A Small-Gain Theorem for Monotone Systems with Multi-Valued Input-State Characteristics

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Abstract

We provide a new global small-gain theorem for feedback interconnections of monotone input-output systems with multi-valued input-state characteristics. This extends a recent small-gain theorem of Angeli and Sontag for monotone systems with singleton-valued characteristics. We prove our theorem using Thieme's convergence theory for asymptotically autonomous systems. An illustrative example is also provided.

Key Words: Monotone control systems, asymptotic equilibria, set-valued input-state characteristics

I. Introduction

The recent extension [1] of the theory of monotone dynamical systems to monotone input-output (i/o) systems has proven to be very useful in analyzing the global behavior of many important dynamics; see for example [1], [2], [3], [4], [5], [6], and see Section II below for the relevant definitions. (See also [10] for a detailed account of monotone dynamical systems.) Of particular interest in this literature are feedback interconnections of subsystems–or "modules"–that are monotone and that possess a unique globally asymptotically stable equilibrium, obviously depending on the particular (constant) input applied. This has lead to the introduction of the notion of *input-state (i/s) characteristics*, which are maps assigning to each constant input value the particular equilibrium point to which solutions converge. In many applications, this

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assignment is exactly the type of quantitative information that is available from experiments (such as gene expression levels, for instance). Monotonicity, on the other hand, may be considered as a qualitative or structural property of an i/o system; see the graphical tests for monotonicity in [2] for example. These two ingredients, monotonicity of the subsystems and existence of characteristics, are key to proving the small-gain theorems in [1], [2], [3], [4]. (For small-gain theorems for nonlinear but not necessarily monotone systems, see [8].)

In practice however, many monotone i/o systems subject to constant inputs possess *several* equilibria and all solutions converge to one of them, although distinct solutions may converge to distinct equilibria. Such systems are sometimes called *multi-stable*. In fact, since monotone i/o systems subject to constant inputs are monotone dynamical systems, this type of global behavior is to be expected (see [10]). This suggests that the notion of an i/s characteristic ought to be generalized to a *multi-valued* map which assigns to each constant input value the set of all possible equilibria to which solutions converge.

This naturally leads to the question of whether the known small-gain theorem for monotone systems in [1] remains valid if instead of the original notion of i/s characteristics, one assumes the existence of multi-valued characteristics for the subsystems. The purpose of our paper is to show that such an extension is indeed possible. In our main result, we prove that a negative feedback interconnection of monotone i/o subsystems with multi-valued characteristics is itself multi-stable, provided that all the solutions of a particular discrete-time inclusion (which is typically of much lower dimension than the subsystems) converge.

Our work provides a significant extension of the Angeli-Sontag monotone control systems theory [1] because [1] requires singleton-valued characteristics and therefore globally asymptotically stable equilibria. For other approaches to proving multi-stability, see [2] (where *positive* feedback interconnections of monotone i/o subsystems are considered and the trajectories converge for *almost all* initial values) and [9] (which is based on density functions and also concludes convergence for almost all initial values). This earlier work does not include ours because for example (a) our results provide global stabilization from all initial values, (b) we do not require any regularity such as singleton-valuedness, differentiability, or non-degeneracy for the i/s characteristics, and (c) our results are intrinsic in the sense that we make no use of Lyapunov or density functions.

This note is organized as follows. In Section II, we provide the necessary definitions and background for monotone control systems, multi-valued characteristics, weakly non-decreasing set-valued maps, and asymptotically autonomous systems. In Section III, we state our small-gain theorem and discuss its relationship to the small-gain theorems in [1], [2], [3]. In Section IV, we prove our theorem and we illustrate our theorem in Section V. We close in Section VI with some suggestions for future research.

II. Background and Motivation

A. Monotonicity and Characteristics

We next provide the relevant definitions for monotone control systems and input-state characteristics. While our monotonicity definitions follow [1], our treatment of characteristics is novel because we allow discontinuous multi-valued characteristics and unstable equilibria. Our general setting is that of an input-output (i/o) system

$$\dot{x} = f(x,u), \;\; y = h(x), \;\; x \in \mathcal{X}, \;\; u \in \mathcal{U}, \;\; y \in \mathcal{Y} \tag{1}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is the closure of its interior and partially ordered, $\mathcal{U}$ and $\mathcal{Y}$ are subsets of partially ordered Euclidean spaces $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{Y}}$ respectively, and f and h are locally Lipschitz on some open set X containing $\mathcal{X}$. We refer to $\mathcal{X}$ as the *state space* of (1), $\mathcal{U}$ as its *input space*, and $\mathcal{Y}$ as its *output space*. In general, $\mathcal{X}$ will not be a linear space, since for example we often take $\mathcal{X} = \mathbb{R}^n_{\geq 0} := \{x \in \mathbb{R}^n : x_i \geq 0 \, \forall i\}$. We use $\preceq$ to denote the partial orders on all our spaces, bearing in mind that the partial orders on our various spaces could differ.

The set of *control functions* (also called *inputs*) for (1), which we denote by $\mathcal{U}_\infty$, consists of all locally essentially bounded Lebesgue measurable functions $\mathbf{u} : \mathbb{R} \to \mathcal{U}$, and we let $t \mapsto \phi(t, x_o, \mathbf{u})$ denote the trajectory of (1) for any given initial value $x_o \in \mathcal{X}$ and $\mathbf{u} \in \mathcal{U}_\infty$. We always assume our dynamics f are *forward complete* and $\mathcal{X}$*-invariant*, which means that $\phi(\cdot, x_o, \mathbf{u})$ is defined on $[0, \infty)$ and valued in $\mathcal{X}$ for all $x_o \in \mathcal{X}$ and $\mathbf{u} \in \mathcal{U}_\infty$. Since we will be considering more than one dynamic, we often use sub- or superscripts to emphasize the state space variable or dynamic, so for example ϕ^f is the flow map for the dynamic f and $\mathcal{Y}_z$ is the output space for an i/o system with state variable z.

