p, and K_r are time varying, the analog of the aggregated weight matrix W from [32, pp.27-28] for time-varying K_p and K_r would be a time-varying aggregated weight matrix and therefore would be beyond the scope of identifying unknown constant parameter matrices. Moreover, [32] does not provide the novel strict Lyapunov functions for the augmented error dynamics for $(e, \tilde{W})$ that we provide here. Therefore, we believe that our work provides potential advantages over earlier methods, which are made possible by our use of the integrator state x_c and our new Lyapunov function constructions. See also [31] for a valuable survey on alternative versions of methods from [32]. □

3.2 Adaptation with Frequency-Limited System Error Dynamics

An important feature of any model reference adaptive control scheme is the system error $e(t)$. This motivated the frequency limited model from [36], which limits the frequency content of the system error dynamics (13) during the transient (or learning) phase, to filter out possible high-frequency oscillations in the error signal $e(t)$. The frequency limiting work [36] (generalized to time-varying systems) uses an $\mathbb{R}^n$-valued low-pass filtered system error $e_{\rm L}(t)$ of $e(t)$, namely,

$$\dot{e}_{\rm L}(t) \;=\; A_r(t)e_{\rm L}(t) + \eta\big(e(t) - e_{\rm L}(t)\big), \quad e_{\rm L}(0) = 0, \tag{17}$$

where $\eta > 0$ is a filter gain, which is a known constant. Since $e_{\rm L}(t)$ is a low-pass filtered system error of $e(t)$, the filter gain η is chosen such that $\eta \le \eta^*$, where $\eta^* > 0$ is a design parameter. Thus, the reference system (12) is modified to be

$$\dot{x}_r(t) \;=\; A_r(t)x_r(t) + B_r c(t) + \kappa\big(e(t) - e_{\rm L}(t)\big) \tag{18}$$

as in [36], where $\kappa > 0$ is a known constant that can be chosen in the system design. Hence, the new system error dynamics for $e = x - x_r$ given by (9), (10), and (18) has the form

$$\begin{aligned} \dot{e}(t) &= A_r(t)e(t) - B(t)\Lambda\tilde{W}^\top(t)\sigma\big(t,x(t)\big) - \kappa\big(e(t) - e_{\rm L}(t)\big) \\ &= \big(A_r(t) - \kappa I_{n\times n}\big)e(t) - B(t)\Lambda\tilde{W}^\top(t)\sigma\big(t,x(t)\big) + \kappa e_{\rm L}(t). \end{aligned} \tag{19}$$

This leads to the dynamics from [36, Section 5] whose generalization to allow time-varying coefficients is

$$\begin{aligned} \dot{e}(t) &= \big(A_r(t) - \kappa I_{n\times n}\big)e(t) - B(t)\Lambda\tilde{W}^\top(t)\sigma\big(t,x(t)\big) + \kappa e_{\rm L}(t), \\ \dot{e}_{\rm L}(t) &= A_r(t)e_{\rm L}(t) + \eta\left(e(t) - e_{\rm L}(t)\right), \quad \dot{\tilde{x}}(t) \;=\; A_r(t)\tilde{x}(t) + \kappa\left(e(t) - e_{\rm L}(t)\right) \end{aligned} \tag{20}$$

where $\tilde{x}(t) = x_r(t) - x_{{\rm r}_i}(t)$ is the deviation between the modified reference signal x_r satisfying (18) and the ideal reference state vector $x_{{\rm r}_i}$ that satisfies $\dot{x}_{{\rm r}_i}(t) = A_r(t)x_{{\rm r}_i}(t) + B_r c(t)$, where $x_{{\rm r}_i}$ has the form $[x_{{\rm pr}_i}^\top \;\; r^\top]^\top$ for a reference trajectory and reference input pair $(x_{{\rm pr}_i}, r)$ for (5), and with $c(t) = E_p(t)x_{{\rm pr}_i}(t) + E_r(t)r(t) - \dot{r}(t)$. Then (20) can be written as

$$\begin{gathered} \dot{q}(t) = H(t)q(t) + \mathcal{G}(t,\tilde{W}(t),x(t)), \text{ where } q = [e^\top \; e_L^\top \; \tilde{x}^\top]^\top, \\ H(t) = \begin{bmatrix} A_r(t) - \kappa I_{n\times n} & \kappa I_{n\times n} & 0_{n\times n} \\ \eta I_{n\times n} & A_r(t) - \eta I_{n\times n} & 0_{n\times n} \\ \kappa I_{n\times n} & -\kappa I_{n\times n} & A_r(t) \end{bmatrix}, \text{ and } \mathcal{G}(t,\tilde{W},x) = \begin{bmatrix} -B(t)\Lambda\tilde{W}^\top\sigma\big(t,x\big) \\ 0_{n\times n} \\ 0_{n\times n} \end{bmatrix}. \end{gathered} \tag{21}$$

For the rest of this subsection, we assume that (11) is driven by the error $e(t) = x(t) - x_r(t)$, where $x_r(t)$ is obtained from (18) (i.e., not (12)). The system (18) captures a desired closed-loop system behavior modified by a mismatch term $\kappa\big(e(t) - e_{\rm L}(t)\big)$ representing the high-frequency content between the uncertain dynamical system and this reference system. Although this implies a modification of the ideal (unmodified) reference system (12) during transient time, this mismatch makes it possible to limit the frequency content of the system error dynamics (19) (as noted in [36, Theorem 6.1]).

4 Theory and Discussions

We next provide our general theoretical results, which are of independent interest and which we apply later in this section to the basic model reference adaptive control from Section 3.1, as well as the frequency-limited model reference control from Section 3.2.

4.1 Key Lemma

While reminiscent of [20, 23], the lemma in this subsection is not a consequence of [23], because the Δ_i's in (23) that are used to transform our nonstrict Lyapunov function $\mathcal{V}$ are of a new type that cancel the effects of the control effectiveness matrices. This 'strictification' process of transforming a nonstrict Lyapunov function into a strict Lyapunov function will be important for our integral ISS and rate of convergence analysis in later subsections. The Δ_i's are analogous to the auxiliary functions in the Matrosov strictification process in [20], but the auxiliary functions here are of a different type, involving a new coupling $J\tilde{\theta}$ that multiplies state components $\tilde{\theta}$ by J. See Proposition 1 and Section 4.3 for ways to exploit the link between the reference input $r(t)$ and the reference trajectory to check our relaxed PE condition

$$\sum_{i=1}^{N}\int_{t-T}^{t}(\mathrm{row}_i G(s,q_r(s)))^\top \mathrm{row}_i G(s,q_r(s))\mathrm{d}s \geq Q \text{ for all } t\geq 0 \tag{22}$$

that is required in the lemma to follow, where N, T, the regressor function G, the reference trajectory q_r, and the positive definite matrix Q will be specified in our lemma. The sufficient conditions for (22) in Section 4.3 are analogous to the L^2 or other sufficient conditions for PE from [26]. Also, see Remark 4 for a formula for the function γ_0 in the strict Lyapunov function formula (25). In our lemma, the existence of the required function P_H follows from [9, Theorem 4.14] as in Section 3.1, the positivity of the eigenvalues of Q follows because Q is positive definite, and the strict Lyapunov function decay condition (27) will be used to conclude tracking and parameter identification in later subsections that apply the lemma to adaptive control.

Lemma 1. *Let the bounded continuous function $H:\mathbb{R}\to\mathbb{R}^{N\times N}$ be such that $\dot{Z}(t)=H(t)Z(t)$ is uniformly globally exponentially stable to 0. Let $P_H:\mathbb{R}\to\mathbb{R}^{N\times N}$ be a bounded C^1 matrix valued function such that there are positive constants c_H and d_H such that the following two conditions hold: (i) The time derivative of $V(t,Z)=Z^\top P_H(t)Z$ along all solutions of $\dot{Z}(t)=H(t)Z(t)$ satisfies $\dot{V}\leq -c_H|Z(t)|^2$ for all $t\geq 0$ and (ii) $V(t,Z)\geq d_H|Z|^2$ holds for all $t\geq 0$ and $Z\in\mathbb{R}^N$. Let $q_r:\mathbb{R}\to\mathbb{R}^N$ be piecewise C^1 and bounded and $|\dot{q}_r|_\infty$ be finite. Let $G:\mathbb{R}^{N+1}\to\mathbb{R}^{N\times p}$ be C^1 and admit a function $\gamma_G\in\mathcal{K}_\infty$ and a constant $c_G>0$ such that $\sup\{\max\{|\nabla G_{ij}(t,Z)|,|G(t,Z)|\}:t\geq 0\}\leq\gamma_G(|Z|)+c_G$ holds for all $Z\in\mathbb{R}^N$ and all entries G_{ij} of G, $\theta=[\theta_1\ \ldots\ \theta_p]^\top\in\mathbb{R}^p$ be an unknown vector that admits known constants $\bar{\theta}_i$ and $\underline{\theta}_i$ such that $\underline{\theta}_i<\theta_i<\bar{\theta}_i$ for each $i\in\{1,2,\ldots,p\}$, $J=\mathrm{diag}\{J_1,\ldots,J_p\}\in\mathbb{D}_+^{p\times p}$ be unknown, and H, G, and q_r be known. Assume that there are a positive definite matrix $Q\in\mathbb{R}^{p\times p}$ and a constant $T>0$ such that (22) holds, and define $\mathcal{S}\subseteq\mathbb{R}^{N+p}$, $\mathcal{V}:[0,\infty)\times\mathcal{S}\to[0,\infty)$, and $\Delta_i:[0,\infty)\times\mathcal{S}\to\mathbb{R}$ for $i=1,2,\ldots,N$ by*

$$\begin{aligned}&\mathcal{S}=\mathbb{R}^N\times\prod_{i=1}^{p}(\underline{\theta}_i-\theta_i,\bar{\theta}_i-\theta_i),\ \ \mathcal{V}(t,q,\tilde{\theta})=q^\top P_H(t)q+\sum_{i=1}^{p}\int_0^{\tilde{\theta}_i}\frac{J_i\ell}{(\bar{\theta}_i-(\ell+\theta_i))(\ell+\theta_i-\underline{\theta}_i)}\mathrm{d}\ell\ ,\ \text{and}\\ &\Delta_i(t,q,\tilde{\theta})=\tfrac{1}{T}(J\tilde{\theta})^\top\left(\int_{t-T}^{t}\left(\int_z^t(\mathrm{row}_i G(\ell,q_r(\ell)))^\top\mathrm{row}_i G(\ell,q_r(\ell))\mathrm{d}\ell\right)\mathrm{d}z\right)J\tilde{\theta}-q_i\mathrm{row}_i G(t,q_r(t))J\tilde{\theta}.\end{aligned}\tag{23}$$

Let $\mathcal{G}_0:\mathbb{R}^N\to\mathbb{R}^N$ be any globally Lipschitz function such that $\mathcal{G}_0(0)=0$. Then we can construct a continuous positive valued increasing function $\gamma_0:[0,\infty)\to(0,\infty)$ and a constant $\underline{v}>0$ such that with the choice

$$\begin{aligned}\mathcal{L}(q,\tilde{\theta})\ &=\ \sup_{t\geq 0}\left\{\mathcal{V}(t,q,\tilde{\theta})\gamma_0(\mathcal{V}(t,q,\tilde{\theta}))\right\}\\ &\quad+\tfrac{NT}{2}\max_i|\mathrm{row}_i G(t,q_r(t))|_\infty^2|J\tilde{\theta}|^2+N|q||J\tilde{\theta}|\max_i|\mathrm{row}_i G(t,q_r(t))|_\infty,\end{aligned}\tag{24}$$

the time derivative of the function

$$\mathcal{V}^\sharp(t,q,\tilde{\theta})=\int_0^{\mathcal{V}(t,q,\tilde{\theta})}\gamma_0(\ell)\mathrm{d}\ell+\sum_{i=1}^{N}\Delta_i(t,q,\tilde{\theta})\tag{25}$$

along all solutions $(q,\tilde{\theta}):[0,\infty)\to\mathcal{S}$ of

$$\left\{\begin{aligned}\dot{q}(t)\ &=\ H(t)q(t)+G(t,\mathcal{G}_0(q(t))+q_r(t))J\tilde{\theta}(t)\\ \dot{\tilde{\theta}}_i(t)\ &=\ -2(\bar{\theta}_i-(\tilde{\theta}_i(t)+\theta_i))(\tilde{\theta}_i(t)+\theta_i-\underline{\theta}_i)q^\top(t)P_H(t)\,\mathrm{col}_i G(t,\mathcal{G}_0(q(t))+q_r(t)),\ \ 1\leq i\leq p\end{aligned}\right.\tag{26}$$

on its state space $\mathcal{S}$ satisfies

$$\dot{\mathcal{V}}^\sharp\leq -c_H|q(t)|^2-\tfrac{\min_i J_i^2}{2T}\lambda_{\min}(Q)|\tilde{\theta}(t)|^2\tag{27}$$

for almost all $t\geq 0$, and such that the inequalities $\underline{v}|(q,\tilde{\theta})|^2\leq\mathcal{V}^\sharp(t,q,\tilde{\theta})\leq\mathcal{L}(q,\tilde{\theta})$ hold on $[0,\infty)\times\mathcal{S}$. □

