Another look at strong invariance

Peter R. Wolenski

Louisiana State University

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Outline

I. Basic Set-up

Suppose $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction satisfying (minimally) the usual **B**asic **A**ssumptions:

$$
\textbf{(BA)} \quad \begin{cases} \textbf{1)}\ \text{gr } F(\cdot) := \{(x,v) : v \in F(x)\} \text{ is closed,} \\ \textbf{2)}\ \forall\, x \in \mathbb{R}^n,\ F(x) \text{ is nonempty, convex, and compact,} \\ \textbf{3)}\ \exists\, r > 0 \text{ so that } \max\{|v| : v \in F(x)\} \le r(1+|x|). \end{cases}
$$

The dynamics, a differential inclusion:

$$
\textbf{(DI)} \qquad \begin{cases} \dot{x}(s) \in F\big(x(s)\big) & \text{a.e. } s \in [t,T] \\ x(t) = x, \end{cases}
$$

Suppose $S \subseteq \mathbb{R}^n$ is closed. The system (S,F) is **Strongly Invariant (SI)** provided $\forall\, x \in S, -\infty \le t < T \le \infty$, one has $x(s) \in S\ \forall s \in [t,T]$ for **all** solutions $x(\cdot)$ of **(DI)**. Weak invariance requires the inclusion for **some** solution $x(\cdot)$ of **(DI)**.

Characterizations of Invariance

Under just the assumptions **(BA)**, weak invariance has both tangential and normal characterizations:

$$F(x) \cap T_S(x) \neq \phi \quad \forall x \in S \qquad \text{(tangential)}$$

$$h(x,\zeta) := \min_{v \in F(x)} \langle v, \zeta \rangle \leq 0 \qquad \forall x \in S, \ \zeta \in N_S^P(x). \qquad \text{(normal)}$$

A characterization of strong invariance requires additional hypotheses. Whatever these assumptions are, the characterizations have the following form:

$$F(x) \subseteq T_S^C(x) \quad \forall x \in S \qquad \text{(tangential)}$$

$$H(x,\zeta) := \max_{v \in F(x)} \langle v, \zeta \rangle \leq 0 \qquad \forall x \in S, \ \zeta \in N_S^P(x). \qquad \text{(normal)}$$

II. Invariance and **HJ** theory

Two main problem types:

MinTime: Given a closed *target* set $\mathcal{K} \subseteq \mathbb{R}^n$, the problem is

$$\min\ (T - t) \quad \text{over } x(\cdot) \text{ satisfying } (\mathbf{DI}) \text{ with } x(T) \in \mathcal{K}.$$

The optimal value $T(x)$ is the **minimum time function**, and satisfies the **HJ** equation (plus boundary conditions)

$$h(x, \zeta) = -1 \quad \forall x \notin \mathcal{K},\ \zeta \in \partial_P T(x).$$

Mayer problem: Given an *endpoint cost* $\ell : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the problem is

$$\min\ \ell\big(x(T)\big) \quad \text{over } x(\cdot) \text{ satisfying } (\mathbf{DI}).$$

The optimal value $V(t, x)$ is the **value function**, and satisfies the **HJ** equation (plus boundary conditions)

$$\tau + h(x, \xi) = 0 \quad \forall (t, x) \in (-\infty, T),\ (\tau, \xi) \in \partial_P V(t, x).$$

HJ lemmas:

We outline the theory for the **Mayer problem** only under the assumptions **(BA)**. We define auxiliary data $\Gamma_\pm : (-\infty, T] \times \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^{n+2}$ by

$$\begin{aligned} \Gamma_+(s,x,r) &= \{1\} \times \{F(x)\} \times \{0\} \\ \Gamma_-(s,x,r) &= \{-1\} \times \{-F(x)\} \times \{0\} \end{aligned}$$

Lemma (Dynamic programming)

Let $S := epi\ \ V(\cdot,\cdot) \subseteq \mathbb{R}^{n+2}$.