We always assume that our partial orders $\preceq$ are induced by distinguished closed nonempty sets K (called *ordering cones*) and we sometimes write $K_{\mathcal{U}}$ to indicate the cone inducing the partial order on the input space $\mathcal{U}$ and similarly for the other partial orders. We always assume K is a pointed convex cone, meaning,

$$aK \subseteq K \;\; \forall a \geq 0, \quad K + K \subseteq K, \quad K \cap (-K) = \{0\}.$$

When we say that a cone K induces a partial order $\preceq$, we mean the following: $x \preceq y$ if and only if $y - x \in K$. This induces a partial order on the set of control functions $\mathcal{U}_\infty$ as follows: $\mathbf{u} \preceq \mathbf{v}$ if and only if $\mathbf{u}(t) \preceq \mathbf{v}(t)$ for Lebesgue almost all (a.a) $t \geq 0$. A function g mapping a partially ordered space into another partially ordered space is called *monotone* provided: $x \preceq y$ implies $g(x) \preceq g(y)$. We say that (1) is *single-input single-output (SISO)* provided $\mathcal{B}_\mathcal{U} = \mathcal{B}_\mathcal{Y} = \mathbb{R}$, taken with the usual order, i.e., the order induced by the cone $K = [0, \infty)$.

Definition 2.1: We say that (1) is *monotone* provided h is monotone and

$$(p \preceq q \text{ and } \mathbf{u} \preceq \mathbf{v}) \;\; \Rightarrow \;\; (\phi(t,p,\mathbf{u}) \preceq \phi(t,q,\mathbf{v}) \; \forall t \geq 0)$$

holds for all $p, q \in \mathcal{X}$ and $\mathbf{u}, \mathbf{v} \in \mathcal{U}_\infty$.

We let $\mathrm{Equil}(f)$ denote the set of all equilibrium pairs for our dynamic f, namely, the set of all input-state pairs $(\bar{u}, \bar{x})$ such that $f(\bar{x}, \bar{u}) = 0$. For each $(\bar{u}, \bar{x}) \in \mathrm{Equil}(f)$, we let $\mathcal{D}^f(\bar{u}, \bar{x})$ denote the *domain of attraction* of $\dot{x} = f(x, \bar{u})$ to $\bar{x}$, namely, the set of all $p \in \mathcal{X}$ for which $\phi(t, p, \bar{u}) \to \bar{x}$ as $t \to +\infty$, where ϕ is the flow map for f. Since we are not assuming our equilibria are stable, the sets $\mathcal{D}^f(\bar{u}, \bar{x})$ are not necessarily open and could even be singletons; see below for an example where $\mathcal{D}^f(\bar{u}, \bar{x})$ is not open. Given $(\bar{u}, \bar{x}) \in \mathrm{Equil}(f)$, we say that f is *static Lyapunov stable at* $(\bar{u}, \bar{x})$ provided the following condition holds for all $\varepsilon > 0$: There exists $\delta = \delta(\bar{u}, \bar{x}, \varepsilon) > 0$ such that for all $x_o \in \mathcal{D}^f(\bar{u}, \bar{x}) \cap \mathcal{B}_\delta(\bar{x})$(= radius δ open ball centered at $\bar{x}$), we have $|\phi(t, x_o, \bar{u}) - \bar{x}| \leq \varepsilon$ for all $t \geq 0$.

Recall the following notions from [12], in which we let $f^{\bar{u}}$ denote the constant input system $f(\cdot, \bar{u})$ for each $\bar{u} \in \mathcal{U}$. Given $\bar{u} \in \mathcal{U}$, we say that two nonempty (but not necessarily distinct) sets $M_1, M_2 \subseteq \mathcal{X}$ are *$f^{\bar{u}}$-chained* provided there exists a value $y \in \mathcal{X} \setminus (M_1 \cup M_2)$ and a trajectory $x : \mathbb{R} \to \mathcal{X}$ for $f^{\bar{u}}$ satisfying $x(0) = y$ whose *α-limit set* $\alpha(x) := \bigcap\{\overline{x((-\infty, -t])} : t \geq 0\}$ lies in M_1 and whose *ω-limit set* $\omega(x) := \bigcap\{\overline{x([t, +\infty))} : t \geq 0\}$ lies in M_2. We say that a finite collection of nonempty sets $M_1, M_2, \ldots, M_r \subseteq \mathcal{X}$ is *$f^{\bar{u}}$-cyclically chained* provided the following holds: If $r = 1$, then M_1 is $f^{\bar{u}}$-chained to itself; and if $r > 1$, then M_i is $f^{\bar{u}}$-chained to M_{i+1} for $i = 1, 2, \ldots, r-1$ and M_r is $f^{\bar{u}}$-chained to M_1. In this case, we call $\{M_i\}$ an *$f^{\bar{u}}$-cycle*.

An $f^{\bar{u}}$***-equilibrium*** is defined to be any point $\bar{x} \in \mathcal{X}$ such that $f(\bar{x}, \bar{u}) = 0$. A set $M \subseteq \mathcal{X}$ is called $f^{\bar{u}}$***-invariant*** provided the flow map ϕ for f satisfies $M = \{\phi(t, x, \bar{u}) : t \geq 0, x \in M\}$. A compact $f^{\bar{u}}$-invariant set $M \subseteq \mathcal{X}$ is called $f^{\bar{u}}$***-isolated compact invariant*** provided there exists an open set $\mathcal{U} \subseteq \mathcal{X}$ such that there is no compact $f^{\bar{u}}$-invariant subset $\tilde{M} \subseteq \mathcal{X}$ satisfying $M \subsetneq \tilde{M} \subseteq \mathcal{U}$ except M. We use the symbol $\rightrightarrows$ to denote a ***set-valued map*** (also called a ***multifunction***), e.g., $F : \mathcal{Z}_1 \rightrightarrows \mathcal{Z}_2$ means that F assigns each $p \in \mathcal{Z}_1$ a nonempty set $F(p) \subseteq \mathcal{Z}_2$.

Definition 2.2: We say that (1) is endowed with a ***static input-state (i/s) characteristic*** $k_x : \mathcal{U} \rightrightarrows \mathcal{X}$ provided:

1) $\mathrm{Graph}(k_x) = \mathrm{Equil}(f)$;
2) $\cup\{\mathcal{D}^f(\bar{u}, \bar{x}) : \bar{x} \in k_x(\bar{u})\} = \mathcal{X}$ for all $\bar{u} \in \mathcal{U}$;
3) f is static Lyapunov stable at each $(\bar{u}, \bar{x}) \in \mathrm{Equil}(f)$; and
4) For each $\bar{u} \in \mathcal{U}$, $k_x(\bar{u})$ consists of $f^{\bar{u}}$-isolated compact invariant $f^{\bar{u}}$-equilibria and contains no $f^{\bar{u}}$-cycles.

In this case, we also call $k_y := h \circ k_x$ an ***input-output (i/o) characteristic*** for (1).