Proof. Letting P_{Hij} denote the (i,j) entry of the matrix valued function P_H for all i and j, it follows that along all solutions of (26) on $\mathcal{S}$, the function $\mathcal{V}$ defined in (23) is such that

$$\begin{aligned}\dot{\mathcal{V}}\ &\leq\ -c_H|q(t)|^2\\ &\quad+\left\{2q^\top(t)P_H(t)G(t,\mathcal{G}_0(q(t))+q_r(t))J\tilde{\theta}(t)-2\sum_{i,j,\ell}q_j(t)P_{Hj\ell}(t)J_i\tilde{\theta}_i(t)G_{\ell i}\big(t,\mathcal{G}_0(q(t))+q_r(t)\big)\right\}\\ &\leq\ -c_H|q(t)|^2,\end{aligned}\tag{28}$$

since the quantity in curly braces in (28) is the zero function and since the ith entry of $J\tilde{\theta}$ is $J_i\tilde{\theta}_i$ for each $i \in \{1, 2, \ldots, p\}$ (since J is diagonal). Here and in the sequel, our inequalities and equalities that include time derivatives are for almost all $t \geq 0$. This ensures the forward completeness property that all solutions of (26) with initial states in $\mathcal{S}$ are defined for all nonnegative times, since (28) implies that the q components of each solution are bounded, and because the barrier terms $(\bar{\theta}_i - (\tilde{\theta}_i + \theta_i))(\tilde{\theta}_i + \theta_i - \underline{\theta}_i)$ in (26) ensure that $(q(t), \tilde{\theta}(t)) \in \mathcal{S}$ for all $t \geq 0$. Also,

$$\begin{aligned} \tfrac{d}{dt}\int_{t-T}^{t}\left(\int_{z}^{t}(\mathrm{row}_i G(\ell, q_r(\ell)))^\top \mathrm{row}_i G(\ell, q_r(\ell))\mathrm{d}\ell\right)\mathrm{d}z \;&=\; T\,\mathrm{row}_i(G(t, q_r(t)))^\top \mathrm{row}_i G(t, q_r(t)) \\ &\quad - \int_{t-T}^{t}(\mathrm{row}_i G(\ell, q_r(\ell)))^\top \mathrm{row}_i G(\ell, q_r(\ell))\mathrm{d}\ell \end{aligned} \tag{29}$$

holds for all $t \geq 0$ and $i \in \{1, 2, \ldots, p\}$, so along all solutions of (26) on $\mathcal{S}$, we also have

$$\begin{aligned} &\sum_{i=1}^{N}\dot{\Delta}_i = \sum_{i=1}^{N}\tfrac{2}{T}(J\tilde{\theta}(t))^\top\left(\int_{t-T}^{t}\left(\int_{z}^{t}(\mathrm{row}_i G(\ell, q_r(\ell)))^\top \mathrm{row}_i G(\ell, q_r(\ell))\mathrm{d}\ell\right)\mathrm{d}z\right)J\dot{\tilde{\theta}}(t) \\ &-\sum_{i=1}^{N}\tfrac{(J\tilde{\theta}(t))^\top}{T}\int_{t-T}^{t}(\mathrm{row}_i G(\ell, q_r(\ell)))^\top \mathrm{row}_i G(\ell, q_r(\ell))\mathrm{d}\ell J\tilde{\theta}(t) \\ &+\left[\sum_{i=1}^{N}(J\tilde{\theta}(t))^\top(\mathrm{row}_i G(t, q_r(t)))^\top \mathrm{row}_i G(t, q_r(t))J\tilde{\theta}(t) - \sum_{i=1}^{N}\dot{q}_i(t)\mathrm{row}_i G(t, q_r(t))J\tilde{\theta}(t)\right] \\ &-\sum_{i=1}^{N}q_i(t)\sum_{j=1}^{p}\tfrac{d}{dt}\big(G_{ij}(t, q_r(t))\big)J_j\tilde{\theta}_j(t) - \sum_{i=1}^{N}q_i\mathrm{row}_i G(t, q_r(t))J\dot{\tilde{\theta}}(t). \end{aligned} \tag{30}$$

Also, the Fundamental Theorem of Calculus applied to the functions $f_{ij}(\ell) = G_{ij}(t, \ell\mathcal{G}_0(q(t)) + q_r(t))$ on the interval $[0, 1]$ for each $t \geq 0$ gives

$$\begin{aligned} G_{ij}(t, \mathcal{G}_0(q(t)) + q_r(t)) = f_{ij}(1) \;&=\; f_{ij}(0) + \int_0^1 f'_{ij}(s)\mathrm{d}s \\ &=\; G_{ij}(t, q_r(t)) + \int_0^1 \nabla_q G_{ij}(t, \ell\mathcal{G}_0(q(t)) + q_r(t))\mathrm{d}\ell\mathcal{G}_0(q(t)) \end{aligned}$$

for all $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, p\}$, where $\nabla_q G_{ij}$ is the gradient with respect to only the last N components of the argument of G_{ij}. Therefore, we can find a function $\mathcal{F}$ such that

$$\dot{q}(t) = H(t)q(t) + G(t, \mathcal{G}_0(q(t)) + q_r(t))J\tilde{\theta}(t) = G(t, q_r(t))J\tilde{\theta}(t) + \mathcal{F}(t, q(t), \tilde{\theta}(t)) \tag{31}$$

holds along all solutions of (26) in $\mathcal{S}$. Moreover, since $\mathcal{G}_0$ is globally Lipschitz and satisfies $\mathcal{G}_0(0) = 0$, we can use our bounds on P_H, θ, $\tilde{\theta}$, and q_r to find a continuous positive valued increasing function $\mathcal{G}_1$ such that

$$\max\left\{\left|\dot{\tilde{\theta}}\right|, |\mathcal{F}_i(t, q, \tilde{\theta})|\right\} \;\leq\; \mathcal{G}_1(|q|)|q| \tag{32}$$

along all solutions of (26) in $\mathcal{S}$ for all indices i; see Remark 4. Replacing $\dot{q}_i(t)$ in (30) by $\mathrm{row}_i G(t, q_r(t))J\tilde{\theta}(t) + \mathcal{F}_i(t, q(t), \tilde{\theta}(t))$ for each i to rewrite the quantity in squared brackets in (30) as

$$-\sum_{i=1}^{N}\mathcal{F}_i(t, q(t), \tilde{\theta}(t))\mathrm{row}_i G(t, q_r(t))J\tilde{\theta}(t), \tag{33}$$

and using (32) in the result, we obtain continuous positive valued increasing functions $\mathcal{G}_2$ and $\mathcal{G}_3$ such that

$$\sum_{i=1}^{N}\dot{\Delta}_i \leq \mathcal{G}_2(|q|)|q|(|q| + |\tilde{\theta}|) - \tfrac{\min_i J_i^2}{T}\lambda_{\min}(Q)|\tilde{\theta}|^2 \leq \mathcal{G}_3(|q|)|q|^2 - \tfrac{\min_i J_i^2}{2T}\lambda_{\min}(Q)|\tilde{\theta}|^2 \tag{34}$$

holds along all solutions of (26) in $\mathcal{S}$, by using Young's inequality to get

$$\mathcal{G}_2(|q|)|q||\tilde{\theta}| \leq \tfrac{\min_i J_i^2\lambda_{\min}(Q)}{2T}|\tilde{\theta}|^2 + \tfrac{T}{2\lambda_{\min}(Q)\min_i J_i^2}\mathcal{G}_2^2(|q|)|q|^2 \tag{35}$$

and then choosing $\mathcal{G}_3(s) = \mathcal{G}_2(s) + T\mathcal{G}_2^2(s)/(2\lambda_{\min}(Q)\min_i J_i^2)$, and also using the fact that the double integral in (30) is bounded by $0.5T^2|\mathrm{row}_i G(t, q_r(t))|_\infty^2$, where the sup is over all $t \geq 0$.

Next note that $\mathcal{V}$ admits a positive definite quadratic lower bound of the form $\underline{c}|(q, \tilde{\theta})|^2$ in $(q, \tilde{\theta})$ on $\mathcal{S}$. To obtain the constant $\underline{c} > 0$, we can apply the relation $\max_{r\in[a,b]}(b - r)(r - a) = (b - a)^2/4$ with the choices

$a = \underline{\theta}_i$, $b = \bar{\theta}_i$, and $r = \ell + \theta_i$ for all $\ell \in (\underline{\theta}_i - \theta_i, \bar{\theta}_i - \theta_i)$ to check that the denominators of the integrands in the formula for $\mathcal{V}$ in (23) are bounded above by $(\bar{\theta}_i - \underline{\theta}_i)^2/4$. This allows us to choose

$$\underline{c} = \min\left\{d_H, \min\left\{\tfrac{2J_j}{(\bar{\theta}_i - \underline{\theta}_i)^2} : 1 \leq j \leq p\right\}\right\}. \tag{36}$$

Then for all $i \in \{1, \dots, N\}$, we can use the triangle inequality $|q||\tilde{\theta}| \leq \frac{1}{2}|(q, \tilde{\theta})|^2$ to obtain

$$\begin{array}{l} |\Delta_i(t, q, \tilde{\theta})| \;\leq\; \frac{T}{2}|J|^2|\mathrm{row}_i G(t, q_r(t))|_\infty^2|\tilde{\theta}|^2 + |J||\mathrm{row}_i G(t, q_r(t))|_\infty|q||\tilde{\theta}| \;\leq\; B_*|(q, \tilde{\theta})|^2, \\ \text{where } B_* = \frac{T}{2}|J|^2|\mathrm{row}_i G(t, q_r(t))|_\infty^2 + \frac{1}{2}|J||\mathrm{row}_i G(t, q_r(t))|_\infty \end{array} \tag{37}$$

for all $(t, q, \tilde{\theta}) \in [0, \infty) \times \mathcal{S}$. Therefore, for any constant $c_* > (2/\underline{c})NB_*c_H$, there is a constant $\underline{v} > 0$ such that the function $\mathcal{G}_4(r) = \mathcal{G}_3(\sqrt{r/\underline{c}}) + c_*$ satisfies

$$\mathcal{G}_4(\mathcal{V}(t, q, \tilde{\theta})) \geq \mathcal{G}_4(\underline{c}|(q, \tilde{\theta})|^2) \geq \mathcal{G}_4(\underline{c}|q|^2) \geq \mathcal{G}_3(|q|) \text{ and} \tag{38}$$

$$\tfrac{1}{c_H}\mathcal{G}_4\left(\tfrac{1}{2}\mathcal{V}(t, q, \tilde{\theta})\right)\tfrac{1}{2}\mathcal{V}(t, q, \tilde{\theta}) + \sum_{i=1}^{N}\Delta_i(t, q, \tilde{\theta}) \geq \tfrac{1}{c_H}c_*\tfrac{1}{2}\underline{c}|(q, \tilde{\theta})|^2 - NB_*|(q, \tilde{\theta})|^2 \geq \underline{v}|(q, \tilde{\theta})|^2 \tag{39}$$

for all $(q, \tilde{\theta}) \in \mathcal{S}$. Choosing $\gamma_0 = \frac{\mathcal{G}_4}{c_H} + 1$ in the formula (25) for $\mathcal{V}^\sharp$ gives

$$\mathcal{V}^\sharp(t, q, \tilde{\theta}) \geq \int_{\mathcal{V}(t,q,\tilde{\theta})/2}^{\mathcal{V}(t,q,\tilde{\theta})}\left(\frac{\mathcal{G}_4(\ell)}{c_H} + 1\right)\mathrm{d}\ell + \sum_{i=1}^{N}\Delta_i(t, q, \tilde{\theta}) \geq \frac{\mathcal{V}(t, q, \tilde{\theta})}{2}\frac{\mathcal{G}_4(\mathcal{V}(t, q, \tilde{\theta})/2)}{c_H} + \sum_{i=1}^{N}\Delta_i(t, q, \tilde{\theta}) \tag{40}$$

so (39) gives $\mathcal{V}^\sharp(t, q, \tilde{\theta}) \geq \underline{v}|(q, \tilde{\theta})|^2$ for all $(q, \tilde{\theta}) \in \mathcal{S}$ and $t \geq 0$. Also, along all solutions of (26) in $\mathcal{S}$,

$$\begin{array}{rcl} \dot{\mathcal{V}}^\sharp & \leq & -\left(\frac{\mathcal{G}_4(\mathcal{V}(t,q,\tilde{\theta}))}{c_H} + 1\right)c_H|q|^2 + \mathcal{G}_4(\mathcal{V}(t, q, \tilde{\theta}))|q|^2 - \frac{\min_i J_i^2\lambda_{\min}(Q)}{2T}|\tilde{\theta}|^2 \\ & = & -c_H|q|^2 - \frac{\min_i J_i^2}{2T}\lambda_{\min}(Q)|\tilde{\theta}|^2, \end{array} \tag{41}$$

by (28) and (34) and (38), so Lemma 1 holds, where the term $\frac{NT}{2}\max_i|\mathrm{row}_i G(t, q_r(t))|_\infty^2|J\tilde{\theta}|^2$ in the formula (24) for $\mathcal{L}$ comes from bounding the double integrals in the Δ_i formulas in (23). □

We can provide sufficient conditions that facilitate checking that our persistency of excitation condition (22) is satisfied for some positive definite matrix Q and some constant $T > 0$. For instance, we prove the following (but see Section 4.3 for other ways to check the PE condition):

Proposition 1. *If $q_r : \mathbb{R} \to \mathbb{R}^N$ is piecewise C^1 and periodic of some period $T > 0$, and if $G : \mathbb{R}^{N+1} \to \mathbb{R}^{N\times p}$ is continuous and has period T in its first argument and is such that there is a $j \in \{1, 2, \dots, N\}$ such that*

$$\int_0^T \mathrm{diag}\{G_{j1}(s, q_r(s)), .., G_{jp}(s, q_r(s))\}R_j(s, q_r(s))\mathrm{d}s \tag{42}$$

is positive definite, where R_j is the $p \times p$ matrix having all of its rows equaling $\mathrm{row}_j G$, then there is a positive definite matrix Q such that (22) holds with this T. □