(a)

$$(S, \Gamma_+) \text{ is weakly invariant.}$$

(b)

$$(S, \Gamma_-) \text{ is strongly invariant.}$$

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Lemma (Comparison)

Suppose $\varphi : (-\infty, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ *is lower semicontinuous,* $\varphi(T,x) = \ell(x)\,\forall x$*, and* $S := epi\ \varphi \subseteq \mathbb{R}^{n+2}$.

(a) *If* (S, Γ_+) *is weakly invariant, then*

$$\varphi(t,x) \leq V(t,x) \qquad \forall (t,x) \in (-\infty, T) \times \mathbb{R}^n.$$

(b) *If* (S, Γ_-) *is strongly invariant, then*

$$\varphi(t,x) \geq V(t,x) \qquad \forall (t,x) \in (-\infty, T) \times \mathbb{R}^n.$$

Lemma (**HJ** inequalities)

Suppose $\varphi : (-\infty, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ *is lower semicontinuous,* $\varphi(T, x) = \ell(x)\, \forall x$*, and* $S := \textit{epi}\ \varphi \subseteq \mathbb{R}^{n+2}$.

(a) (S, Γ_+) *is weakly invariant if and only if*

$$\tau + h(x, \xi) \leq 0 \quad \forall (\tau, \xi) \in \partial_P \varphi(t, x), \forall (t, x) \in (-\infty, T) \times \mathbb{R}^n;$$

(b) (S, Γ_-) *is strongly invariant if and only if*

$$\tau + h(x, \xi) \geq 0 \quad \forall (\tau, \xi) \in \partial_P \varphi(t, x), \forall (t, x) \in (-\infty, T] \times \mathbb{R}^n.$$

Everything is valid under only **(BA)** except the last part (b).

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III. Results with Lipschitz assumptions

The usual assumption utilized to characterize strong invariance (either tangentially or normally) is a local Lipschitz assumption that is to hold in a *neighborhood* U of S. This is equivalent to: $\forall r > 0$, $\exists k_r$ so that

$$\Big| H(x,\zeta) - H(y,\zeta) \Big| \le k_r \cdot \|x-y\| \cdot \|\zeta\| \quad \forall\, x, y \in U \cap r\mathbb{B}, \zeta \in \mathbb{R}^n. \quad \textbf{(Lip)}$$

Recall the characterization is

$$(S, F) \text{ is Strongly Invariant} \qquad \textbf{(SI)}$$

$$\Updownarrow$$

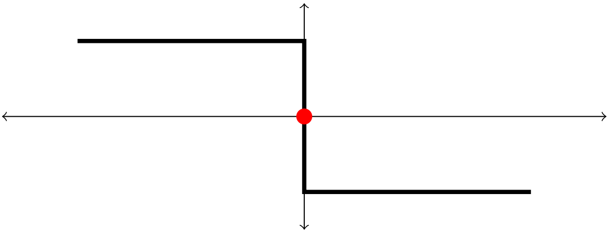
$$H(x,\zeta) := \max_{v \in F(x)} \langle v, \zeta \rangle \le 0 \quad \forall x \in S, \zeta \in N_S^P(x). \qquad \textbf{(HJ}\le\textbf{)}$$

Both directions need assumptions beyond (BA)

Two elementary examples

We let $n = 1$ and $S = \{0\}$.

(SI) holds, but (HJ$\leq$) does not:

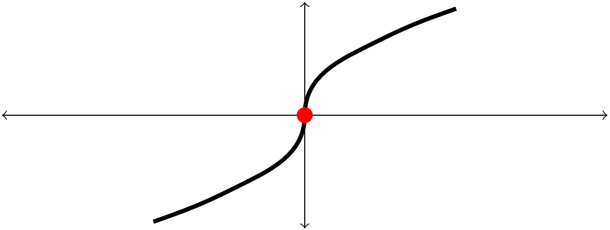


gr F and $S = \{0\}$

$$H(0,1) = \max_{v \in [-1,1]} \langle v, 1 \rangle = 1 = H(0,-1) > 0$$

(HJ$\leq$) holds, but (SI) does not:

With $F(x) = \{\mathrm{sgn}\,(x)\sqrt{|x|}\}$, clearly **(HJ$\leq$)** is satisfied, and nonuniquess of the ODE implies the system is not **(SI)**.