This definition reduces to the usual singleton-valued i/s characteristic definition in [1] when $\mathrm{Card}\{k_x(\bar{u})\} = 1$ for all $\bar{u} \in \mathcal{U}$. We will not use the static Lyapunov stability property in the proof of our small-gain theorem *per se*, but we still include it to make our definition of i/s characteristics include the singleton-valued characteristic definition in [1]. Condition 3 in our definition is not implied by the other conditions in the definition, even if f has no controls, since it is well-known that f could admit an unstable globally attractive equilibrium; see for example [7, pp. 191-4]. Condition 2 in the definition says for each $\bar{u} \in \mathcal{U}$ and each initial state, the corresponding $f^{\bar{u}}$-trajectory asymptotically approaches some state $\bar{x} \in k_x(\bar{u})$ (where $\bar{x}$ can in principle depend on the initial state of the trajectory). The stipulation in the static Lyapunov stability definition that $x_o \in \mathcal{D}^f(\bar{u}, \bar{x}) \cap \mathcal{B}_\delta(\bar{x})$ is motivated by the fact that our domains of attraction $\mathcal{D}^f(\bar{u}, \bar{x})$ may or may not be open, even if there are no controls. Condition 4 is needed to apply the theory of asymptotically autonomous systems; see Section II-C for the relevant definitions and details.

Remark: Condition 4, and in particular the "no cycles" part, may be hard to check in practice, at least if the system dimension is higher than 2, but can often be checked using monotonicity arguments. Consider for instance a monotone system $\dot{x} = f(x)$ having two f-isolated compact invariant equilibria p and q and assume that $p \prec\prec q$ (where the latter means that $q - p$

belongs to the interior of the order cone K, which is assumed to be nonempty). Then there exist neighborhoods N_p and N_q of p and q respectively such that $n_p \prec\prec n_q$ for all $n_p \in N_p$ and $n_q \in N_q$. We show that $\{p, q\}$ cannot be an f-cycle. Suppose it was a cycle. Then there exist points y and z such that $\alpha(y) = \{p\}$, $\omega(y) = \{q\}$ and $\alpha(z) = \{q\}$, $\omega(z) = \{p\}$. It follows in particular that there exists $T > 0$ large enough such that $n_p := \phi(-T, y) \in N_p$ and $n_q := \phi(-T, z) \in N_q$. Consider the strictly ordered initial conditions $n_p \prec\prec n_q$ for the monotone system $\dot{x} = f(x)$. Since $\omega(n_p) = \{q\}$ and $\omega(n_q) = \{p\}$, there exists $\tilde{T} > 0$ large enough so that $\phi(\tilde{T}, n_p) \in N_q$ and $\phi(\tilde{T}, n_q) \in N_p$ and thus $\phi(\tilde{T}, n_q) \prec\prec \phi(\tilde{T}, n_p)$, which contradicts monotonicity of the system. The same argument can be used to rule out cycles containing more than two equilibria, if we assume that the equilibria are totally ordered by $\prec\prec$ (that is, either $p \prec\prec q$ or $q \prec\prec p$ whenever p and q are distinct equilibria).

B. Weakly Non-Decreasing Set-Valued Maps

A basic property of singleton-valued i/s characteristics k_x is that they are non-decreasing in the relevant partial orders, in the sense that the following holds for all $u, v \in \mathcal{U}_x$: $u \preceq v$ implies $k_x(u) \preceq k_x(v)$; see [1] for the elementary proof. It is therefore natural to inquire about whether set-valued i/s characteristics posses some analogous (but more general) order-preserving property. This motivates the following definition and lemma:

Definition 2.3: Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be partially ordered Euclidean spaces and $F : \mathcal{Z}_1 \rightrightarrows \mathcal{Z}_2$ be any set-valued map. We say that F is *weakly non-decreasing* provided the following holds for all $p, q \in \mathcal{Z}_1$ such that $p \preceq q$: For each $k_p \in F(p)$ and $k_q \in F(q)$, there exist $r_p \in F(p)$ and $r_q \in F(q)$ such that $r_p \preceq k_q$ and $k_p \preceq r_q$.

Lemma 2.4: If k_x is an i/s characteristic for (1) and (1) is monotone, then k_x is weakly non-decreasing.

Proof: Let $p, q \in \mathcal{U}_x$ be such that $p \preceq q$, let $k_p \in k_x(p)$ and $k_q \in k_x(q)$, and let ϕ denote the flow map of f. The corresponding trajectories for the constant inputs satisfy $\phi(t, k_q, p) \preceq \phi(t, k_q, q) = k_q$ for all $t \geq 0$, and $\phi(t, k_q, p) \to r_p$ for some $r_p \in k_x(p)$ as $t \to +\infty$, so $r_p \preceq k_q$ follows because ordering cones are closed. The other order inequality is proved similarly. ■

Definition 2.3 reduces to non-decreasingness in the relevant orders when F is singleton-valued. We are especially interested in solution sequences w_k satisfying discrete set-valued inclusions $w_{k+1} \in F(w_k)$ for all $k \in \mathbb{N}$ where F is weakly non-decreasing. To further motivate our study of weakly non-decreasing multifunctions, let us first assume that $F : [0,1] \to [0,1]$ is a singleton-valued and non-decreasing map in the usual orders (that is, $F(x) \le F(y)$ when $x \le y$). Then it is obvious that every solution of $x_{k+1} = F(x_k)$ converges. Indeed, either $x_0 \le F(x_0)$ and then $x_0 \le F(x_0) \le F^2(x_0) \le \ldots \le F^k(x_0)$ for all $k \in \mathbb{N}$, so the sequence $\{F^k(x_0)\}$ must converge since it is bounded above by 1; or else $F(x_0) \le x_0$, which leads to a non-increasing sequence $\{F^k(x_0)\}$. That converges as well since it is bounded below by 0. On the other hand, this simple dynamical behavior will not occur in general for *multi-valued*, weakly non-decreasing maps.

To see why, consider the following simple example. Assume that $F : [0,1] \rightrightarrows [0,1]$ is a multi-valued map whose graph consists of the union of three straight line segments: one connecting $A = (0,0)$ with $B = (1/2, 1/4)$, a second connecting B to $C = (1/4, 1/2)$ (of slope -1), and a third connecting C with $D = (1,1)$. This "inverted Zorro map" is illustrated in Figure 1 below and is weakly non-decreasing in the usual orders. Then the inclusion $x_{k+1} \in F(x_k)$ has periodic points of period 2. For instance, the periodic sequence $\{1/2, 1/4, 1/2, 1/4, ...\}$ is a solution of the inclusion. In fact, to every initial condition $x_0 \in [1/4, 1/2]$ corresponds a periodic sequence of period 2 satisfying the inclusion, namely $\{x_0, 3/4 - x_0, x_0, 3/4 - x_0, ...\}$ (since $3/4 - x \in F(x)$ for all $x \in [1/4, 1/2]$).