Proof. First note that since q_r has period T and since $(\mathrm{row}_i G(s, q_r(s)))^\top\mathrm{row}_i G(s, q_r(s))$ is nonnegative definite for all i and s, we can lower bound the left side of (22) by

$$\begin{array}{rcl} \sum_{i=1}^{N}\int_{t-T}^{t}(\mathrm{row}_i G(s, q_r(s)))^\top\mathrm{row}_i G(s, q_r(s))\mathrm{d}s & = & \sum_{i=1}^{N}\int_0^T(\mathrm{row}_i G(s, q_r(s)))^\top\mathrm{row}_i G(s, q_r(s))\mathrm{d}s \\ & \geq & \int_0^T(\mathrm{row}_j G(s, q_r(s)))^\top\mathrm{row}_j G(s, q_r(s))\mathrm{d}s. \end{array} \tag{43}$$

Since we also have $\mathrm{diag}\{G_{j1}(s, q_r(s)), .., G_{jp}(s, q_r(s))\}R_j(s, q_r(s)) = (\mathrm{row}_j G(s, q_r(s)))^\top\mathrm{row}_j G(s, q_r(s))$ for all $s \in [0, T]$, the result follows. □

Remark 3. *See Section 4.3 for a way to apply a Poincaré fixed point argument to ensure that a periodic reference input can generate a periodic reference trajectory, which will make it possible to apply Proposition 1. Since all rows of R_j in (42) are equal to the jth row of G, it follows that for each choice of $s \in [0, T]$, the integrand in (42) is a matrix of rank at most 1. However, (42) requires its integral over $[0, T]$ to be a positive definite matrix. See Section 5 for an example where this positive definiteness condition holds.* □

Remark 4. *We can construct a formula for the function γ_0 in (24)-(25). This can be done by finding formulas for the functions $\mathcal{G}_i$ for $i = 1, 2, 3, 4$ from the proof of Lemma 1, as follows. To find a formula for $\mathcal{G}_1$, first note that we can use the Fundamental Theorem of Calculus to write the function $\mathcal{F}$ from (31) as*

$$\mathcal{F}(t, q, \tilde{\theta}) = H(t)q + [G(t, \mathcal{G}_0(q) + q_r(t)) - G(t, q_r(t))]\, J\tilde{\theta} = H(t)q + \mathcal{M}(t, q)J\tilde{\theta}, \tag{44}$$

where the $N \times p$ matrix $\mathcal{M}(t, q)$ has the ij entry

$$\mathcal{M}_{ij}(t, q) = \textstyle\int_0^1 \nabla_q G_{ij}(t, q_r(t) + \ell\mathcal{G}_0(q))\mathrm{d}\ell\mathcal{G}_0(q)$$

for all $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, p\}$, where ∇_q indicates the gradient with respect to the last N components of the argument of G as before. Hence, for each $i \in \{1, \ldots, N\}$, we obtain

$$\mathcal{F}_i(t, q, \tilde{\theta}) = (\mathrm{row}_i H(t))q + \sum_{j=1}^{p} \left(\textstyle\int_0^1 \nabla_q G_{ij}(t, q_r(t) + \ell\mathcal{G}_0(q))\mathcal{G}_0(q)\mathrm{d}\ell\right) J_j\tilde{\theta}_j. \tag{45}$$

Also, along all solutions of (26) in $\mathcal{S}$, we can use the relation $\max\{(\bar{\theta}_i - \ell)(\ell - \underline{\theta}_i) : \ell \in [\underline{\theta}_i, \bar{\theta}_i]\} = \frac{1}{4}(\bar{\theta}_i - \underline{\theta}_i)^2$ to obtain

$$\left|\dot{\tilde{\theta}}_i(t)\right| \le \frac{(\bar{\theta}_i - \underline{\theta}_i)^2}{2} |q(t)||P_H|_\infty |\mathrm{col}_i G(t, \mathcal{G}_0(q(t)) + q_r(t))| \tag{46}$$

for $i = 1, 2, \ldots, p$, which gives

$$\left|\dot{\tilde{\theta}}(t)\right| \le \tfrac{1}{2}|P_H|_\infty |q(t)| \sqrt{\sum_{i=1}^{p} (\bar{\theta}_i - \underline{\theta}_i)^4 \left|\mathrm{col}_i G(t, \mathcal{G}_0(q(t)) + q_r(t))\right|^2} \tag{47}$$

for all $t \ge 0$. Therefore, we can use

$$\mathcal{G}_1(r) = \max\left\{\tfrac{1}{2}|P_H|_\infty \sum_{i=1}^{p} (\bar{\theta}_i - \underline{\theta}_i)^2 \left(\gamma_G(\overline{\mathcal{G}}_0 r + |q_r|_\infty) + c_G\right), |H|_\infty + p\bar{\mathcal{L}}\overline{\mathcal{G}}_0 \left(\gamma_G(\overline{\mathcal{G}}_0 r + |q_r|_\infty) + c_G\right)\right\}, \tag{48}$$

where $\bar{\mathcal{L}} = \max_i J_i(\bar{\theta}_i - \underline{\theta}_i)$, $\overline{\mathcal{G}}_0$ is any global Lipschitz constant for $\mathcal{G}_0$, γ_G and c_G are from the statement of Lemma 1, and we used the subadditivity of the square root.

To find $\mathcal{G}_2$, first note that along all solutions of (26), our formula (30) and our bounds from (32) give

$$\begin{array}{rcl}\sum_{i=1}^{N} \dot{\Delta}_i & \le & N\frac{2}{T}|J\tilde{\theta}|\frac{T^2}{2}|\mathrm{row}_i G(t, q_r(t))|_\infty^2 |J|\mathcal{G}_1(|q|)|q| + \sum_{i=1}^{N} |\mathrm{row}_i G(t, q_r(t))|_\infty |J\tilde{\theta}|\mathcal{G}_1(|q|)|q| \\ & & +N|q||J||\tilde{\theta}|\max_i \sum_{j=1}^{p} |\frac{d}{dt}G_{ij}(t, q_r(t))|_\infty + |q|\sum_{i=1}^{N} |\mathrm{row}_i G(t, q_r(t))|_\infty |J|\mathcal{G}_1(|q|)|q| - \frac{\min_i J_i^2 \lambda_{\min}(Q)|\tilde{\theta}|^2}{T},\end{array} \tag{49}$$

(using the formula (33) for the quantity in squared brackets in (30) as before) which allows us to choose

$$\begin{array}{rcl}\mathcal{G}_2(r) & = & NT|J|^2 \max_i |\mathrm{row}_i G(t, q_r(t))|_\infty^2 \mathcal{G}_1(r) \\ & & +2\sum_{i=1}^{N} |\mathrm{row}_i G(t, q_r(t))|_\infty |J|\mathcal{G}_1(r) + N|J|\max_i \sum_{j=1}^{p} |\frac{d}{dt}G_{ij}(t, q_r(t))|_\infty.\end{array} \tag{50}$$

With the preceding choice of $\mathcal{G}_2$, and with $\underline{c}$ and B_* defined in (36) and (37), we can then choose

$$\gamma_0 = \tfrac{\mathcal{G}_4}{c_H} + 1, \text{ where } \mathcal{G}_4(r) = \mathcal{G}_3\left(\sqrt{\tfrac{r}{\underline{c}}}\right) + \max\left\{1, \tfrac{3}{\underline{c}}NB_* c_H\right\} \text{ and } \mathcal{G}_3(r) = \mathcal{G}_2(r) + \tfrac{T\mathcal{G}_2^2(r)}{2\lambda_{\min}(Q)\min_i J_i^2}. \tag{51}$$

The preceding formulas will be useful for computing rates of exponential convergence in Section 4.6 below. □

4.2 Applying Lemma 1 to Basic Model Reference Adaptive Control

We apply the theory from the preceding subsection to the basic model reference adaptive controller from Section 3.1; see Section 4.4 for an application to the frequency limited model from Section 3.2. We use two central ideas in this subsection. First, we arrange the entries of $W \in \mathbb{R}^{\mathcal{N} \times m}$ as a column vector

$\theta = [\theta_1 \ \ \ldots \ \ \theta_{\mathcal{N}m}]^\top \in \mathbb{R}^{\mathcal{N}m}$, where $\mathcal{N} = s + m$. Second, we replace the update law (11) by update laws of the form

$$\dot{\hat{\theta}}_i(t) = -2(\bar{\theta}_i - \hat{\theta}_i(t))(\hat{\theta}_i(t) - \underline{\theta}_i)e^\top P_A(t)\mathrm{col}_i G(t, x(t)) \tag{52}$$

where the $\hat{\theta}_i$'s will be estimates of the θ_i's, $\bar{\theta}_i$ and $\underline{\theta}_i$ are known constants such that $\underline{\theta}_i < \theta_i < \bar{\theta}_i$ for each i, $x = e + x_r$, P_A will be defined in terms of a suitable quadratic Lyapunov function for the system $\dot{Z}(t) = A_r(t)Z(t)$, and G will be defined in terms of B and σ from Section 3.1.[3] In our theorems, we will specify known bounds

$$\underline{W}_{ij} < W_{ij} < \bar{W}_{ij} \text{ for all } i \in \{1, \ldots, \mathcal{N}\} \text{ and } j \in \{1, \ldots, m\}. \tag{53}$$

for the entries of an unknown weight matrix W. Our result will follow from this theorem with $\mathcal{N} = s + m$, by noting that the dynamics for the errors $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$ in (55) agree with (52).

Theorem 1. *Let the bounded continuous matrix valued function $A_r : \mathbb{R} \to \mathbb{R}^{n\times n}$ be such that $\dot{Z}(t) = A_r(t)Z(t)$ is uniformly globally exponentially stable to 0, $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}^{\mathcal{N}}$ be C^1 and admit a function $\sigma_* \in \mathcal{K}_\infty$ and a constant $\sigma_{**} \geq 0$ such that* $\sup\{\max\{|\nabla\sigma_i(t, z)|, |\sigma(t, z)|\} : t \geq 0\} \leq \sigma_*(|z|) + \sigma_{**}$ *for all $i \in \{1, \ldots, \mathcal{N}\}$ and $z \in \mathbb{R}^n$, and $x_r : \mathbb{R} \to \mathbb{R}^n$ be a piecewise C^1 function such that x_r is bounded and $|\dot{x}_r|_\infty$ is finite, where A_r, σ, and x_r are known. Let $\tilde{W} : \mathbb{R} \to \mathbb{R}^{\mathcal{N}\times m}$ be piecewise C^1, $W \in \mathbb{R}^{\mathcal{N}\times m}$ be unknown, and $\bar{W}_{ij}$ and $\underline{W}_{ij}$ be known constants such that (53) holds. Let $\Lambda = \mathrm{diag}\{\Lambda_1, \ldots, \Lambda_m\} \in \mathbb{D}_+^{m\times m}$ be unknown, and let $B : \mathbb{R} \to \mathbb{R}^{n\times m}$ be C^1 and bounded. Define $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n\times(m\mathcal{N})}$ and the $m\mathcal{N}$ dimensional vectors θ, $\bar{\theta}$, $\underline{\theta}$, and $\tilde{\theta}$ by*

$$\begin{array}{l} G(t, z) = -\big[B(t)\sigma_1(t, z) \quad B(t)\sigma_2(t, z) \quad \ldots \quad B(t)\sigma_{\mathcal{N}}(t, z)\big], \ \ \theta_{\ell+(p-1)m} = W_{p\ell}, \ \ \bar{\theta}_{\ell+(p-1)m} = \bar{W}_{p\ell}, \\ \underline{\theta}_{\ell+(p-1)m} = \underline{W}_{p\ell}, \ \ \textit{and} \ \ \tilde{\theta}_{\ell+(p-1)m} = \tilde{W}_{p\ell} \ \ \textit{for all} \ \ p \in \{1, 2, \ldots, \mathcal{N}\} \ \textit{and} \ \ell \in \{1, 2, \ldots, m\}. \end{array} \tag{54}$$

Assume that there are a constant $T > 0$ and a positive definite Q such that (22) holds with the choice of G from (54) and $q_r = x_r$. Choose a bounded C^1 function $P_A : \mathbb{R} \to \mathbb{R}^{n\times n}$ that admits constants $c_A > 0$ and $d_A > 0$ such that the time derivative of $V_A(t, Z) = Z^\top P_A(t)Z$ satisfies $\dot{V}_A \leq -c_A|Z(t)|^2$ along all solutions of $\dot{Z}(t) = A_r(t)Z(t)$ for all $t \geq 0$ and $V_A(t, Z) \geq d_A|Z|^2$ for all $t \geq 0$ and $Z \in \mathbb{R}^n$. Then the system

$$\left\{ \begin{array}{rcl} \dot{e}(t) & = & A_r(t)e(t) - B(t)\Lambda\tilde{W}^\top(t)\sigma\big(t, x_r(t) + e(t)\big) \\ \dot{\tilde{\theta}}_i(t) & = & -2(\bar{\theta}_i - (\tilde{\theta}_i(t) + \theta_i))(\tilde{\theta}_i(t) + \theta_i - \underline{\theta}_i)e^\top(t)P_A(t)\mathrm{col}_i G(t, x_r(t) + e(t)), \ \ i = 1, 2, \ldots, m\mathcal{N} \end{array} \right. \tag{55}$$

is globally asymptotically stable to 0 on its state space $\mathcal{S} = \mathbb{R}^n \times \prod_{i=1}^{m\mathcal{N}}(\underline{\theta}_i - \theta_i, \bar{\theta}_i - \theta_i)$. □