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Properties of Lipschitz differential inclusions

Assume F is locally Lipschitz.

Main ingredient to proving **(SI)**$\Rightarrow$**(HJ$\leq$)**:

Proposition

For each $v \in F(x)$, there exists a trajectory of **(DI)** *with $\dot{x}(0) = v$.*

In fact much more is true: $x(\cdot)$ can be chosen to belong to C^{1+}.

Main ingredient to proving **(HJ$\leq$)**$\Rightarrow$**(SI)**:

Proposition

The set of C^1 trajectories of **(DI)** *are dense (w.r.t. the sup-norm) in the set of all trajectories.*

Both are demonstrably false more generally

IV. One-sided Lipschitz multifunctions

$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is **One - Sided Lipschitz (OSL)** provided

$$H(x, x-y) - H(y, x-y) \leq k \cdot \|x-y\|^2 \quad \forall x, y \in \mathbb{R}^n. \qquad \textbf{(OSL)}$$

Contrast with

$$\Big| H(x,\zeta) - H(y,\zeta) \Big| \leq k \cdot \|x-y\| \cdot \|\zeta\| \quad \forall x, y, \zeta \in \mathbb{R}^n. \qquad \textbf{(Lip)}$$

Special case: $F(x) = D(x) + G(x)$ where $D(\cdot)$ is dissipative (i.e. (OSL) with $k = 0$) and $G(\cdot)$ is Lipschitz.

. . . .

More special: $F(x) = -\partial g(x) + G(x)$ where $g(\cdot) : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex.

Proposition (T. Donchev, V. Rios, PW)

Suppose F is **(OSL)**. *Then (S, F) is* **(SI)** *if and only if*

$$\lim_{y \to_\zeta x} H(y, \zeta) \leq 0 \quad \forall x \in S, \zeta \in N_S^P(x).$$

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V. Sufficiency for **(SI)** for non-Lip systems

We are given a closed set $S \subseteq \mathbb{R}^n$, an open set $U \subseteq \mathbb{R}^n$ with $S \subseteq U$ and a multifunction $F : U \rightrightarrows \mathbb{R}^n$ satisfying (at the minimum) **(BA)**.

Recall the **HJ** characterization of strong invariance with **(Lip)** dynamics:

Theorem (Clarke '76, Krastanov '86)

Suppose $F(\cdot)$ *is Lipschitz on* U *. Then* (S, F) *is* **(SI)** *if and only if*

$$H(x, \zeta) \leq 0 \quad \forall x \in S, \zeta \in N_S^P(x). \qquad \textbf{(HJ}\leq\textbf{)}$$

Theorem (R. Barnard, PW, 2013)

Suppose the **Hamilton-Jacobi Projection Property (HJPP)** *holds:*

$$H\big(x, x - proj_S(x)\big) \leq k\, d_S(x)^2 \quad \forall x \in U. \qquad \textbf{(HJPP)}$$

Then for each solution $x(\cdot)$ *of* **(DI)** *defined on* $[0, T]$ *and lying within* U**,** **the Growth Estimate (GE)** *holds:*

$$d_S\big(x(T)\big) \leq e^{kT} d_S\big(x(0)\big). \qquad \textbf{(GE)}$$

Remarks

$$H(x,\zeta) \le 0 \quad \forall x \in S,\ \zeta \in N_S^P(x). \qquad \textbf{(HJ}\le\textbf{)}$$

$$H\big(x, x - \mathrm{proj}_S(x)\big) \le k\, d_S(x)^2 \quad \forall x \in U. \qquad \textbf{(HJPP)}$$