These periodic sequences are caused by the fact that the slope of the middle line segment of the graph of F is -1. Any slight decrease of this slope will destroy the periodic points and leads to solutions that converge to one of the fixed points. For example, for arbitrary $\epsilon > 0$ we can define F_ϵ as the map whose graph consists of three straight line segments connecting A to B, B to $E = ((1+2\epsilon)/(4+4\epsilon), 1/2)$ (so the slope of this line segment is $-1-\epsilon$), and E to D. Then every solution of the inclusion $x_{k+1} \in F_\epsilon(x_k)$ will converge to one of the three fixed points of F. In fact, each solution sequence of this inclusion converges to either 0 or 1, except for the constant sequence at the middle fixed point $\tilde{x} = (3+2\epsilon)/(4(2+\epsilon))$. To see why, notice that if $x_o > 1/2$, then $(x_k, F_\epsilon(x_k))$ remains on the segment $\overline{ED}$, so $x_k \uparrow 1$ by the argument for the singleton-valued case. Similarly, if $x_o < (1+2\epsilon)/(4+4\epsilon)$, then $(x_k, F_\epsilon(x_k))$ remains on $\overline{AB}$ so $x_k \downarrow 0$ again by the singleton-valued case; while if x_k stays in $[(1+2\epsilon)/(4+4\epsilon), 1/2]$, then $x_{k+1} = -(1+\epsilon)x_k + \frac{3}{4} + \frac{\epsilon}{2}$ for all k. Then either $x_k \equiv \tilde{x}$, or else $|x_{k+1} - x_k| = (1+\epsilon)^k |x_1 - x_o| \to +\infty$ as $k \to +\infty$ which is impossible. Therefore, either x_k stays at $\tilde{x}$, or else x_k exits $[(1+2\epsilon)/(4+4\epsilon), 1/2]$ and then converges to either 0 or 1, as claimed.

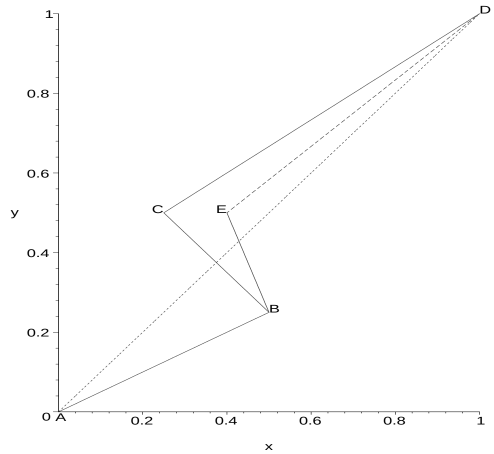

Fig. 1. The inverted Zorro map F (ABCD) and its perturbation F_ϵ with $\epsilon = 1.5$ (ABED) from Section II-B.

C. Asymptotically Autonomous Systems

We will be especially interested in dynamics for which the asymptotic behavior under constant inputs is known. We will then obtain information about the trajectories for not-necessarily constant inputs using the theory of asymptotically autonomous systems. Before turning to this theory, first recall the following "Converging-Input Converging-State" (CICS) Property. This property was shown in [11] and was used in [1] to study the stability of interconnected monotone systems. We use the CICS property at the very end of the proof of our main result (on p.13).

Lemma 2.5: Let $\bar{u} \in \mathcal{U}$, and let $\bar{x}$ be an asymptotically stable equilibrium point for $f^{\bar{u}}$. Let $\mathcal{K}$ be a compact subset of $\mathcal{D}^f(\bar{u}, \bar{x})$. If $x : [0, \infty) \to \mathcal{X}$ is a $\mathcal{K}$-recurrent trajectory of f for some continuous input $u : [0, \infty) \to \mathcal{U}$, and if $u(t) \to \bar{u}$ as $t \to +\infty$, then $x(t) \to \bar{x}$ as $t \to +\infty$.
Here $\mathcal{K}$-recurrent means for each $T > 0$, there exists $t > T$ such that $x(t) \in \mathcal{K}$. One of the requirements of asymptotic stability of $\bar{x}$ (in addition to the convergence condition) is the following stability property: For each $\varepsilon > 0$, there exists $\delta > 0$ such that $|\phi(t, \xi, \bar{u}) - \bar{x}| \leq \varepsilon$ for all $\xi \in \mathcal{B}_\delta(\bar{x})$ and $t \geq 0$. The proof of the CICS property in [11] uses the fact that $\mathcal{D}^f(\bar{u}, \bar{x})$ is open, which follows from the assumption that $\bar{x}$ is a *stable* equilibrium.

However, in our more general setting where the i/s characteristics are multi-valued, the domains of attraction will not necessarily be open, so the CICS property does not apply. Instead, we prove our result using the theory of asymptotically autonomous systems developed by Thieme in [12]. To this end, we first note that Condition 2 from our definition of i/s characteristics implies the following equilibrium condition (EC) from [12]:

(EC) For each $\bar{u} \in \mathcal{U}$, the ω-limit set of any pre-compact $f^{\bar{u}}$-trajectory on $[0, \infty)$ consists of an $f^{\bar{u}}$-equilibrium.

By an *asymptotically autonomous system*, we mean a system $\dot{x} = H(t, x)$ that admits a second dynamic $\dot{x} = \bar{H}(x)$ (called a *limiting dynamic*) such that $H(t, x) \to \bar{H}(x)$ as $t \to +\infty$ locally uniformly in x. For example, if $u \in \mathcal{U}_\infty$ is continuous and $\bar{u} \in \mathcal{U}$ is such that $u(t) \to \bar{u}$ as $t \to +\infty$, then for our locally Lipschitz dynamic f, we know $\dot{x} = H(t, x) := f(x, u(t))$ is asymptotically autonomous with limiting dynamic $\dot{x} = \bar{H}(x) := f(x, \bar{u})$. Using this observation, the following is then immediate from [12, Corollary 4.3] and our i/s characteristic definition:

Lemma 2.6: Assume (1) is endowed with an i/s characteristic. Let $\bar{u} \in \mathcal{U}$ and $u : [0, \infty) \to \mathcal{U}$ be any locally Lipschitz function for which $u(t) \to \bar{u}$ as $t \to +\infty$. Let $x : [0, \infty) \to \mathcal{X}$ be any bounded trajectory for (1) and this input $u(t)$. Then $x(t)$ converges towards an $f^{\bar{u}}$-equilibrium as $t \to +\infty$.