Proof. First note that for all $z \in \mathbb{R}^n$ and $i = 1, 2, \ldots, n$ and $t \geq 0$, we have

$$\mathrm{row}_i\left(B(t)\Lambda\tilde{W}^\top(t)\sigma(t, z)\right) = \sum_{\ell=1}^{m}\sum_{p=1}^{\mathcal{N}} B_{i\ell}(t)\Lambda_\ell\tilde{W}_{p\ell}(t)\sigma_p(t, z) = -\mathrm{row}_i G(t, z)J\tilde{\theta}(t), \tag{56}$$

where $J = \mathrm{diag}\{J_1, \ldots, J_{m\mathcal{N}}\}$ is defined by $J_{\ell+(p-1)m} = \Lambda_\ell$ for all $p \in \{1, 2, \ldots, \mathcal{N}\}$ and $\ell \in \{1, 2, \ldots, m\}$. To check the second equality in (56), note that if $1 \leq i \leq n$, then the left side of this second equality is

$$\begin{array}{rcl} \sum_{\ell=1}^{m}\sum_{p=1}^{\mathcal{N}} B_{i\ell}(t)J_{\ell+(p-1)m}\tilde{\theta}_{\ell+(p-1)m}\sigma_p(t, z) & = & -\sum_{\ell=1}^{m}\sum_{p=1}^{\mathcal{N}} G_{i,\ell+(p-1)m}(t, z)J_{\ell+(p-1)m}\tilde{\theta}_{\ell+(p-1)m} \\ & = & -\mathrm{row}_i G(t, z)J\tilde{\theta}, \end{array} \tag{57}$$

which proves (56). The result follows from Lemma 1 with $H = A_r$, $P_H = P_A$, $\mathcal{G}_0(q) = q = e$, $p = m\mathcal{N}$, $q_r = x_r$, and $N = n$. □

Remark 5. *In model reference adaptive control, $\tilde{\theta}_i$ is the error $\hat{\theta}_i - \theta_i$ between the estimate $\hat{\theta}_i$ and θ_i, where $\theta_{\ell+(p-1)m} = W_{p\ell}$ for all p and ℓ and W is the aggregated weight matrix. Theorem 1 says to initialize $\hat{\theta}_i$ in an interval $(\underline{\theta}_i, \bar{\theta}_i)$ containing θ_i. This can be done, since the $\underline{\theta}_i$'s and $\bar{\theta}_i$'s are known. Under the assumptions of Theorem 1, (55) is uniformly globally asymptotically stable to 0 on $\mathcal{S}$ (by a variant of the integral ISS proof in [23], which yields a positive definite function α_0 such that the $\mathcal{V}^\sharp$ from Lemma 1 satisfies $\dot{\mathcal{V}}^\sharp \leq -\alpha_0(\mathcal{V}^\sharp)$ along all solutions of (26)). The C^1 of x_r holds if $c(t)$ is continuous, by the structure of (12). Then x_r and $\dot{x}_r$ will be bounded if x_r is periodic. In the next subsection we provide ways to ensure that x_r is periodic.* □

[3]The C^1 requirements in Theorem 1 are needed to apply Lemma 1, whose proof needed gradients of entries of G. The existence of P_A in Theorem 1 follows from [9, Theorem 4.14], in the same way that we can find the function P_H in Lemma 1.

4.3 Checking the PE Condition for Basic Model Reference Adaptive Control

To apply Proposition 1 to model reference adaptive control problems, note that in the context of Section 3.1, if the reference input $r(t)$ from the reference system (5) is of some period $T > 0$ and is piecewise C^1, with (A_p, B_p) and $K = [K_p\ \ K_r]$ also having the same period T, and if we use the notation

$$\mathcal{H}(s,a) = \int_s^a \Phi_{A_c}(a,\ell)B_p(\ell)K_r(\ell)r(\ell)\mathrm{d}\ell \tag{58}$$

for all real values s and a where $A_c = A_p + B_pK_p$ and Φ_{A_c} is the fundamental matrix for the system

$$\dot{Z}(t) = A_c(t)Z(t) \tag{59}$$

(as defined, e.g., in [30, Appendix C.4]), then the unique maximal solution of the reference system (5) that satisfies $x_{pr}(0) = (I_{n_p\times n_p} - \Phi_{A_c}(T,0))^{-1}\mathcal{H}(0,T)$ will also have period T.[4] This follows by finding a fixed point of the corresponding Poincaré map, because applying variation of parameters to (5) gives

$$\begin{aligned} x_{pr}(T) &= \Phi_{A_c}(T,0)x_{pr}(0) + \mathcal{H}(0,T) \;=\; -(I_{n_p\times n_p} - \Phi_{A_c}(T,0))x_{pr}(0) + x_{pr}(0) + \mathcal{H}(0,T) \\ &= -(I_{n_p\times n_p} - \Phi_{A_c}(T,0))(I_{n_p\times n_p} - \Phi_{A_c}(T,0))^{-1}\mathcal{H}(0,T) + x_{pr}(0) + \mathcal{H}(0,T) \;=\; x_{pr}(0), \end{aligned} \tag{60}$$

and so also $x_{pr}(t+T) = \Phi_{A_c}(t+T,T)x_{pr}(T)+\mathcal{H}(T,t+T) = \Phi_{A_c}(t+T,T)x_{pr}(0)+\mathcal{H}(0,t) = \Phi_{A_c}(t,0)x_{pr}(0)+\mathcal{H}(0,t) = x_{pr}(t)$ for all $t \in \mathbb{R}$, since the periodicity of B_p, K_r, and r gives $\mathcal{H}(T,t+T) = \mathcal{H}(0,t)$. Hence, x_{pr} has period T, so x_r will also have period T, so we can choose $q_r = x_r$ in Proposition 1 if we choose

$$x_r(0) = \big((I_{n_p\times n_p} - \Phi_{A_c}(T,0))^{-1}\mathcal{H}(0,T), r(0)\big) \tag{61}$$

as the initial state for x_r from (12). When A_c is a constant Hurwitz matrix, we can write $\Phi_{A_c}(t,s) = e^{A_c(t-s)}$ for all real t and s to find formulas for the required initial state $x_r(0)$ to ensure that x_r has period T.

Using the linearity of (59), we can compute the matrix $\Phi_{A_c}(T,0)$ in the formula for $x_{pr}(0)$, because its ith column is the solution $\phi(T,0,e_i)$ of (59) with the initial state $Z(0) = e_i$ evaluated at T, where e_i is the ith standard basis vector, but in general, an explicit formula for Φ_{A_c} may not be available when A_c is time varying. One strategy for verifying the PE condition when A_c is time varying is to use a dynamic extension to compute the Φ_{A_c} values that are required for the initial condition $x_{pr}(0)$, using the fact that $\Phi_{A_c}(t,s) = \alpha_{A_c}(t)\beta_{A_c}(s)$ for all real t and s, where α_{A_c} and β_{A_c} are the unique solutions of the matrix differential equations $\dot\alpha_{A_c}(t) = A_c(t)\alpha_{A_c}(t)$ and $\dot\beta_{A_c}(t) = -\beta_{A_c}(t)A_c(t)$ that satisfy $\alpha_{A_c}(0) = \beta_{A_c}(0) = I_{n_p\times n_p}$. To verify the preceding formula for the fundamental solution, it suffices to notice that $\omega(t) = \alpha_{A_c}(t)\beta_{A_c}(t)$ satisfies $\dot\omega(t) = A_c(t)\omega(t) - \omega(t)A_c(t)$ for all $t \neq 0$ and $\omega(0) = I_{n_p\times n_p}$ and therefore ω must be identically equal to $I_{n_p\times n_p}$ on $\mathbb{R}$ by standard uniqueness results, and then to notice that $\alpha_{A_c}(t) = \Phi_{A_c}(t,0)$, and finally invoke the semigroup property to get $\Phi_{A_c}(t,s) = \Phi_{A_c}(t,0)\Phi_{A_c}(0,s) = \alpha_{A_c}(t)\Phi_{A_c}^{-1}(s,0) = \alpha_{A_c}(t)\alpha_{A_c}^{-1}(s) = \alpha_{A_c}(t)\beta_{A_c}(s)$.

The preceding discussion produces the following way to check the PE condition (22) for the special case where G has the form from (54) and A_p, B_p, and K have the same period T and σ has period T in t. First, choose a piecewise C^1 period T reference input r. Second, choose the initial state $x_{pr}(0)$ such that $x_{pr}(t)$ and so also $x_r(t)$ will be periodic of period T, by the argument above. Finally, check whether the matrix

$$\sum_{i=1}^{N} \int_0^T (\mathrm{row}_i G(s, x_{pr}(s), r(s)))^\top \mathrm{row}_i G(s, x_{pr}(s), r(s))\mathrm{d}s \tag{62}$$

is positive definite. Positive definiteness of (62) is sufficient for our PE condition to hold. This is a more user friendly way to check for the PE condition, because we do not need to find a positive definite Q such that (22) holds. Our sufficient condition for PE is analogous to the L^2 or other sufficient conditions from [26] for PE for simpler model reference adaptive controls for linear systems without unknown weight functions.

On the other hand, we can also check the PE condition in nonperiodic cases by using numerical methods to check that

$$\inf\left\{\sum_{i=1}^{N}\int_{t-T}^{t} x^\top J_*(s)x\mathrm{d}s : x \in \mathbb{R}^p, |x| = 1, t \geq 0\right\} > 0$$

[4] Invertibility of $I_{n_p\times n_p} - \Phi_{A_c}(T,0)$ follows because if $\Phi_{A_c}(T,0)v = v$ for a $v \in \mathbb{R}^{n_p}\setminus\{0\}$, then since $\Phi_{A_c}(T,0) = \Phi_{A_c}(kT,(k-1)T)$ for all integers $k \geq 1$ (which follows because A_c has period T and from changing variables in the Peano-Baker formula for the fundamental solution from [30, Appendix C.4]), the semigroup property of the fundamental solution gives $\Phi_{A_c}(kT,0)v = v$ for all $k \in \mathbb{N}$, contradicting the uniform global exponential stability of (59) (which would imply that $\lim_{k\to\infty}\Phi_{A_c}(kT,0)v = 0$).

for a pair (x_{pr}, r) consisting of a reference trajectory and reference input for (5) and a constant $T > 0$, where J_* is the integrand in (62), in which case the PE condition is satisfied by $Q = q_0 I_{p\times p}$ where q_0 is the value of the inf on the left side. We illustrate our methods for verifying our PE condition in Section 5 below.

4.4 Application to Frequency-Limited Model Reference Adaptive Control

We next apply Lemma 1 to the model from Section 3.2, by converting W into a vector to identify the weight and control effectiveness matrix entries and choosing $N = 3n$, $q_r = [x_{\mathrm{r}_i}^\top \ 0_{1\times n} \ 0_{1\times n}]^\top$, $q = [e^\top \ e_L^\top \ \tilde{x}^\top]^\top$, $\mathcal{G}_0(q) = [(e+\tilde{x})^\top \ 0_{1\times n} \ 0_{1\times n}]^\top$, and a G that we specify below that will be a function of only t and the first n components of q. We use the fact that in Section 3.2, we have $x_{\mathrm{r}_i} + e + \tilde{x} = x_{\mathrm{r}_i} + x - x_r + x_r - x_{\mathrm{r}_i} = x$. Here we assume that the weight matrix W is constant; see Section 4.5 for time-varying weight matrices. In what follows, we again specify known bounds

$$\underline{W}_{ij} < W_{ij} < \bar{W}_{ij} \text{ for all } i \in \{1,\dots,\mathcal{N}\} \text{ and } j \in \{1,\dots,m\}. \tag{63}$$

for the entries of an unknown aggregated weight matrix W, and (66) represents the augmented error dynamics, including the tracking and parameter estimation error, which agrees with (21) with analogs of the update laws (52) and $\mathcal{N} = s + m$.