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- If F is Lipschitz on U, then **(HJPP)** $\Leftrightarrow$ **(HJ$\le$)**.
- Obviously, **(GE)** $\Rightarrow$ **(SI)**.
- As previously mentioned even if one only wants the **(SI)** property, it is not enough to consider just the values of the multifunction on S; the values on a neighborhood U of S play a significant role.
- It does not appear that **(GE)** will imply **(HJPP)**, and in order to obtain a necessary condition, one still needs further structure beyond **(BA)**.
- For non-Lipschitz maps F, the assumptions on H_F and H_{-F} will likely be different, and using the invariance approach to **(HJ)** requires a characterization involving H_{-F}.

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VI. Stratified systems

Stratified domains: The state space is partitioned into a finite collection $\{\mathcal{M}_1, \ldots, \mathcal{M}_M\}$ of smooth manifolds embedded in $\mathbb{R}^N$ such that

1. $\mathbb{R}^N = \bigcup_{i=1}^M \mathcal{M}_i$; $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for all $i \neq j$.
2. If $\overline{\mathcal{M}}_i \cap \mathcal{M}_j \neq \emptyset$, then $\mathcal{M}_j \subseteq \overline{\mathcal{M}}_i$.
3. Each $\overline{\mathcal{M}}_i$ is **proximally smooth** of radius $\delta > 0$;
4. Each $\overline{\mathcal{M}}_i$ is **relatively wedged**.

$\overline{\mathcal{M}}$ Proximally smooth: The distance function $d_{\overline{\mathcal{M}}}(x) := \inf_{y \in \overline{\mathcal{M}}} ||x - y||$ is differentiable on $\{\mathcal{M} + \delta\mathbb{B}\} \setminus \overline{\mathcal{M}}$. **One consequence:** The Clarke normal cone $\mathcal{N}_{\overline{\mathcal{M}}}(x)$ is the proximal one, and therefore has closed graph.

$\overline{\mathcal{M}}$ relatively wedged: The dimension of the relative interior of the tangent cone $\mathcal{T}_{\overline{\mathcal{M}}}(x)$ is the dimension of $\mathcal{M}$ for all $x \in \overline{\mathcal{M}}$.