If one drops the "no cycle" part of condition 4 in Definition 2.2, then the conclusion of the above Lemma does not necessarily hold; see [12] for an example.

III. Statement and Discussion of Small-Gain Theorem

We turn next to our small-gain theorem, which generalizes [1, Theorem 3]. The main novelty of our result lies in its applicability to cases where one of the interconnected systems has a multi-valued i/s characteristic, but see Remark 3 below for a further extension for cases where *both* subsystems have multi-valued i/s characteristics. In what follows, an *equilibrium of a discrete inclusion* $w_{k+1} \in F(w_k)$ is defined to be any value $\bar{w}$ such that $\bar{w} \in F(\bar{w})$; the set of all equilibria for this inclusion is denoted by $\mathcal{E}(F)$. A multi-function F is called *locally bounded* provided it maps bounded sets into bounded sets. We say that a continuous time dynamics F has a *pointwise globally attractive set* S provided each maximal trajectory $\zeta(t)$ for F asymptotically approaches some point in S (which could in principle depend on the specific trajectory) as $t \to +\infty$.

Theorem 1: Consider the following interconnection of two SISO dynamic systems:

$$\begin{aligned} \dot{x} &= f_x(x, w), \quad y = h_x(x) \\ \dot{z} &= f_z(z, y), \quad w = h_z(z) \end{aligned} \tag{2}$$

with $\mathcal{U}_x = \mathcal{Y}_z$ and $\mathcal{U}_z = \mathcal{Y}_x$. Assume the following:

1) The first system is monotone when its input w and output y are ordered by the "standard order" induced by the positive real semi-axis.
2) The second system is monotone when its input y is ordered by the standard order and its output w is ordered by the opposite order (induced by the negative real semi-axis).
3) The respective static i/s characteristics k_x and k_z exist with k_x singleton-valued and k_z locally bounded.
4) Each trajectory of (2) is bounded; and each solution sequence $\{v_k\}$ of $v_{k+1} \in (k_y \circ k_w)(v_k)$ converges.

Then (2) has the pointwise globally attractive set $\cup\{\{k_x(\bar{w})\} \times (k_z \circ k_y)(\bar{w}) : \bar{w} \in \mathcal{E}(k_w \circ k_y)\}$. In this setting, $k_y = h_x \circ k_x$ and $k_w = h_z \circ k_z$.

Our theorem differs from the small-gain theorem [1, Theorem 3] mainly in that (a) we replaced the single valuedness of k_z with local boundedness of k_z, (b) we replaced the discrete system $w_{k+1} = (k_w \circ k_y)(w_k)$ from [1] with a discrete inclusion, and (c) we conclude that (2) is attracted to a set of equilibrium points rather than a single point as in [1]. Moreover, in contrast to [2], our theorem gives *global* convergence of the interconnection from all initial values.

Remark 2: Assumption 4 of our theorem is equivalent to the following: $4'$. *Each trajectory of (2) is bounded; and* $\{k_y(w_k)\}$ *converges for each solution sequence* $\{w_k\}$ *of* $w_{k+1} \in (k_w \circ k_y)(w_k)$. In fact, if Assumption 4 holds and w_k is any solution of $w_{k+1} \in (k_w \circ k_y)(w_k)$, then $k_y(w_k)$ converges because $v_k = k_y(w_k)$ is a solution sequence for $v_{k+1} \in (k_y \circ k_w)(v_k)$. Conversely, if Assumption $4'$ holds, and if v_k is any solution sequence of $v_{k+1} \in (k_y \circ k_w)(v_k)$, then we can inductively find a new sequence r_k such that $v_{k+1} \equiv k_y(r_k)$ and $r_{k+1} \in (k_w \circ k_y)(r_k)$ for all k, so v_k converges. On the other hand, it could be that Assumption 4 holds but that there exists a divergent sequence w_k for $w_{k+1} \in (k_w \circ k_y)(w_k)$. See Remark 4 for an example where this occurs. However, if the trajectories of (2) are bounded, and if each solution of $w_{k+1} \in (k_w \circ k_y)(w_k)$ converges, then Assumption $4'$ (or equivalently Assumption 4) holds because k_y is continuous (by the arguments from [1, Proposition V.5] and our assumption that k_x is singleton valued).

IV. Proof of Small-Gain Theorem

The following key lemma generalizes [1, Proposition V.8] to systems with multi-valued characteristics. In it, we set $u_{\inf} := \liminf_{t \to +\infty} u(t)$ and $u_{\sup} := \limsup_{t \to +\infty} u(t)$ for any continuous scalar function u on $[0, \infty)$.

Lemma 4.1: Under the hypotheses of Theorem 1, if $(x(t), z(t))$ is any trajectory of (2) and $\zeta \in \omega(z)$, then there exist $k_- \in k_z(y_{\inf})$ and $k_+ \in k_z(y_{\sup})$ such that $k_- \preceq \zeta \preceq k_+$.

Proof: We only prove the existence of k_- since the proof of the existence of k_+ is similar. Set $\mu = y_{\inf}$ and let ξ be the initial value for $z(t)$. Let $t_j \to +\infty$ and $\mu_j \to \mu$ be sequences such that $\mu_j \in \mathcal{U}_z$ and $y(t) \geq \mu_j$ for all $t \geq t_j$ and all j. We have the following for all $t \geq t_j$ and $j \in \mathbb{N}$:

$$z(t) \;=\; \phi(t, \xi, y) \;=\; \phi(t - t_j, \phi(t_j, \xi, y), y(\cdot + t_j)) \;\succeq\; \phi(t - t_j, \phi(t_j, \xi, y), \mu_j), \tag{3}$$

where ϕ is the flow map for f_z and the last order inequality follows from the monotonicity of the z-subsystem. Therefore, if $z(s_l) \to \zeta$ for some sequence $s_l \to +\infty$, then we can set $t = s_l$ in (3) and use the closedness of order cones to find values $v_j \in k_z(\mu_j)$ such that

$$\zeta \succeq \lim_{l \to \infty} \phi(s_l - t_j, \phi(t_j, \xi, y), \mu_j) = v_j \quad \forall j \in \mathbb{N}. \tag{4}$$