Theorem 2. *Let the bounded continuous function $A_r : \mathbb{R} \to \mathbb{R}^{n\times n}$ be such that $\dot{Z}(t) = A_r(t)Z(t)$ is uniformly globally exponentially stable to 0, η and κ be known positive constants, $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}^{\mathcal{N}}$ be C^1 and admit a function $\sigma_* \in \mathcal{K}_\infty$ and a constant $\sigma_{**} \geq 0$ such that $\sup\{\max\{|\nabla\sigma_i(t,z)|, |\sigma(t,z)|\} : t \geq 0\} \leq \sigma_*(|z|) + \sigma_{**}$ for all $i \in \{1,\dots,\mathcal{N}\}$ and $z \in \mathbb{R}^n$, and $x_{\mathrm{r}_i} : \mathbb{R} \to \mathbb{R}^n$ be piecewise C^1 and such that $|x_{\mathrm{r}_i}|_\infty$ and $|\dot{x}_{\mathrm{r}_i}|_\infty$ are finite, where A_r, σ, and x_{r_i} are known. Let $\tilde{W} : \mathbb{R} \to \mathbb{R}^{\mathcal{N}\times m}$ be piecewise C^1, $W \in \mathbb{R}^{\mathcal{N}\times m}$ be unknown, and $\bar{W}_{ij}$ and $\underline{W}_{ij}$ be known constants such that (63) holds. Let $\Lambda = \mathrm{diag}\{\Lambda_1,\dots,\Lambda_m\} \in \mathbb{D}_+^{m\times m}$ be unknown, and let $B : \mathbb{R} \to \mathbb{R}^{n\times m}$ be bounded and C^1. Let $H : \mathbb{R} \to \mathbb{R}^{3n\times 3n}$ and $B^* : \mathbb{R} \to \mathbb{R}^{3n\times m}$ be defined by*

$$H = \begin{bmatrix} A_r - \kappa I_{n\times n} & \kappa I_{n\times n} & 0_{n\times n} \\ \eta I_{n\times n} & A_r - \eta I_{n\times n} & 0_{n\times n} \\ \kappa I_{n\times n} & -\kappa I_{n\times n} & A_r \end{bmatrix} \text{ and } B^* = \begin{bmatrix} B \\ 0_{n\times m} \\ 0_{n\times m} \end{bmatrix}. \tag{64}$$

Define the function $G : \mathbb{R}^{n+1} \to \mathbb{R}^{(3n)\times(m\mathcal{N})}$ and the $m\mathcal{N}$ dimensional vectors θ, $\bar{\theta}$, $\underline{\theta}$, and $\tilde{\theta}$ by

$$\begin{gathered} G(t,z) = -\big[B^*(t)\sigma_1(t,z) \quad B^*(t)\sigma_2(t,z) \quad \dots \quad B^*(t)\sigma_{\mathcal{N}}(t,z)\big], \quad \theta_{\ell+(p-1)m} = W_{p\ell}, \\ \bar{\theta}_{\ell+(p-1)m} = \bar{W}_{p\ell}, \quad \underline{\theta}_{\ell+(p-1)m} = \underline{W}_{p\ell}, \quad \text{and} \quad \tilde{\theta}_{\ell+(p-1)m} = \tilde{W}_{p\ell} \end{gathered} \tag{65}$$

for all $p \in \{1,2,\dots,\mathcal{N}\}$ and $\ell \in \{1,2,\dots,m\}$. Assume there exist a constant $T > 0$ and a positive definite matrix Q such that (22) holds with the G from (65) and $q_r = x_{\mathrm{r}_i}$. Let q_a and q_c denote the first n and last n components of $q \in \mathbb{R}^{3n}$, respectively. Then the system $\dot{Z}(t) = H(t)Z(t)$ is uniformly globally exponentially stable[5] to 0 on $\mathbb{R}^{3n}$ and for any bounded C^1 function $P_H : \mathbb{R} \to \mathbb{R}^{3n\times 3n}$ that admits constants $c_H > 0$ and $d_H > 0$ such that the time derivative of $V_H(t,Z) = Z^\top P_H(t)Z$ satisfies $\dot{V}_H \leq -c_H|Z(t)|^2$ along all solutions of $\dot{Z}(t) = H(t)Z(t)$ for all $t \geq 0$ and such that $V_H(t,Z) \geq d_G|Z|^2$ for all $t \geq 0$ and $Z \in \mathbb{R}^{3n}$, the dynamics

$$\left\{ \begin{array}{rcl} \dot{q}(t) & = & H(t)q(t) - B^*(t)\Lambda\tilde{W}^\top(t)\sigma(t, x_{\mathrm{r}_i}(t) + q_a(t) + q_c(t)) \\ \dot{\tilde{\theta}}_i(t) & = & -2(\bar{\theta}_i - (\tilde{\theta}_i(t)+\theta_i))(\tilde{\theta}_i(t)+\theta_i - \underline{\theta}_i)q^\top(t)P_H(t)\mathrm{col}_i G(t, x_{\mathrm{r}_i}(t) + q_a(t) + q_c(t)), \\ & & i = 1,2,\dots,m\mathcal{N} \end{array} \right. \tag{66}$$

is globally asymptotically stable to 0 on its state space $\mathcal{S}^\sharp = \mathbb{R}^{3n} \times \prod_{i=1}^{m\mathcal{N}}(\underline{\theta}_i - \theta_i, \bar{\theta}_i - \theta_i)$. □

Proof. To check that $\dot{Z}(t) = H(t)Z(t)$ is uniformly globally exponentially stable to 0, we write its state as $Z = (Z_a, Z_b, Z_c)$ with Z_a, Z_b, and Z_c each valued in $\mathbb{R}^n$. Then $\dot{Z}_a - \dot{Z}_b = (A_r(t) - (\kappa+\eta)I_{n\times n})(Z_a - Z_b)$, $\dot{Z}_a = A_r(t)Z_a - \kappa(Z_a - Z_b)$, $\dot{Z}_b = A_r(t)Z_b + \eta(Z_a - Z_b)$, and $\dot{Z}_c = A_r(t)Z_c + \kappa(Z_a - Z_b)$, so $Z_a - Z_b \to 0$ exponentially, so the same is true for Z, because of the uniform global exponential stability of the systems

[5]Notice that we are not introducing a new exponential stability assumption to make our theory work. This is because H from (21) already produces the required exponential stability property, so the required matrix valued function P_H exists by [9, Theorem 4.14] and can be computed using the dynamic extension method that we discussed in Section 4.3 above.

$\dot{X}(t) = A_r(t)X(t)$ and $\dot{X}(t) = (A_r(t) - (\kappa + \eta)I_{n\times n})X(t)$ to zero. Next, note that, for all $z \in \mathbb{R}^n$ and $i = 1, 2, \ldots, 3n$ and $t \geq 0$, we have

$$\mathrm{row}_i\big(B^*(t)\Lambda\tilde{W}^\top\sigma(t,z)\big) = \sum_{\ell=1}^{m}\sum_{p=1}^{\mathcal{N}} B^*_{i\ell}(t)\Lambda_\ell\tilde{W}_{p\ell}\sigma_p(t,z) = -\mathrm{row}_i G(t,z)J\tilde{\theta}(t) \tag{67}$$

where $J = \mathrm{diag}\{J_1, \ldots, J_{m\mathcal{N}}\}$ is defined by $J_{\ell+(p-1)m} = \Lambda_\ell$ for all $p \in \{1, 2, \ldots, \mathcal{N}\}$ and $\ell \in \{1, 2, \ldots, m\}$. To check the second equality in (67), first note that both sides of the second equality in (67) are 0 if $n < i \leq 3n$ since the corresponding rows of B^* and G are zero; while if $1 \leq i \leq n$, then the requirement follows from the calculation we provided in (57). Since $\dot{Z}(t) = H(t)Z(t)$ is uniformly globally exponentially stable to 0, the theorem now follows as a special case of Lemma 1 with $p = m\mathcal{N}$, $N = 3n$, $\mathcal{G}_0(q) = [(q_a + q_c)^\top \ 0_{1\times n} \ 0_{1\times n}]^\top$, $q_r = [x_{r_i}^\top \ 0_{1\times n} \ 0_{1\times n}]^\top$, and G only depending on t and on the first n components of $q \in \mathbb{R}^{3n}$. □

Remark 6. *Section 4.3 on ways to check our PE condition also applies to frequency limited model reference adaptive control, except that instead of applying the Poincaré fixed point argument to x_r as was done in the more basic model reference adaptive control case, the argument would be applied to x_{r_i}. For instance, given a periodic reference input $r(t)$ for the first n_p components x_{pr_i} of the x_{r_i} dynamics, one can specify the initial state $x_{\mathrm{pr}_i}(0)$ to ensure that x_{pr_i} has the same period as $r(t)$. However, we do not require any periodicity in Theorem 2, so our work also applies to cases where neither r nor the vector fields are periodic.* □

4.5 Extension

When Lemma 1 is applied to model reference adaptive control, we choose $\tilde{\theta} = \hat{\theta} - \theta$, where θ is an unknown parameter vector and $\hat{\theta}$ is its estimate. When θ is a constant vector in $\mathbb{R}^p$, we choose the dynamics

$$\dot{\hat{\theta}}_i(t) = -2(\bar{\theta}_i - \hat{\theta}_i(t))(\hat{\theta}_i(t) - \underline{\theta}_i)q^\top(t)P_H(t)\mathrm{col}_i G(t, \mathcal{G}_0(q(t)) + q_r(t)) \tag{68}$$

for the update law for the ith component of θ for each $i \in \{1, 2, \ldots, p\}$ when each component $\hat{\theta}_i$ of the estimator is valued in an interval $(\underline{\theta}_i, \bar{\theta}_i)$ that is known to contain the true θ_i value, and where P_H and G satisfy the requirements from Lemma 1.

However, it may be the case that θ is an unknown time-varying vector that is represented as

$$\theta(t) = \sum_{j=1}^{L} \omega_j b_j(t) \tag{69}$$

with $L \geq 1$ known basis functions $b_j : \mathbb{R} \to \mathbb{R}^p$ and unknown constant weights $\omega_j \in \mathbb{R}$, but with the b_j's not necessarily periodic (e.g., in artificial neural network expansions). Then the dynamics (68) for the update law is no longer valid. However, in this case, we can write the parameter estimator as

$$\hat{\theta}(t) = \sum_{j=1}^{L} \hat{\omega}_j(t) b_j(t) \tag{70}$$

and then the goal is to choose dynamics for the weight estimates $\hat{\omega}_j(t)$ to drive both the parameter estimation error vector $\tilde{\omega}(t) = \hat{\omega}(t) - \omega$ and the state error q to 0. Then we can combine (69)-(70) to obtain

$$\tilde{\theta}(t) = \sum_{j=1}^{L} \tilde{\omega}_j(t) b_j(t) = B_\omega(t)\tilde{\omega}(t), \tag{71}$$

where B_ω has b_j as its jth column for all j. By substituting (71) into the q subsystem of (26), we obtain $\dot{q}(t) = H(t)q(t) + G(t, \mathcal{G}_0(q(t)) + q_r(t))JB_\omega(t)\tilde{\omega}(t)$. If in addition B_ω is C^1 and $J = I_{p\times p}$, then we can apply the method from Lemma 1 with $G(t, \mathcal{G}_0(q(t)) + q_r(t))$ and θ in (26) replaced by $G(t, \mathcal{G}_0(q(t)) + q_r(t))B_\omega(t)$ and ω respectively, and with $\mathrm{row}_i G(s, q_r(s))$ in the PE condition (22) replaced by $\mathrm{row}_i G(s, q_r(s))B_\omega(s)$ for each i, which makes it possible to identify the unknown weights ω_i, for any number L of basis functions. This contrasts with [17, Section 7], where the unknown time-varying parameter was a linear combination of time-varying basis functions with constant weights, but where it was only possible to identify the unknown weights when the number of basis functions was $L = 1$.

4.6 Rates of Exponential Convergence

The strictness of the strict Lyapunov function $\mathcal{V}^\sharp$ in Lemma 1 also allows us to prove exponential convergence of the combined state $(q, \tilde{\theta})$ to 0, and to compute formulas for rates of exponential convergence. We next provide formulas for the exponential decay rates that depend on the selected compact subset $\mathcal{C} \subseteq \mathcal{S}$ of initial conditions (where $\mathcal{S}$ is our state space from Lemma 1) and on the θ_i's and J_i's; see Remark 7 for a way to modify them to make them independent of the θ_i's and J_i's.

For any compact neighborhood $\mathcal{C} \subseteq \mathcal{S}$ of the origin, and along any solution of (26) for any initial state $(q(0), \tilde{\theta}(0)) \in \mathcal{C}$, we can integrate (28) on any interval $[0, t]$ to obtain

$$\mathcal{V}\big(t, q(t), \tilde{\theta}(t)\big) \;\leq\; \mathcal{V}\big(0, q(0), \tilde{\theta}(0)\big) \;\leq\; |\mathcal{V}|_{\mathcal{C}}, \tag{72}$$

where we use $|\mathcal{V}|_{\mathcal{C}}$ to denote the supremum of $\mathcal{V}$ over $[0, \infty) \times \mathcal{C}$. Since the continuity of $\mathcal{V}$ on the compact set $\mathcal{C}$ ensures that the right side of (72) is finite, and since $\inf\{\mathcal{V}(t, q, \tilde{\theta}) : t \geq 0\} \to \infty$ as $(q, \tilde{\theta}) \to \text{boundary}(\mathcal{S})$ or as $|q| \to \infty$, it follows that the sublevel set $L_{\mathcal{C}} = \{(q, \tilde{\theta}) \in \mathcal{S} : \sup_{t \geq 0} \mathcal{V}(t, q, \tilde{\theta}) \leq |\mathcal{V}|_{\mathcal{C}}\}$ is compact and forward invariant for (26). By the compactness of $L_{\mathcal{C}}$ and the continuity of γ_0 from Lemma 1 and the quadratic upper bounds on $\mathcal{V}$ and on the Δ_i's in $(q, \tilde{\theta})$ on $L_{\mathcal{C}}$, we can build a constant $\bar{w} > 0$ such that

$$\bar{w}\mathcal{V}^\sharp(t, q, \tilde{\theta}) \leq c_{H}|q|^2 + \tfrac{\min_i J_i^2}{2T}\lambda_{\min}(Q)|\tilde{\theta}|^2 \tag{73}$$

for all $(t, q, \tilde{\theta}) \in [0, \infty) \times L_{\mathcal{C}}$. Hence, our decay estimate (27) gives $\dot{\mathcal{V}}^\sharp \leq -\bar{w}\mathcal{V}^\sharp(t, q, \tilde{\theta})$ along all solutions of (26) with initial states in $\mathcal{C}$. This gives $\mathcal{V}^\sharp(t, q(t), \tilde{\theta}(t)) \leq e^{-\bar{w}t}\mathcal{V}^\sharp(0, q(0), \tilde{\theta}(0))$, hence the estimate

$$|(q(t), \tilde{\theta}(t))| \leq e^{-\bar{w}t/2}\sqrt{\frac{c_{H}}{\underline{v}\bar{w}}|q(0)|^2 + \frac{\lambda_{\min}(Q)\min_i J_i^2}{2T\underline{v}\bar{w}}|\tilde{\theta}(0)|^2} \tag{74}$$

for all $t \geq 0$ along all solutions of (26) starting in $\mathcal{C}$, by our quadratic lower bound $\underline{v}|(q, \tilde{\theta})|^2$ on $\mathcal{V}^\sharp$ from Lemma 1 and (73). When $q(0) = 0$, it gives an exponential convergence rate of $\tilde{\theta}$ to 0. This proves the following, where T satisfies our PE condition from Lemma 1 as before:

Proposition 2. *Let the requirements from Lemma 1 hold. Then for each compact subset $\mathcal{C}$ of $\mathcal{S}$, the constant*

$$\bar{w} = \inf\left\{\frac{c_{H}|q|^2 + \frac{\min_i J_i^2}{2T}\lambda_{\min}(Q)|\tilde{\theta}|^2}{\mathcal{V}^\sharp(t, q, \tilde{\theta})} : t \geq 0, (q, \tilde{\theta}) \in L_{\mathcal{C}} \setminus \{0\}\right\} \tag{75}$$

is such that the exponential convergence estimate (74) holds for all solutions of (26) with initial states $(q(0), \tilde{\theta}(0)) \in \mathcal{C}$ and all $t \geq 0$, where $L_{\mathcal{C}} = \{(q, \tilde{\theta}) \in \mathcal{S} : \sup_{t \geq 0}\mathcal{V}(t, q, \tilde{\theta}) \leq |\mathcal{V}|_{\mathcal{C}}\}$. □

For each choice of $\mathcal{C}$, the right side of (75) is a positive constant, because of the positive definite quadratic upper bound for $\mathcal{V}^\sharp$ on $L_{\mathcal{C}}$ in $(q, \tilde{\theta})$ (which ensures that the set in curly braces in (75) has a positive lower bound) and $\bar{w}$ can be computed by numerical methods, using the formula (25) for $\mathcal{V}^\sharp$ and the formulas from Remark 4. We can also obtain formulas for positive lower bounds on the right side of (75), using:

Proposition 3. *Let the requirements from Lemma 1 hold and $\mathcal{C}$ be any compact subset of $\mathcal{S}$. Let $L_{\mathcal{C}} = \{(q, \tilde{\theta}) \in \mathcal{S} : \sup_{t \geq 0}\mathcal{V}(t, q, \tilde{\theta}) \leq |\mathcal{V}|_{\mathcal{C}}\}$ and $|\mathcal{V}|_{\mathcal{C}} = \sup\{\mathcal{V}(t, q, \tilde{\theta}) : t \geq 0, (q, \tilde{\theta}) \in \mathcal{C}\}$. Then, with the choices of P_{H}, G, γ_0, and c_{H} from Lemma 1, we can construct constants $\mu_i \in (0, 1)$ such that (i) the inequalities*

$$\mu_i(\underline{\theta}_i - \theta_i) \leq \tilde{\theta}_i \leq \mu_i(\bar{\theta}_i - \theta_i), \;\; 1 \leq i \leq p \tag{76}$$

hold for all $\tilde{\theta}$ such that there exists a q such that $(q, \tilde{\theta}) \in L_{\mathcal{C}}$ and such that (ii) with the choices

$$\begin{aligned} w_* \;&=\; \max\{\bar{w}_a, \bar{w}_b\}, \; \textit{where } \bar{w}_a = \gamma_0(|\mathcal{V}|_{\mathcal{C}})|P_{H}|_\infty + \tfrac{N|J|}{2}\max_i|\text{row}_i G(t, q_r(t))|_\infty \textit{ and} \\ \bar{w}_b \;&=\; \gamma_0(|\mathcal{V}|_{\mathcal{C}})\max_i J_i\tfrac{1}{2(1-\mu_i)^2(\bar{\theta}_i - \theta_i)(\theta_i - \underline{\theta}_i)} \\ &\quad + \tfrac{N}{2}\big(T|J|^2\max_i|\text{row}_i G(t, q_r(t))|_\infty^2 + |J|\max_i|\text{row}_i G(t, q_r(t))|_\infty\big), \end{aligned} \tag{77}$$

we have

$$\mathcal{V}^\sharp(t, q, \tilde{\theta}) \leq \bar{w}_a|q|^2 + \bar{w}_b|\tilde{\theta}|^2 \;\textit{ for all } t \geq 0 \textit{ and } (q, \tilde{\theta}) \in L_{\mathcal{C}}, \tag{78}$$

and the constant $\bar{w}$ from (75) satisfies $\bar{w} \geq \frac{1}{w_*}\min\{c_{H}, \frac{1}{2T}\lambda_{\min}(Q)\min_i J_i^2\}$. □

Proof. First choose any constants $\mu_i \in (0,1)$, and notice that $\underline{\theta}_i - \theta_i < \tilde{\theta}_i < \bar{\theta}_i - \theta_i$ hold for all i and all $\tilde{\theta}$ such that $(q, \tilde{\theta}) \in L_{\mathcal{C}}$, by our definition of $\mathcal{S}$ from (23). If there exists an i and a pair $(q, \tilde{\theta}) \in L_{\mathcal{C}}$ such that

$$\bar{\theta}_i - \theta_i > \tilde{\theta}_i > \mu_i(\bar{\theta}_i - \theta_i), \tag{79}$$

then (72), the formula for $\mathcal{V}$, and a partial fraction decomposition yield a constant $\chi_i < |\mathcal{V}|_{\mathcal{C}}$ such that

$$\begin{aligned} |\mathcal{V}|_{\mathcal{C}} &\geq \int_0^{\tilde{\theta}_i} \frac{J_i \ell}{(\bar{\theta}_i - (\ell + \theta_i))(\ell + \theta_i - \underline{\theta}_i)} \mathrm{d}\ell = \frac{J_i}{\bar{\theta}_i - \underline{\theta}_i} \left\{ (\underline{\theta}_i - \theta_i) \ln\left(\frac{\tilde{\theta}_i + \theta_i - \underline{\theta}_i}{\theta_i - \underline{\theta}_i}\right) + (\theta_i - \bar{\theta}_i) \ln\left(\frac{\bar{\theta}_i - \theta_i - \tilde{\theta}_i}{\bar{\theta}_i - \theta_i}\right) \right\} \\ &> \frac{J_i}{\bar{\theta}_i - \underline{\theta}_i} \left\{ (\underline{\theta}_i - \theta_i) \ln\left(\frac{\bar{\theta}_i - \underline{\theta}_i}{\theta_i - \underline{\theta}_i}\right) + (\theta_i - \bar{\theta}_i) \ln(1 - \mu_i) \right\} = \chi_i + \frac{J_i}{\bar{\theta}_i - \underline{\theta}_i} (\theta_i - \bar{\theta}_i) \ln(1 - \mu_i), \end{aligned} \tag{80}$$

where χ_i depends on J_i, $\bar{\theta}_i$, $\underline{\theta}_i$, and θ_i, and we used the fact that for all i and all $\tilde{\theta}_i \in (\underline{\theta}_i - \theta_i, \bar{\theta}_i - \theta_i)$, the denominator of the ith integrand in the formula for $\mathcal{V}$ is positive. The last inequality (80) would imply that

$$\mu_i < 1 - \exp\left(\frac{(|\mathcal{V}|_{\mathcal{C}} - \chi_i)(\bar{\theta}_i - \underline{\theta}_i)}{J_i(\theta_i - \bar{\theta}_i)}\right), \tag{81}$$

by subtracting χ_i from both sides of (80) and using the fact that $\theta_i - \bar{\theta}_i < 0$ to reverse the direction of the inequality that is obtained from (80). On the other hand, if there exists an i and a pair $(q, \tilde{\theta}) \in L_{\mathcal{C}}$ such that $\underline{\theta}_i - \theta_i < \tilde{\theta}_i < \mu_i(\underline{\theta}_i - \theta_i)$, then the same partial fraction decomposition that we used in (80) gives

$$|\mathcal{V}|_{\mathcal{C}} > \frac{J_i}{\bar{\theta}_i - \underline{\theta}_i} \left\{ (\underline{\theta}_i - \theta_i) \ln(1 - \mu_i) + (\theta_i - \bar{\theta}_i) \ln\left(\frac{\bar{\theta}_i - \underline{\theta}_i}{\bar{\theta}_i - \theta_i}\right) \right\} = \mathcal{Y}_i + \frac{J_i}{\bar{\theta}_i - \underline{\theta}_i} (\underline{\theta}_i - \theta_i) \ln(1 - \mu_i), \tag{82}$$

where the constants $\mathcal{Y}_i < |\mathcal{V}|_{\mathcal{C}}$ also depend on J_i, $\bar{\theta}_i$, $\underline{\theta}_i$, and θ_i, and (82) would require that

$$\mu_i < 1 - \exp\left(\frac{(|\mathcal{V}|_{\mathcal{C}} - \mathcal{Y}_i)(\bar{\theta}_i - \underline{\theta}_i)}{J_i(\underline{\theta}_i - \theta_i)}\right). \tag{83}$$

Hence, our constants $\mu_i \in (0,1)$ will satisfy our requirements of part (i) of the proposition if we choose

$$\mu_i = \max\{b_i, c_i\}, \text{ where } b_i = 1 - \exp\left(\frac{(|\mathcal{V}|_{\mathcal{C}} - \chi_i)(\bar{\theta}_i - \underline{\theta}_i)}{J_i(\theta_i - \bar{\theta}_i)}\right) \text{ and } c_i = 1 - \exp\left(\frac{(|\mathcal{V}|_{\mathcal{C}} - \mathcal{Y}_i)(\bar{\theta}_i - \underline{\theta}_i)}{J_i(\underline{\theta}_i - \theta_i)}\right). \tag{84}$$

For each $\tilde{\theta}$ that admits a point q such that $(q, \tilde{\theta}) \in L_{\mathcal{C}}$, and for each index i, we have $-\mu_i(\theta_i - \underline{\theta}_i) \leq \ell \leq \mu_i(\bar{\theta}_i - \theta_i)$ for all ℓ such that $\ell \in [0, \tilde{\theta}_i]$ when $\tilde{\theta}_i \geq 0$, and for all $\ell \in [\tilde{\theta}_i, 0]$ when $\tilde{\theta}_i < 0$ (because $-\mu_i(\theta_i - \underline{\theta}_i) < 0 < \mu_i(\bar{\theta}_i - \theta_i)$ for all i). This provides the lower bound $(1 - \mu_i)^2(\bar{\theta}_i - \theta_i)(\theta_i - \underline{\theta}_i)$ for the denominator in the ith integrand in the formula (23) for $\mathcal{V}$. Hence, our formula for $\mathcal{V}$ from Lemma 1 gives

$$\begin{aligned} &\mathcal{V}(t, q, \tilde{\theta}) \leq |P_H|_\infty |q|^2 + \max_i J_i \frac{1}{2(1 - \mu_i)^2(\bar{\theta}_i - \theta_i)(\theta_i - \underline{\theta}_i)} |\tilde{\theta}|^2 \\ &\text{and } |\Delta_i(t, q, \tilde{\theta})| \leq \tfrac{T}{2} |J\tilde{\theta}|^2 |\mathrm{row}_i G(t, q_r(t))|_\infty^2 + \tfrac{|J|}{2} |\mathrm{row}_i G(t, q_r(t))|_\infty |(q, \tilde{\theta})|^2 \end{aligned} \tag{85}$$

on $[0, \infty) \times L_{\mathcal{C}}$, where we used the inequalities $|q_i||\tilde{\theta}| \leq 0.5(q_i^2 + |\tilde{\theta}|^2)$ to upper bound $|\Delta_i|$ for all i. Therefore, the bound (78) follows from (72) and the structure (25) of the function $\mathcal{V}^\sharp$, so the last conclusion of the proposition follows from the structure of the right side of the formula (75) for $\bar{w}$. □

Remark 7. *The preceding choices of the μ_i's and $\mathcal{V}$'s formula imply that our $\bar{w}_a$ and $\bar{w}_b$ formulas depend on the θ_i's and J_i's. However, given any known constants $\bar{\theta}_a$ and $\underline{\theta}_a$ and any positive constants $\underline{J}$ and $\bar{J}$ such that $\underline{\theta}_i < \underline{\theta}_a \leq \theta_i \leq \bar{\theta}_a < \bar{\theta}_i$ and $\underline{J} \leq J_i \leq \bar{J}$ for all i, we can minimize the formulas for $\bar{w}$ over all values $\theta_i \in [\underline{\theta}_a, \bar{\theta}_a]$ and over all values $J_i \in [\underline{J}, \bar{J}]$. This provides lower bounds for $\bar{w}$ that are independent of the θ_i's and J_i's, hence estimates of the exponential decay rates that are independent of the θ_i's and J_i's.* □

4.7 Robustness

Another advantage of the strict Lyapunov function from Lemma 1 is that it can be used to prove robustness properties that do not follow from using nonstrict or weak Lyapunov functions. For instance, we can prove the following integral input-to-state stable (or integral ISS) result that generalizes the integral ISS results from [23, Section 4.5] that were confined to a model of a brushless DC motor turning a mechanical load (but see Proposition 4 for an alternative ISS result that ensures boundedness of solutions under bounded perturbations and under additional assumptions); see [1] for background on integral ISS and ISS.