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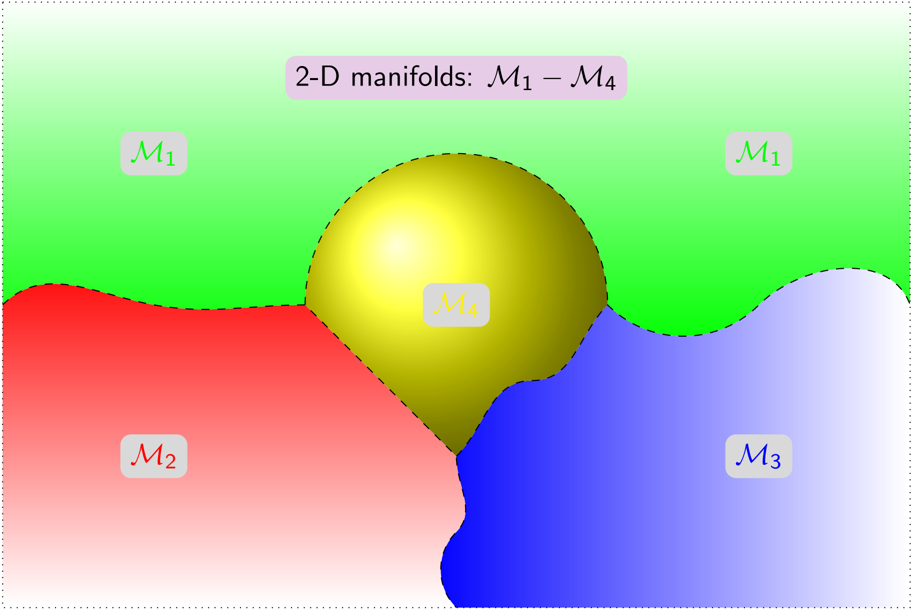
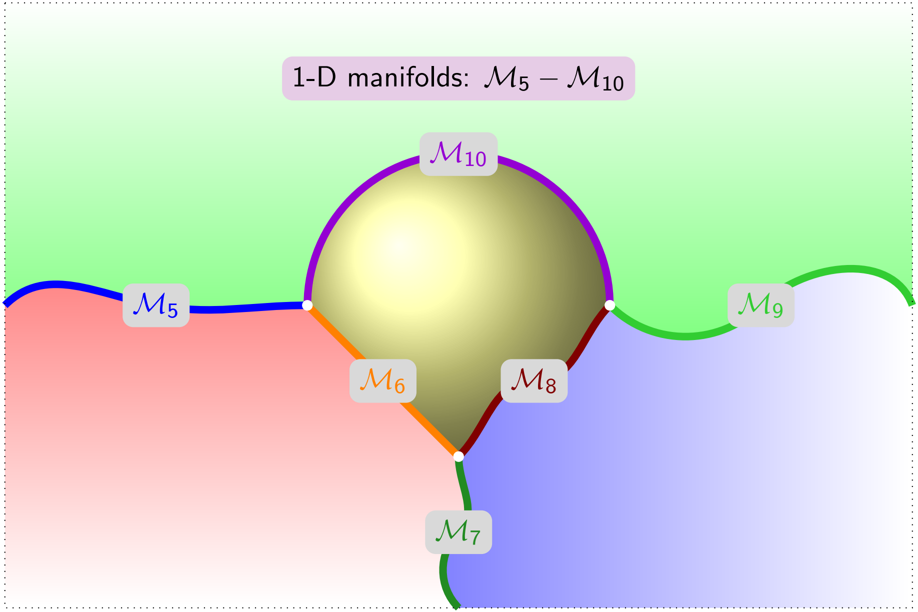
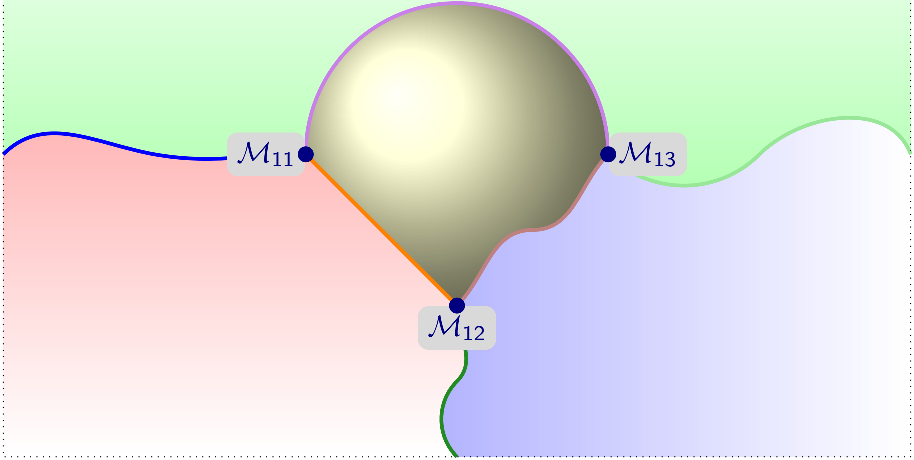

2-D manifolds: $\mathcal{M}_1 - \mathcal{M}_4$
$\mathcal{M}_1$
$\mathcal{M}_1$
$\mathcal{M}_4$
$\mathcal{M}_2$
$\mathcal{M}_3$


1-D manifolds: $\mathcal{M}_5 - \mathcal{M}_{10}$
$\mathcal{M}_{10}$
$\mathcal{M}_5$
$\mathcal{M}_9$
$\mathcal{M}_6$
$\mathcal{M}_8$
$\mathcal{M}_7$