Since k_z is assumed to be locally bounded and has a closed graph (by the continuity of the dynamic f_z in all arguments), we can find $k_- \in k_z(\mu)$ such that $\zeta \succeq v_j \to k_-$, possibly by passing to a subsequence without relabelling. This proves the desired inequality. ■

Returning to the proof of our small-gain theorem, notice that since the output w is ordered by the negative real semi-axis, and since k_z is weakly non-decreasing (by Lemma 2.4), it follows that

$$\max_{k_p \in k_w(p)} \min_{k_q \in k_w(q)} (k_p - k_q)(p - q) \;\leq\; 0 \quad \forall p, q \in \mathcal{U}_z. \tag{5}$$

In other words, for each $p, q \in \mathcal{U}_z$ and $k_p \in k_w(p)$, we can find $k_q \in k_w(q)$, such that $k_p - k_q$ and $p - q$ have opposite signs. Also, k_y is continuous and non-decreasing, as shown in [1, Propositions V.5 and V.8] and Lemma 2.4. Choose any initial value ξ for the interconnection (2), and let $(x(t), z(t))$ denote the corresponding trajectory for (2) starting at ξ. This trajectory is defined on $[0, \infty)$ since we are assuming our trajectories are bounded. Set $w_+ = w_{\sup}$, $w_- = w_{\inf}$, and similarly define $y_\pm$. Let z_+ (resp., z_-) $\in \omega(z)$ be such that $w_- = h_z(z_+)$ (resp., $w_+ = h_z(z_-)$). These limits exist because h_z is continuous and $z(t)$ is bounded in the closed set $\mathcal{X}_z$. By Lemma 4.1, we can find $k_+ \in k_z(y_+)$ and $k_- \in k_z(y_-)$ such that $k_- \preceq z_-$ and $z_+ \preceq k_+$. Setting $r_+^{(0)} = h_z(k_+)$ and $r_-^{(0)} = h_z(k_-)$ and recalling that w reverses order gives

$$k_w(y_+) \ni r_+^{(o)} \leq w_- \leq w_+ \leq r_-^{(o)} \in k_w(y_-). \tag{6}$$

Since we are assuming k_x is singleton-valued, the proof of [1, Theorem 3] gives

$$k_y(w_-) \le y_- \le y_+ \le k_y(w_+). \tag{7}$$

Combining (6) and (7) and recalling that k_y is non-decreasing gives

$$\begin{aligned}(k_y \circ k_w)(y_+) \ni k_y(r_+^o) =: s_+^{(1)} &\le k_y(w_-) \le y_- \\ &\le y_+ \le k_y(w_+) \le s_-^{(1)} := k_y(r_-^{(o)}) \in (k_y \circ k_w)(y_-).\end{aligned} \tag{8}$$

In summary,

$$(k_y \circ k_w)(y_+) \ni s_+^{(1)} \le y_- \le y_+ \le s_-^{(1)} \in (k_y \circ k_w)(y_-). \tag{9}$$

Since $y_+ \le s_-^{(1)}$ and $r_+^{(0)} \in k_w(y_+)$, we can use (5) to find $r_+^{(1)} \in k_w(s_-^{(1)}) \subseteq k_w(k_y \circ k_w)(y_-)$ such that $r_+^{(1)} \le r_+^{(0)}$. Since k_y is non-decreasing, (8) therefore gives

$$y_- \ge k_y(r_+^{(0)}) \ge k_y(r_+^{(1)}) =: s_-^{(2)} \in (k_y \circ k_w)^2(y_-). \tag{10}$$

Similarly, since $y_- \ge s_+^{(1)}$ and $r_-^{(0)} \in k_w(y_-)$, we can use (5) to find $r_-^{(1)} \in k_w(s_+^{(1)}) \subseteq k_w(k_y \circ k_w)(y_+)$ such that $r_-^{(o)} \le r_-^{(1)}$. Hence, (8) also gives

$$y_+ \le k_y(r_-^{(0)}) \le k_y(r_-^{(1)}) =: s_+^{(2)} \in (k_y \circ k_w)^2(y_+). \tag{11}$$

Combining (10) and (11) gives

$$(k_y \circ k_w)^2(y_-) \ni s_-^{(2)} \le y_- \le y_+ \le s_+^{(2)} \in (k_y \circ k_w)^2(y_+).$$

Recalling (9) and proceeding inductively gives sequences $\{s_\pm^{(r)}\}$ satisfying the following for all $j \in \mathbb{N}$:

$$(k_y \circ k_w)^{2j}(y_-) \ni s_-^{(2j)} \le y_- \le y_+ \le s_+^{(2j)} \in (k_y \circ k_w)^{2j}(y_+) \tag{12}$$

$$(k_y \circ k_w)^{2j-1}(y_+) \ni s_+^{(2j-1)} \le y_- \le y_+ \le s_-^{(2j-1)} \in (k_y \circ k_w)^{2j-1}(y_-). \tag{13}$$

Notice that

$$s_\pm^{(j)} \in (k_y \circ k_w)^{j-1}(s_\pm^{(1)}) \ \forall j \in \mathbb{N}. \tag{14}$$

Therefore, Assumption 4 from our theorem provides $\bar{r}_\pm$ such that $s_\pm^{(j)} \to \bar{r}_\pm$ as $j \to +\infty$. Letting $j \to +\infty$ in (12) shows that $\bar{r}_- \le \bar{r}_+$. On the other hand, letting $j \to +\infty$ in (13) gives $\bar{r}_+ \le \bar{r}_-$. Thus,

$$\bar{r}_+ = \bar{r}_- = y_+ = y_- =: \bar{y}.$$

Applying Lemma 2.6 to the z-subsystem $f = f_z$ and the input $u(t) = y(t) \to \bar{y}$ shows that $z(t) \to \bar{z}$ for some $\bar{z} \in k_z(\bar{y})$. Since h_z is continuous, $w(t)$ converges as well; i.e., $w_+ = w_- =: \bar{w}$. Therefore, $\bar{w} = h_z(\bar{z}) \in k_w(\bar{y})$ and (7) gives $\bar{y} = k_y(\bar{w})$. It follows that $\bar{w} \in (k_w \circ k_y)(\bar{w})$, so $\bar{w} \in \mathcal{E}(k_w \circ k_y)$. Therefore, our theorem will follow once we show that $(x(t), z(t))$ converges to some point in $\{k_x(\bar{w})\} \times (k_z \circ k_y)(\bar{w})$ as $t \to +\infty$. To this end, first note that $x(t) \to k_x(\bar{w})$ as $t \to +\infty$ as a consequence of the CICS property (namely Lemma 2.5 above) applied to the x-subsystem $f = f_x$ and the input $u(t) = w(t) \to \bar{w}$, because we are assuming that k_x is singleton-valued. Since $\bar{z} \in k_z(\bar{y}) = k_z(k_y(\bar{w}))$, this completes the proof of the theorem.