Corollary 1. *If the assumptions of Lemma 1 are satisfied, then with the notation from Lemma 1, we can construct functions $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}_\infty$, and $\beta \in \mathcal{KL}$ such that for all measurable essentially bounded functions $\delta : [0,\infty) \to \mathbb{R}^N$, the following is true: All solutions $(q, \tilde{\theta}) : [0,\infty) \to \mathcal{S}$ of the dynamics*

$$\left\{ \begin{array}{rcl} \dot{q}(t) & = & H(t)q(t) + G(t, \mathcal{G}_0(q(t)) + q_r(t))J\tilde{\theta}(t) + \delta(t) \\ \dot{\tilde{\theta}}_i(t) & = & -2(\bar{\theta}_i - (\tilde{\theta}_i(t) + \theta_i))(\tilde{\theta}_i(t) + \theta_i - \underline{\theta}_i)q^\top(t)P_H(t)\,\mathrm{col}_i G(t, \mathcal{G}_0(q(t)) + q_r(t)), \quad 1 \le i \le p \end{array} \right. \tag{86}$$

on its state space $\mathcal{S} = \mathbb{R}^N \times \prod_{i=1}^{p} (\underline{\theta}_i - \theta_i, \bar{\theta}_i - \theta_i)$ satisfy

$$\alpha(|(q(t), \tilde{\theta}(t))|) \ \le\ \beta\big(\mathcal{L}(q(0), \tilde{\theta}(0)), t\big) + \textstyle\int_0^t \gamma(|\delta(\ell)|)\mathrm{d}\ell \tag{87}$$

for all $t \ge 0$, where $\mathcal{L}$ is the modulus (24) from the conclusions of Lemma 1. □

Proof. We use the functions $\mathcal{V}$ and $\mathcal{V}^\sharp$ and other notation from Lemma 1. First note that since $G(t, q_r(t))$ and $\tilde{\theta}$ are bounded along all solutions of (86) in $\mathcal{S}$, we can find a constant $\mathcal{L}_* > 0$ and enlarge the positive valued increasing function γ_0 from the formula (25) for $\mathcal{V}^\sharp$ so that

$$\mathcal{L}_*\gamma_0(\mathcal{V}(t, q, \tilde{\theta})) \ \ge\ \sum_{i=1}^{N} |\mathrm{row}_i G(t, q_r(t))|_\infty |J\tilde{\theta}| \quad \text{and} \tag{88}$$

$$\mathcal{V}^\sharp(t, q, \tilde{\theta}) \ \ge\ \tfrac{1}{2}\textstyle\int_0^{\mathcal{V}(t,q,\tilde{\theta})} \gamma_0(\ell)\mathrm{d}\ell \tag{89}$$

hold for all $(q, \tilde{\theta}) \in \mathcal{S}$ and $t \ge 0$, using the boundedness of the right side of (88), the positive definite quadratic lower bound in $(q, \tilde{\theta})$ on $\mathcal{V}$, and the quadratic bounds on the functions Δ_i in the variable $(q, \tilde{\theta})$. This can be done by adding a large enough positive constant to the formula for γ_0 from the proof of Lemma 1 without relabeling (and without changing the control design) in order to satisfy (89), where the added constant will depend on the choices of q_r and is needed because γ_0 was not constructed to take the requirement (89) into account. Also, along all solutions of (86) in $\mathcal{S}$, we can use the triangle inequality to obtain

$$2|q(t)||P_H(t)||\delta(t)| \ =\ \left\{\sqrt{c_H}|q(t)|\right\}\left\{\tfrac{2|P_H(t)||\delta(t)|}{\sqrt{c_H}}\right\} \ \le\ \tfrac{1}{2}c_H|q(t)|^2 + 2\tfrac{|P_H(t)|^2|\delta(t)|^2}{c_H} \tag{90}$$

and so also

$$\dot{\mathcal{V}} \ \le\ -c_H|q(t)|^2 + 2|q(t)||P_H(t)||\delta(t)| \ \le\ -\tfrac{c_H}{2}|q(t)|^2 + \tfrac{2|P_H(t)|^2}{c_H}|\delta(t)|^2 \tag{91}$$

for all $t \ge 0$. The first inequality in (91) can be obtained from the calculations that gave the decay condition (28) in the unperturbed case where $\delta = 0$, combined with the fact that δ is added to the right side of $\dot{q}$ in the perturbed dynamics (86). It follows from our decay estimate on $\mathcal{V}^\sharp$ from Lemma 1 (with c_H replaced by $c_H/2$) that along all solutions of (86) in $\mathcal{S}$, the choice $\lambda_0 = \min\{\frac{1}{2}c_H, \frac{1}{2T}\min_j J_j^2 \lambda_{\min}(Q)\}$ gives

$$\begin{array}{rcl} \dot{\mathcal{V}}^\sharp & \le & -\lambda_0|(q(t), \tilde{\theta}(t))|^2 + \gamma_0(\mathcal{V}(t, q(t), \tilde{\theta}(t)))\,\frac{2|P_H(t)|^2|\delta(t)|^2}{c_H} + \sum_{i=1}^{N} |\mathrm{row}_i G(t, q_r(t))|_\infty |J\tilde{\theta}(t)||\delta(t)| \\ & \le & -\lambda_0|(q(t), \tilde{\theta}(t))|^2 + \gamma_0(\mathcal{V}(t, q(t), \tilde{\theta}(t)))\left[\frac{2|P_H(t)|^2|\delta(t)|^2}{c_H} + \mathcal{L}_*|\delta(t)|\right], \end{array} \tag{92}$$

where the second inequality in (92) followed from (88).

Next, we use the preceding bounds and decay estimates to build a useful decay condition on the function $\mathcal{V}^{\sharp\sharp} = \mathcal{H}(V^\sharp)$ where the function $\mathcal{H} \in \mathcal{K}_\infty$ is defined by

$$\mathcal{H}(\ell) = \mathcal{M}_0^{-1}(2\ell)\ , \ \text{where } \mathcal{M}_0(r) = \int_0^r \gamma_0(\ell)\mathrm{d}\ell. \tag{93}$$

To this end, first note that our lower bound in (89) gives $2V^\sharp(t, q, \tilde{\theta}) \ge \mathcal{M}_0(\mathcal{V}(t, q, \tilde{\theta}))$ and therefore also

$$\mathcal{H}'(\mathcal{V}^\sharp) = \frac{2}{\mathcal{M}_0'(\mathcal{M}_0^{-1}(2\mathcal{V}^\sharp))} \le \frac{2}{\mathcal{M}_0'(\mathcal{V})} = \frac{2}{\gamma_0(\mathcal{V})} \tag{94}$$

along all solutions of (86) in $\mathcal{S}$, where we used the fact that $\mathcal{M}_0' = \gamma_0$ is increasing and positive valued. Combining (92)-(94), it follows that along all solutions of (86) in $\mathcal{S}$, we have

$$\dot{\mathcal{V}}^{\sharp\sharp} \quad \leq \quad -\lambda_0|(q(t),\tilde{\theta}(t))|^2\mathcal{H}'(\mathcal{V}^{\sharp}(t,q(t),\tilde{\theta}(t))) + \gamma_*(|\delta(t)|), \tag{95}$$

where the function $\gamma_* \in \mathcal{K}_\infty$ is defined by $\gamma_*(\ell) = 2(2|P_H|_\infty^2\ell^2/c_H + \mathcal{L}_*\ell)$. By separately considering points in $\mathcal{S}$ that are close to or far from the origin and using the positive definite quadratic lower bound $\underline{v}|(q,\tilde{\theta})|^2$ for $\mathcal{V}^\sharp$, we can construct a continuous positive definite function $\rho : [0,\infty) \to [0,\infty)$ such that

$$\rho(\mathcal{V}^{\sharp\sharp}(t,q,\tilde{\theta})) \leq \lambda_0|(q,\tilde{\theta})|^2\mathcal{H}'(\mathcal{V}^\sharp(t,q,\tilde{\theta})) \tag{96}$$

for all $(q,\tilde{\theta}) \in \mathcal{S}$, which gives the decay estimate

$$\dot{\mathcal{V}}^{\sharp\sharp} \quad \leq \quad -\rho(\mathcal{V}^{\sharp\sharp}(t,q(t),\tilde{\theta}(t))) + \gamma_*(|\delta(t)|) \tag{97}$$

along all solutions of (86) in $\mathcal{S}$. Then standard integral ISS arguments [1] provide functions $\alpha_0 \in \mathcal{K}_\infty$, $\beta_0 \in \mathcal{KL}$, and $\gamma \in \mathcal{K}_\infty$ such that

$$\alpha_0\big(\mathcal{V}^{\sharp\sharp}(t,q(t),\tilde{\theta}(t))\big) \leq \beta_0\big(\mathcal{V}^{\sharp\sharp}(0,q(0),\tilde{\theta}(0)),t\big) + \textstyle\int_0^t \gamma(|\delta(\ell)|)\mathrm{d}\ell \tag{98}$$

along all solutions of (86) in $\mathcal{S}$. Hence, the corollary follows by choosing $\alpha(\ell) = \alpha_0(\mathcal{H}(\underline{v}\ell^2))$ and $\beta(s,t) = \beta_0(\mathcal{H}(s),t)$ in the integral ISS condition (87), where $\underline{v}$ is the positive constant from Lemma 1. $\square$

A notable feature of Corollary 1 is that it does not place any additional conditions on G (beyond what was already assumed in Lemma 1) or on the lengths of the intervals $(\underline{\theta}_i, \bar{\theta}_i)$ that are known to contain the unknown parameter values θ_i. However, the integral ISS property (87) from Corollary 1 allows cases where bounded uncertainties $\delta(t)$ can produce an unbounded $q(t)$. This motivates the following proposition, which provides sufficient conditions under which all solutions of (86) are bounded over $[0,\infty)$ when δ is bounded.

Proposition 4. *Let the assumptions of Lemma 1 hold. Assume that $\mathcal{G}_0$ admits the global Lipschitz constant $\bar{\mathcal{G}}_0$, and that there is a constant $\bar{G} > 0$ such that $|G(t,q_1) - G(t,q_2)| \leq \bar{G}|q_1 - q_2|$ holds for all $t \geq 0$, $q_1 \in \mathbb{R}^N$, and $q_2 \in \mathbb{R}^N$, and that with the notation of Lemma 1, we have*

$$\bar{G}\bar{\mathcal{G}}_0|P_H|_\infty|J||\theta_d| < \frac{c_H}{2}, \tag{99}$$

where $\theta_d = [\bar{\theta}_1 - \underline{\theta}_1 \ \ \ldots \ \ \bar{\theta}_p - \underline{\theta}_p]^\top \in \mathbb{R}^p$. Let $\delta : [0,\infty) \to \mathbb{R}^N$ be a measurable locally essentially bounded function. Set

$$\begin{array}{l} \epsilon = c_H - 2\bar{G}\bar{\mathcal{G}}_0|P_H|_\infty|J||\theta_d|, \ c = \frac{\epsilon}{2|P_H|_\infty}, \ \text{and} \\ d(t) = \frac{1}{2\epsilon}\left(2|P_H|_\infty((\bar{G}|q_r|_\infty + \sup_{t\geq 0}|G(t,0)|)|J||\theta_d| + |\delta|_{[0,t]})\right)^2 \end{array} \tag{100}$$

for all $t \geq 0$. Then, in terms of the notation from Lemma 1, the inequality

$$|q(t)| \leq e^{-ct/2}\sqrt{\frac{|P_H|_\infty}{d_H}}|q(0)| + \sqrt{\frac{d(t)}{cd_H}} \tag{101}$$

holds along all solutions $(q,\tilde{\theta}) : [0,\infty) \to \mathcal{S}$ of (86) for all $t \geq 0$. $\square$

Proof. Along all solutions of (86) for all $t \geq 0$, we have $|\tilde{\theta}(t)| \leq |\theta_d|$ because the structure of the $\tilde{\theta}$ dynamics in (86) ensures that each $\tilde{\theta}_i(t)$ stays in the interval $(\underline{\theta}_i - \theta_i, \bar{\theta}_i - \theta_i)$ for each initial state $(q(0),\tilde{\theta}(0)) \in \mathcal{S}$ for (86) and because $\theta_i \in (\underline{\theta}_i, \bar{\theta}_i)$ for each i. Also, along all solutions of (86) in $\mathcal{S}$, the triangle inequality gives $|G(t,\mathcal{G}_0(q(t)) + q_r(t))| - |G(t,0)| \leq |G(t,\mathcal{G}_0(q(t)) + q_r(t)) - G(t,0)| \leq \bar{G}(|\mathcal{G}_0(q(t))| + |q_r(t)|) \leq \bar{G}(\bar{\mathcal{G}}_0|q(t)| + |q_r(t)|)$ for all $t \geq 0$. Hence, along all solutions of (86) in its state space $\mathcal{S}$ for our fixed choice of δ, the function $V_P(t,q) = q^\top P_H(t)q$ satisfies

$$\begin{array}{rcl} \dot{V}_P & \leq & -c_H|q(t)|^2 + 2q^\top(t)P_H(t)G(t,\mathcal{G}_0(q(t)) + q_r(t))J\tilde{\theta}(t) + 2q^\top(t)P_H(t)\delta(t) \\ & \leq & -c_H|q(t)|^2 + 2|q(t)||P_H(t)|(\bar{G}\bar{\mathcal{G}}_0|q(t)| + \bar{G}|q_r|_\infty + |G(t,0)|)|J||\theta_d| + 2|q(t)||P_H(t)||\delta|_{[0,t]} \\ & = & -\epsilon|q(t)|^2 + \{|q(t)|\}\{2|P_H(t)|((\bar{G}|q_r|_\infty + |G(t,0)|)|J||\theta_d| + |\delta|_{[0,t]})\} \\ & \leq & -\frac{\epsilon}{2}|q(t)|^2 + d(t) \ \ \leq \ \ -cV_P(t,q(t)) + d(t) \end{array} \tag{102}$$

for all $t \geq 0$, where the third inequality used Young's inequality $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ with a and b being the terms in curly braces in (102). Applying an integrating factor to the last inequality from (102) gives $d_H|q(t)|^2 \leq V_P(t, q(t)) \leq e^{-ct}V_P(0, q(0)) + d(t)/c \leq e^{-ct}|P_H|_\infty|q(0)|^2 + d(t)/c$. Then (101) follows by dividing the previous inequalities by d_H and then using the subadditivity of the square root. □