0-D manifolds: $\mathcal{M}_{11} - \mathcal{M}_{13}$



The dynamics

Associated to each manifold $\mathcal{M}_i$ is a multifunction $F_i : \mathcal{M}_i \rightrightarrows \mathbb{R}^N$ with

$$F_i(x) \subseteq \mathcal{T}_{\mathcal{M}_i}(x) \quad \text{whenever} \quad x \in \mathcal{M}_i,$$

and a Lipschitz extension $\overline{F}_i(\cdot)$ to $\overline{\mathcal{M}}_i$. The **basic velocity** multifunction $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is then defined by

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Goal: Characterize (SI) in stratified systems.

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Various models

- State Constraints that are truly **hard** constraints of varying dimensions. (Hasnaa Zidani)
- Reflected problems. (Oana Serea)
- Network problems. (Fabio Camilli)

A Structural Condition:

$$G(x) \cap \mathcal{T}_{\mathcal{M}_i}(x) = F_i(x) \quad \text{when } x \in \mathcal{M}_i.$$

This (essentially) says that you cannot be just off a highway and move in a way that is not allowed on the highway:

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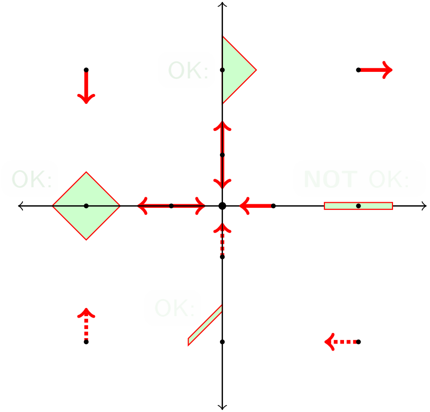

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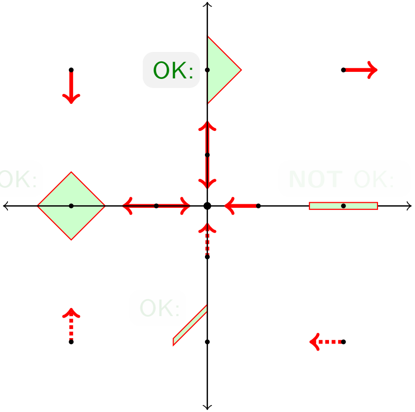

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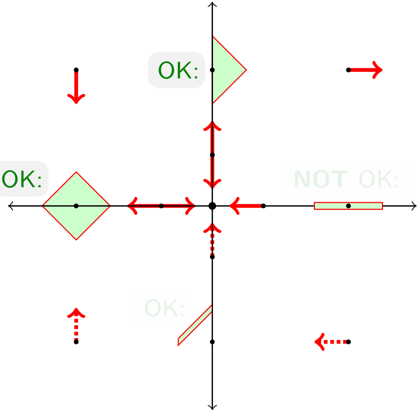

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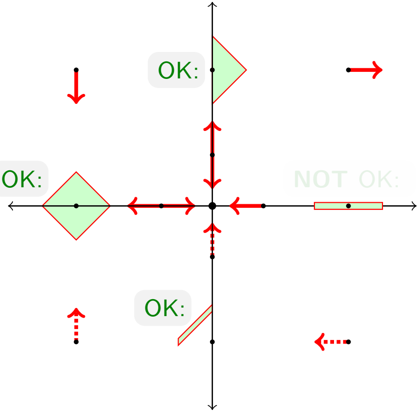

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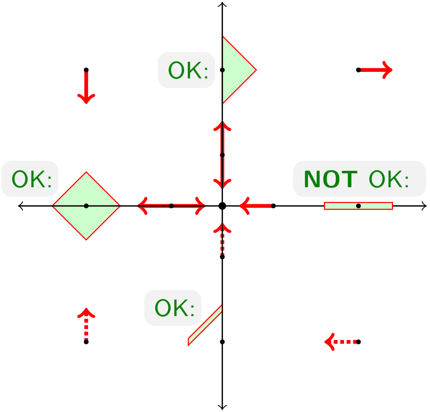