Remark 3: One can extend our theorem to cases where k_x and k_z are *both* multi-valued. For example, our theorem remains true if we replace its Assumption 3 by:

$3'$. *The respective i/s characteristics k_x and k_z exist and are locally bounded.*

In this case the conclusion of the theorem is that our interconnection (2) has the pointwise globally attractive set $\cup\{k_x(\bar{w}) \times (k_z \circ k_y)(\bar{w}) : \bar{w} \in \mathcal{E}(k_w \circ k_y)\}$. The proof of this alternative formulation is similar to the proof we gave above and proceeds by a repeated application of

$$\min_{k_p \in k_y(p)} \max_{k_q \in k_y(q)} (k_p - k_q)(p - q) \;\geq\; 0 \quad \forall p, q \in \mathcal{U}_x. \tag{15}$$

Condition (15) follows because h_x is monotone and k_x is weakly non-decreasing. We leave the details of the proof of this more general version of our theorem to the reader.

V. ILLUSTRATION

We next illustrate our theorem using the interconnection

$$\begin{aligned} \dot{x} &= -x + 5 + w, \quad y = x \\ \dot{z} &= -P(z) + y, \quad\;\; w = \tfrac{1}{1+z^2} \end{aligned} \tag{16}$$

evolving on $[0, \infty) \times [0, \infty)$, where $P(z) = z(2z^2 - 9z + 12)$. We order x and z by the usual cone $[0, \infty)$. This dynamic satisfies Conditions 1-2 from Theorem 1. Replacing w with $\frac{1}{1+w^2}$ in (16) gives the planar positive feedback system

$$\begin{aligned} \dot{x} &= -x + 5 + \tfrac{1}{1+w^2}, \quad y = x \\ \dot{z} &= -P(z) + y, \qquad\quad w = z. \end{aligned} \tag{17}$$

If we use superscripts o to label the characteristics of our original interconnection (16), and if we use k_x and so on to denote the characteristics of (17), then $k_x^o(\frac{1}{1+w^2}) \equiv k_x(w)$ and $k_z^o \equiv k_z$.

Also, if $u_{k+1} \in (k_w^o \circ k_y^o)(u_k)$ with $u_k > 0$ for all k, then $w_{k+1} \in (k_w \circ k_y)(w_k)$ for all k when the w_k's are chosen to satisfy

$$\frac{1}{1+w_k^2} = u_k$$

for all $k \in \mathbb{N}$. Moreover, since the output w in (16) is always positive, $(k_w^o \circ k_y^o)(0) \subseteq (0,\infty)$, so $u_k > 0$ for all $k \geq 1$ along all solution sequences $\{u_k\}$ of $u_{k+1} \in (k_w^o \circ k_y^o)(u_k)$. Therefore, if each solution sequence $\{w_k\}$ for $w_{k+1} \in (k_w \circ k_y)(w_k)$ converges, then each solution sequence $\{u_k\}$ for $u_{k+1} \in (k_w^o \circ k_y^o)(u_k)$ converges as well, which implies the required convergence of solutions of $v_{k+1} \in (k_y^o \circ k_w^o)(v_k)$ by Remark 2. The fact that Condition 3 will also hold for the original interconnection (16) will then follow because (16) has the same trajectories as (17).

It therefore remains to show that (17) satisfies Condition 3 from our theorem, that all its trajectories are bounded, and that each solution of $w_{k+1} \in (k_w \circ k_y)(w_k)$ converges. To this end, first note that since the outputs of both subsystems in (17) are also their states, i/s and i/o characteristics coincide for (17)-if they exist–so we can define

$$k_1 = k_x = k_y, \quad k_2 = k_z = k_w$$

wherever the characteristics exist. The characteristic of the first subsystem in (17) is the singleton-valued function

$$k_1(w) = 5 + \frac{1}{1+w^2}, \quad w \in \mathbb{R}_+,$$

while the characteristic for the second subsystem is multi-valued and only determined implicitly as follows: $k_2(y) = \{z \in \mathbb{R} : P(z) = y\}$ for $y \in \mathbb{R}_+$. A bifurcation analysis of the scalar system $\dot{z} = -P(z) + y$, treating $y \in \mathbb{R}_+$ as a bifurcation parameter, shows that $k_2(y)$ is a characteristic which is

1) single-valued if $y \in [0,4)$ or if $y \in (5,\infty)$.
2) triple-valued if $y \in (4,5)$.
3) double-valued if $y = 4$ or 5: $k_2(4) = \{1/2, 2\}$ and $k_2(5) = \{1, 5/2\}$.

There are two saddle-node bifurcations, one at $y = 4$ and the other at $y = 5$. The four defining properties of a characteristic (see Definition 2.2) can indeed be readily verified: For each $y \in \mathbb{R}_+$, the system $\dot{z} = -P(z) + y$ has a finite number of isolated compact equilibria and no cycles (since the system is scalar), and every solution converges to one of the equilibria. It is also not hard to see that k_2 is locally bounded. In order to apply Theorem 1, we only need to verify that (17) satisfies Condition 4 of our theorem.

To check that the trajectories of (17) (or equivalently of (16)) are bounded, it suffices to verify the following: *Claim (G): If* $(x(t), z(t))$ *is any trajectory of (16) defined on some interval* $[0, T]$, *then there is a compact set* D *depending only on* $(x(0), z(0))$ *(and not on* T*) such that* $(x(t), z(t)) \in D$ *for all* $t \in [0, T]$. Boundedness will follow from (G) by standard results for extendability of solutions of ODE's. To prove (G), first note that the boundedness of w on $[0, T]$ and the variations of parameters formula gives

$$|y(t)| = |x(t)| \leq |x(0)| + 6$$

for all $t \in [0, T]$. Pick $\tilde{z} > 5/2$ such that $\tilde{z} = P^{-1}(|x(0)| + 6)$ which exists because P is one-to-one above $5/2$. It follows that if $t \in [0, T)$ is such that $z(t) > \tilde{z}$, then

$$(z(t)) \geq P(\tilde{z}) = |x(0)| + 6 \geq y(t),$$

so $\dot{z}(t) \leq 0$. Therefore, $z(t)$ stays below $\tilde{z}$ on $[0, T]$. Since $\tilde{z}$ depends only on $x(0)$, Claim (G) follows.