Remark 8. *Condition (101) is an ISS property for the q dynamics, with (θ_d, δ) playing the role of the uncertainty in the ISS condition. An essential ingredient used in Proposition 4 is that the known intervals $(\underline{\theta}_i, \bar{\theta}_i)$ containing the unknown components θ_i of the parameter vector are not required to be symmetric intervals $(-\bar{\theta}_i, \bar{\theta}_i)$ around 0. In [16], these intervals around the θ_i's were required to be symmetric around 0. However, had we required the intervals to be symmetric around 0 here, then we would have had $\underline{\theta}_i = -\bar{\theta}_i$ for each i, and then our smallness condition (99) on the norm $|\theta_d|$ of the difference vector θ_d would have become a requirement that the θ_i's are sufficiently small. By contrast, there is no smallness requirement on the θ_i's in Proposition 4. This is one motivation for our using barrier terms in our parameter update laws that do not require symmetric known intervals around the unknown parameter components.* □

5 Illustrative Numerical Example

To illustrate the value of our methods, consider the uncertain pitch dynamics of a helicopter during hover flight [10, Example 9.1], which produces the system

$$\dot{q} = M_q(t)q + M_\delta\left(u + \theta \tanh\left(\tfrac{360}{\pi}q\right)\right), \tag{103}$$

where $M_q(t) = -0.61 + \Delta_q(t)$, the bounded continuous function Δ_q is assumed to be known, and M_δ (which represents the elevator effectiveness) and θ are unknown constants but M_δ is known to be negative. The work [10] was confined to the case of a constant vehicle pitch damping M_q, but our more general choice of M_q will illustrate our ability to cover time-varying dynamics. We apply both Theorem 1 based on more basic model reference adaptive control and Theorem 2 based on frequency limited model reference adaptive control. We assume that the true values are $M_\delta = -6.65$ and $\theta = -0.01$. If we pick $x_p = q$, $A_p = M_q$, $B_p = -1$, $\Lambda = -M_\delta$, $W_p = -M_\delta\theta$, and $\sigma_p(x_p) = \tanh\left(\frac{360}{\pi}x_p\right)$, then (103) can be written in the form (1) with the structure (2) of δ_p. Using our notation from Section 3.1, we choose $E_r = 0$ and $E_p = 1$, which produce the integral state dynamics $\dot{x}_c = x_p - c$, to obtain the dynamics (7). For our simulation, we choose

$$r(t) = \tfrac{\pi}{18}\sin\left(\tfrac{2\pi t}{T}\right)\ \text{rad} \tag{104}$$

with period $T = 25$ (but see below for an example where $r(t)$ is discontinuous). By using linear quadratic regulator theory, $K(t) = [K_p(t)\ \ K_r(t)]$ is set to $[1.2263 + \Delta_q(t)\ \ 1]$. It produces the constant Hurwitz matrix

$$A_r(t) = \begin{bmatrix} A_p + B_pK_p & B_p \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1.8363 & -1 \\ 1 & 0 \end{bmatrix}. \tag{105}$$

We solve the Lyapunov equation $A_r^\top P_A + P_A A_r + R = 0$ with $R = I_{2\times 2}$ to obtain the positive definite matrix

$$P_A = \begin{bmatrix} 0.5446 & 0.5 \\ 0.5 & 1.4627 \end{bmatrix} \tag{106}$$

that is required by our parameter update law (52). We select the known bounds $\bar{\theta}_1 = 0.5$, $\underline{\theta}_1 = -0.5$, $\bar{\theta}_2 = 20$, and $\underline{\theta}_2 = -15$ for the unknown ideal parameters. For frequency limited model reference adaptive control, η and κ are set to 2 and 1, respectively.

To apply Proposition 1, we must guarantee that the solution of the reference model in (12) is periodic, so we calculate the initial state of the reference trajectory as in Section 4.3. For simplicity, we choose Δ_q in the M_q formula to be 0, but analogous reasoning applies for any bounded continuous period T choice of Δ_q. For our choice $r(t)$ above and the basic model reference adaptive control, this produces $x_r(0) = [x_{pr}(0)\ \ r(0)]^\top = [0.0127694\ \ 0]^\top$, and for this initial state, the reference solution $x_r(t)$ in radians has period $T = 25$. If we now choose $j = 1$ and $q_r = x_r$, then one can check that the PE condition (22) is satisfied, since the matrix (42) from Proposition 1 (with G as defined in (54) and B and σ as defined in Section 3.1, so the integrand in the

PE condition is $\sigma(t, x_r(t))^\top \sigma(t, x_r(t))$ with the choice $\sigma(t, x_r(t)) = [\sigma_p(x_{pr}(t)) \quad -1.2263x_{pr}(t) - r(t)])$ has the eigenvalues 24.2268 and 0.0158414. Therefore, Theorem 1 ensures tracking and parameter identification for (103). This is demonstrated in Figure 2. Similarly, using Remark 6, Theorem 2 also guarantees tracking and parameter identification for (103). This is presented in Figure 3. As seen in Figures 2-3, both controls achieve tracking and parameter identification performance. However, the control has approximately 5% less oscillation in Figure 3 owing to the frequency limited approach.

Note that a time-varying choice of Δ_q calls for a time-varying choice of K_p, but the unknown (aggregated) weight matrix $W^\top$ will be constant even if M_q is time varying. Moreover, although r is a component of x_r, we only require x_r to be piecewise C^1, which allows discontinuous choices of r. For instance, if we replace the reference input (104) by $r(t) = (\pi/18)J_a(t)\sin(2\pi t/T)$ in the previous example, where J_a is the period 5 function taking the value 1 on $[0, 2.5)$ and -1 on $[2.5, 5)$ and keep all of the other model parameter the same, then the new $x_{pr}(0)$ that is required to produce a period $T = 25$ reference trajectory is $x_{pr}(0) = 0.011466$, and our PE condition is again satisfied, since the matrix (42) from Proposition 1 with $j = 1$ has the eigenvalues 24.8205 and 0.160102 and therefore is positive definite.

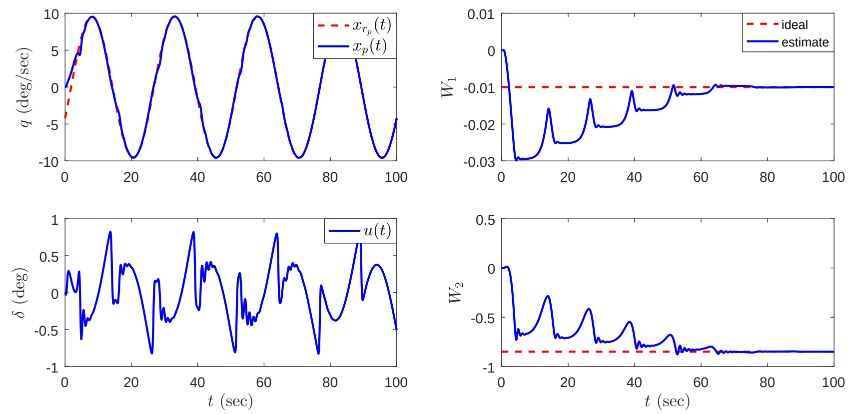

Figure 2: Pitch Rate Tracking and Parameter Estimation Performance Based on Theorem 1. x_{r_p} is First Component of Reference Trajectory x_r.

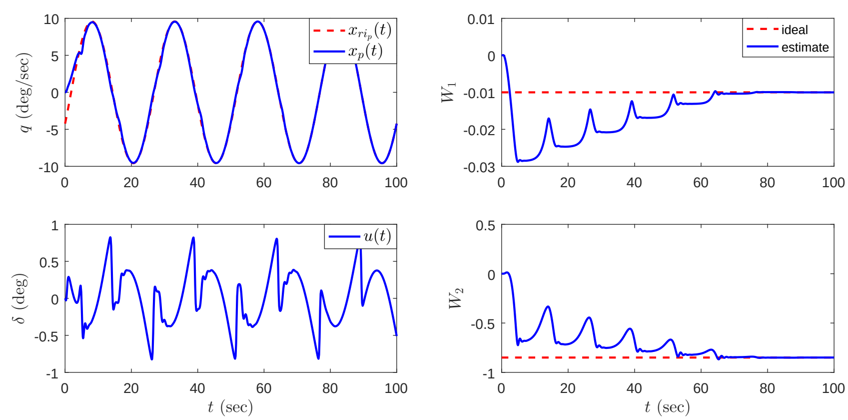

Figure 3: Pitch Rate Tracking and Parameter Estimation Performance Based on Theorem 2. x_{ri_p} is First component of x_{r_i}.

Although the preceding example produced a scalar valued dynamics, the results of our paper apply in arbitrary dimensions, and therefore can be applied to higher-order systems. For instance, consider the generic delta wing rock dynamic model

$$\left\{ \begin{array}{rcl} \dot{\varphi} & = & p \\ \dot{p} & = & \theta_1\varphi + \theta_2 p + (\theta_3|\varphi| + \theta_4|p|)p + \theta_5\varphi^3 + \theta_6\delta_a \end{array} \right. \tag{107}$$

from [10, Section 9.5] where φ is the aircraft roll angle (rad), p is the roll rate (rad/s), δ_a denotes the differential aileron (rad, which is the control input), and the θ_i's are parameters. The system (107) can be written in the form (1)-(2) with the choices $n_p = 2$, $m = 1$, $s = 3$, $\Lambda = \theta_6$, $u = \delta_a$,

$$x_p = \begin{bmatrix} \varphi \\ p \end{bmatrix},\ A_p = \begin{bmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ W_p^\top = \begin{bmatrix}\theta_3 & \theta_4 & \theta_5\end{bmatrix},\ \text{and } \sigma_p(x_p) = \begin{bmatrix}|\varphi|p & |p|p & \varphi^3\end{bmatrix}^\top. \quad (108)$$

Then with the preceding choices, and with the choices $\theta_1 = -0.018$ and $\theta_2 = 0.015$ from [10, Section 9.5], and with $K_p = [-\theta_1 - 1 \quad -\theta_2 - 1]$, $K_r = 1$, $n_c = 1$, $A_c = A_p + B_p K_p$, $T = 25$, and the reference input $r(t) = (\pi/18) J_a(t) \sin(2\pi t/T)$ with J_a as defined in the preceding paragraph, we can choose the initial state $x_r(0)$ according to the formula (61) to obtain a period T reference trajectory $x_r = [x_{pr}^\top \ r]^\top$. Also, with G as defined in (54) (in terms of the B and σ as defined in Section 3.1 with $K = [K_p \ K_r]$), $N = 4$, and the preceding $x_r(0)$, we can then use Mathematica to check that the matrix (62) is positive definite with the preceding choices, so our PE condition is satisfied and the approach of Theorem 1 again applies. Moreover, (62) is still positive definite if we replace the preceding values for θ_1 and θ_2 by $\theta_1 = -0.036$ and $\theta_2 = 0.03$ respectively, or by $\theta_1 = -0.009$ and $\theta_2 = 0.0075$ respectively (and keep all of the other choices the same as before). This illustrates the applicability of our work to higher-order systems and for a range of possible parameter values.

6 Conclusions and Future Work

We built a new class of barrier strict Lyapunov functions for classes of time-varying adaptive systems, which enabled us to prove robustness and rate of convergence results for globally asymptotic tracking and parameter convergence for model reference adaptive control systems. The unknown parameters that we identify are unknown weight and control effectiveness matrices. The ISS and other robustness properties that we proved are important features that were not available in the model reference adaptive control literature, and our strict Lyapunov function construction made it possible to provide formulas for exponential convergence rates. In addition to basic model reference adaptive control, we applied our methods to the frequency-limited model reference adaptive control framework from [36], which can improve adaptive transient response. By choosing the initial state for the reference trajectory to be a fixed point of a suitable Poincaré map, we can check our relaxed PE condition by computing eigenvalues. Our prior work on adaptive control for 3D curve tracking converted barrier Lyapunov functions into Lyapunov-Krasovskii functionals that can be used to prove convergence of parameter estimates and trajectory tracking under input delays, and a similar conversion can be done for the models in the present paper. However, since the Lyapunov-Krasovskii functional conversion approach would impose bounds on the delays and require the delays to be constant, we hope to combine the present work with our research [21, 22] on sequential predictors to compensate for arbitrarily long time-varying input delays. We also hope to cover systems with outputs, through interconnections of our adaptive control design with observers for unmeasured states and sequential predictors [34].

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