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Trajectories of stratified systems

The stratified dynamical system employs the differential inclusion

$$(\mathbf{DI})_{\mathbf{G}} \quad \begin{cases} \dot{x}(t) \in G\big(x(t)\big) & \text{a.e. } t \in [0, T] \\ x(0) = x, \end{cases}$$

which satisfies **(BA)** but is generally not Lipschitz. DI theory with G has existence and closure properties, but F contains all the relevant velocities:

Proposition

An arc $x(\cdot)$ satisfies $(\mathbf{DI})_{\mathbf{G}}$ if and only if it satisfies

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Essential velocities

Suppose $(\mathcal{M}, \Gamma)$ satisfies **(BA)** and is Lipschitz . Then for all $v \in \Gamma(x)$, there exists a trajectory $x(\cdot)$ of **(DI)** with $\dot{x}(0) = v$; i.e. every velocity at every point *matters* or is *essential* to the flow. This is clearly not the case for stratified systems.

Which velocities of a stratified system are essential?

Given $x \in \mathcal{M}_i$ and $v \in G(x)$, when does there exist a trajectory $x(\cdot)$ of $\textbf{(DI)}_G$ for which $\dot{x}(0) = v$?

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A penetration result

Suppose $\mathcal{M} \subseteq \mathbb{R}^N$ is a manifold, $\Gamma : \mathcal{M} \rightrightarrows \mathbb{R}^n$ is a Lipschitz multifunction, and $x \in \overline{\mathcal{M}}$. When is there a trajectory $x(\cdot)$ of $(\mathrm{DI})_\Gamma$ for which $x(t) \in \mathcal{M}$ for small t?

Theorem (Clarke-PW, '95)

Suppose $\mathcal{C} \subseteq \mathbb{R}^N$ is closed and $\Gamma : \mathcal{C} + \delta\mathbb{B} \rightrightarrows \mathbb{R}^n$ is Lipschitz . Assume $\mathcal{C}$ is epi-Lipschitz at $x \in \mathcal{C}$ and

$$v \in \Gamma(x) \bigcap int\, \mathcal{T}_{\mathcal{C}}(x) \neq \emptyset.$$

Then $\exists C^1$ trajectory $x(\cdot)$ of $(\mathbf{DI})_\Gamma$ with $\dot{x}(0) = v$ and $x(t) \in \mathcal{C}$ for small t.

Corollary

Given any manifold $\mathcal{M} \subseteq \mathbb{R}^N$ with $\overline{\mathcal{M}}$ proximally smooth and $\Gamma : \overline{\mathcal{M}} \rightrightarrows \mathbb{R}^n$ Lipschitz, if $x \in \overline{\mathcal{M}}$ is such that $dim\, \mathcal{T}_{\overline{\mathcal{M}}}(x) = dim\, \mathcal{M}$ and $\exists v \in \Gamma(x) \bigcap r\text{-}int\, \mathcal{T}_{\overline{\mathcal{M}}}(x)$, then there is a C^1 trajectory $x(\cdot)$ of $(\mathbf{DI})_\Gamma$ for which $\dot{x}(0) = v$ and $x(t) \in \mathcal{M}$ for all small t.

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The essential multifunction $G^\sharp : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ *is given by*

$$G^\sharp(x) := \bigcup_{i,\, x \in \overline{\mathcal{M}}_i} \left[\overline{F}_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x) \right].$$

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This is where the proximally smooth and wedge properties play a crucial role - they allow for a complete understanding and description to the tangent cone.