Next consider the discrete inclusion $w_{k+1} \in (k_2 \circ k_1)(w_k)$ and notice that it reduces to a discrete equation $w_{k+1} = (k_2 \circ k_1)(w_k)$ because $k_1(w) > 5$ and $k_2(y)$ is single-valued when $y > 5$. Notice also that for all $w_0 \in \mathbb{R}_+$, the discrete equation gives $w_k > 5/2$ for all $k \geq 1$. In particular, the interval $(5/2, \infty)$ is forward invariant for the discrete equation. Finally, since $|k_1'(w)|$ is decreasing for $w \geq 5/2$, elementary calculus shows that

$$|k_2'(k_1(w))k_1'(w)| \ \leq \ \frac{|k_1'(5/2)|}{P'(k_2 \circ k_1(w))} \ \leq \ \frac{|k_1'(5/2)|}{P'(5/2)} \ = \ \frac{5}{(1 + 25/4)^2}\frac{2}{9} \ < \ 1 \ \ \forall w \geq 5/2,$$

so $k_2 \circ k_1$ is a contraction mapping on $[5/2, \infty)$, hence the discrete equation has a unique globally attractive fixed point $\bar{w}$. Therefore, we know from Remark 2 that (17) satisfies Conditions 3-4 of our theorem, as claimed. Since

$$\mathcal{E}(k_w^o \circ k_y^o) = \left\{ \frac{1}{1 + \bar{w}^2} : \bar{w} \in \mathcal{E}(k_2 \circ k_1) \right\},$$

we conclude that our original interconnection (16) has the unique globally attractive equilibrium

$$\left\{ \left(5 + \frac{1}{1 + \bar{w}^2}, k_2 \left(5 + \frac{1}{1 + \bar{w}^2} \right) \right) \right\}.$$

Figure 2 below illustrates this.

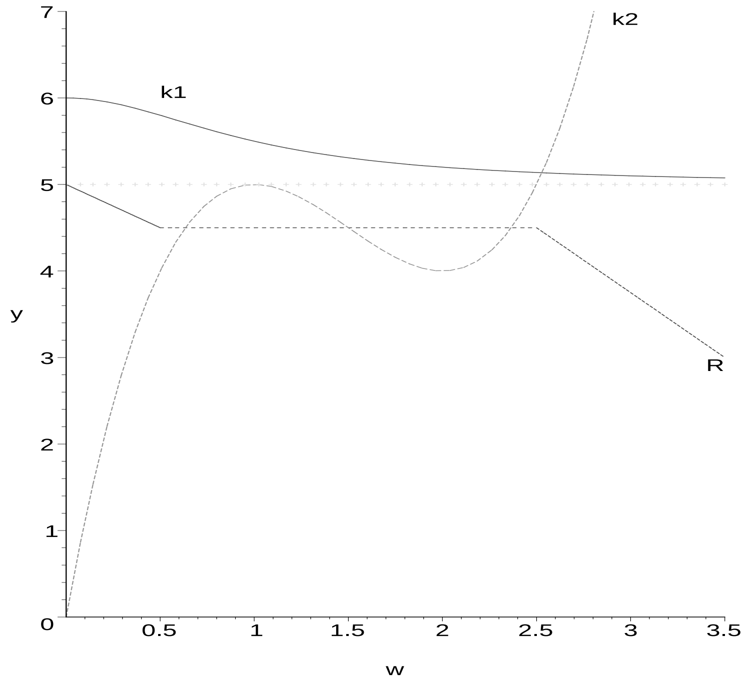

Fig. 2. Characteristics $k_1(w)$, $k_2(y)$ and $R(w)$ from Section V.

Remark 4: In the preceding example, the inclusion $w_{k+1} \in (k_w \circ k_y)(w_k)$ had a unique equilibrium, but our theory applies to examples where $\mathcal{E}(k_w \circ k_y)$ has more than one element as well. One such example is constructed by modifying the interconnection (17) in the following way: replace the x-subsystem with $\dot{x} = -x + R(w)$ where R consists of the line segments in the wy-plane joining $(0,5)$ to $(.5,4.5)$, $(.5,4.5)$ to $(2.5,4.5)$, and $(2.5,4.5)$ to $(3.5,3)$. With this change we get $\text{Card}\{\mathcal{E}(k_w \circ k_y)\} = 3$, and the conclusion of our theorem remains true because $\{k_y(w_k)\}$ converges for each solution sequence $\{w_k\}$ of $w_{k+1} \in (k_w \circ k_y)(w_k)$; see Remark 2. In fact, if $w_o \in [.5, 2.5]$, then $k_y(w_o) = 4.5$, so $w_k \in \mathcal{E}(k_w \circ k_y)$ for all $k \in \mathbb{N}$, which gives $k_y(w_k) = 4.5$ for all $k \in \mathbb{N}$. If $w_o \in [0, .5]$, then $k_y(w_o) \in [4.5, 5]$, which gives $w_1 \in [.5, 2.5]$, so $k_y(w_k) \equiv 4.5$ for all $k \geq 2$ as before. Finally, if $w_o > 5/2$, then $k_y(w_o) \leq 4.5$, so $w_1 \in k_w \circ k_y(w_o) \in [0, 5/2]$, so $k_y(w_k) = 4.5$ for $k \geq 3$, by the previous two cases. On the other hand, one can find non-periodic divergent solution sequences of $w_{k+1} \in (k_w \circ k_y)(w_k)$ when $w_o \in [1/2, 5/2]$. The detailed analysis of this more complicated example is similar to the analysis of (17) and is left to the reader. Note that the convergence of the iterations in the preceding remark follows because $R(w)$ is a horizontal line, at least locally where it meets the other characteristic.

VI. Conclusion

We presented a new small-gain theorem for interconnections of monotone i/o systems with set-valued i/s characteristics. This corresponds to situations where the trajectory for a given constant input can converge to several possible equilibria, depending on the initial value for the trajectory. A key ingredient in the proof of our small-gain theorem is the theory of asymptotically autonomous systems, which requires in particular that the equilibria of the subsystems in the interconnection contain no chains. This suggests the question of how one might extend our theory to cases where the sets of equilibria of the subsystems are more general, e.g., where they contain chains or limit cycles. Research on this question is ongoing.

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