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*If (S, G) is **(SI)**, then*

$$H_{G^\sharp}(x,\zeta) := \max_{v \in G^\sharp(x)} \langle v, \zeta \rangle \leq 0 \quad \forall x \in S, \zeta \in N_S^P(x).$$

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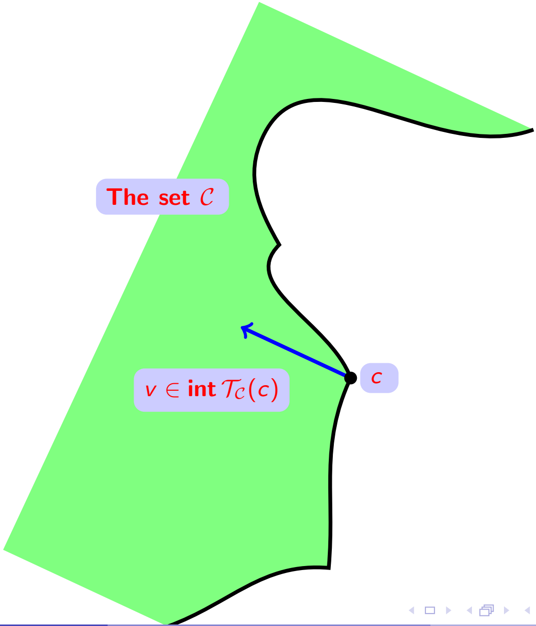
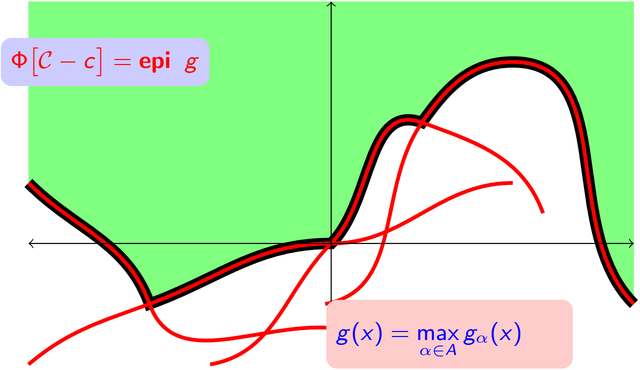
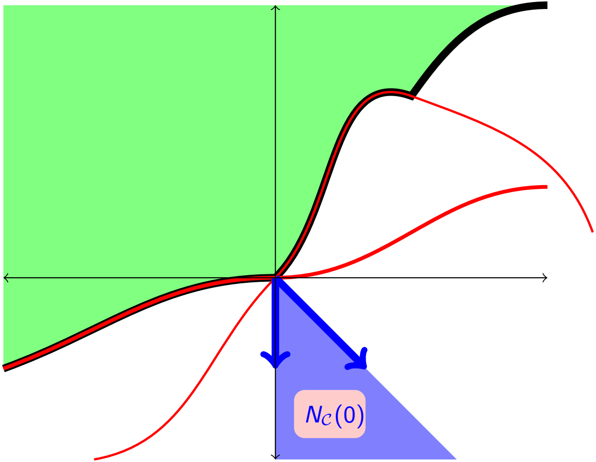
Proximally smooth + wedged sets



Wedge Property $\Rightarrow$ $\exists$ rotation Φ that converts $\mathcal{C}$ into an epigraph of a Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$:



Proximally smooth $\Rightarrow$ $g(\cdot)$ is semiconcave.


$N_{\mathcal{C}}(0)$

Final Remarks

- It is still open to find a good sufficient condition for **(SI)**.
- Stratified systems seems to be an attractive model for many real-life scenarios. The system is discontinuous, but it has structure that can and should be exploited.
- The basic philosophy is that the subsystem on each strata is very "nice" but the overall system does not fit any existing theory. The issue is then to find how to patch together the pieces.
- This can be quite challenging, but there is a lot to be done: necessary conditions, HJ theory, feedback synthesis, regularity theory, numerics, etc.

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Thank you for your attention!

And congratulations and many thanks to the tributees, Hèléne and Hector, for their many contributions to this exciting area of mathematics.