Extension of Razumikhin’s Theorem for Time-Varying Systems with Delay

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Proving GAS of Nonlinear Time Delay Systems

Delay: arise from latencies, communication time lags, and time consuming information gathering, often too long to ignore,...

Indirect methods: flow maps usually not expressible in explicit closed form, hard to measure decay of solutions,...

Lyapunov-Krasovskii functionals: infinite dimensional domains, sum of a Lyapunov function for undelayed system + integral,...

Razumikhin functions: finite dimensional domains, can be hard to satisfy their negative decay rate requirements,...
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Nonlinear Time-Varying Systems

\[ \dot{x} = F(t, x_t), \text{ where } x_t(\theta) = x(t + \theta) \text{ for all } \theta \in [-\tau, 0] \]
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(1)

Assumption 1: There exist a function \( V : [0, \infty) \times \mathbb{R}^n \rightarrow [0, \infty) \) that is \( C^1 \) on \( ([0, \infty) \times \mathbb{R}^n) \setminus \{0\} \), and \( \alpha_1 \) and \( \alpha_2 \) in \( K_\infty \), such that

\[ \alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \text{ on } [0, \infty) \times \mathbb{R}^n \]  

(2)

and such that there are bounded piecewise continuous functions \( a : \mathbb{R} \rightarrow \mathbb{R} \) and \( b : \mathbb{R} \rightarrow [0, \infty) \) such that

\[ \frac{d}{dt} V(t, x(t)) \leq a(t) V(t, x(t)) + b(t) \sup_{\ell \in [t-\tau, t]} V(\ell, x(\ell)) \]  

(3)

holds along all trajectories of (1).
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Alternative to Standard Razumikhin Theorem

\[
\frac{d}{dt} V(t, x(t)) \leq a(t)V(t, x(t)) + b(t) \sup_{\ell \in [t-\tau, t]} V(\ell, x(\ell))
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\]

**Assumption 2:** There exist constants $\beta > 0$ and $\varpi > 0$ and a bounded piecewise continuous function $\epsilon$ such that the function

\[
\kappa(t) = \sup_{\ell \in [t-\tau, t]} \int_\ell^t (-\epsilon(s) - a(s) - b(s))ds \tag{4}
\]

is such that

\[
\left| \int_0^t (\epsilon(\ell) + a(\ell) + b(\ell))d\ell \right| \leq \beta
\]

and

\[
(e^{\kappa(t)} - 1) b(t) - \epsilon(t) \leq -\varpi
\]

hold for all $t \geq 0$. □
Alternative to Standard Razumikhin Theorem

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\frac{d}{dt} V(t, x(t)) \leq a(t) V(t, x(t)) + b(t) \sup_{\ell \in [t-\tau, t]} V(\ell, x(\ell)) \quad (3)
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Assumption 2: There exist constants \( \beta > 0 \) and \( \varpi > 0 \) and a bounded piecewise continuous function \( \epsilon \) such that the function

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and \((e^{\kappa(t)} - 1) b(t) - \epsilon(t) \leq -\varpi\) \quad (5)

hold for all \( t \geq 0 \).

\[ \square \]

Theorem: Under the preceding assumptions, (1) is UGAS to 0. \[ \square \]
Ideas of Proof of Theorem

\[ \left| \int_{0}^{t} (\epsilon(\ell) + a(\ell) + b(\ell)) d\ell \right| \leq \beta \]

and

\[ (e^{\kappa(t)} - 1) b(t) - \epsilon(t) \leq -\varpi \] (5)
Ideas of Proof of Theorem

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Pick a constant \( r > 0 \) such that \( re^{\tau(\bar{\epsilon} + \bar{a} + \bar{b})} \leq \frac{\varpi}{2} \), where \( \bar{\epsilon} \), \( \bar{a} \), and \( \bar{b} \) are upper bounds on \( |\epsilon| \), \( |a| \), and \( b \), respectively.
Ideas of Proof of Theorem

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Pick a constant \( r > 0 \) such that \( r e^{\tau(\bar{\epsilon} + \bar{a} + \bar{b})} b \leq \frac{\varpi}{2} \), where \( \bar{\epsilon}, \bar{a}, \) and \( \bar{b} \) are upper bounds on \( |\epsilon|, |a|, \) and \( b, \) respectively.

\[ U(t, x) = \exp \left( - \int_0^t (\epsilon(\ell) + a(\ell) + b(\ell))d\ell \right) V(t, x) \]

\[ e^{-\beta \alpha_1(|x|)} \leq U(t, x) \leq e^{\beta \alpha_2(|x|)} \] on \([0, \infty) \times \mathbb{R}^n\) \( (R1) \)

\[ U(t, x(t)) \geq \frac{1}{r+1} \sup_{\ell \in [t-\tau, t]} \sup_{\ell \in [t-\tau, t]} U(\ell, x(\ell)) \] \( (R2) \)

\[ \implies \frac{d}{dt} (U(t, x(t)) \leq -\frac{\varpi}{2} U(t, x(t)) \]
Ideas of Proof of Theorem

\[ \left| \int_{0}^{t} (\epsilon(\ell) + a(\ell) + b(\ell)) d\ell \right| \leq \beta \]

and \( (e^{\kappa(t)} - 1) b(t) - \epsilon(t) \leq -\varpi \)

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Pick a constant \( r > 0 \) such that \( re^{\tau(\overline{\epsilon} + \overline{a} + \overline{b})} b \leq \frac{\varpi}{2} \), where \( \overline{\epsilon} \), \( \overline{a} \), and \( \overline{b} \) are upper bounds on \( |\epsilon| \), \( |a| \), and \( b \), respectively.

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\[ e^{-\beta \alpha_1(|x|)} \leq U(t, x) \leq e^{\beta \alpha_2(|x|)} \text{ on } [0, \infty) \times \mathbb{R}^n \]  

(R1)

\[ U(t, x(t)) \geq \frac{1}{r + 1} \sup_{\ell \in [t-\tau, t]} \sup_{\ell \in [t-\tau, t]} U(\ell, x(\ell)) \]

(R2)

\[ \Rightarrow \quad \frac{d}{dt} (U(t, x(t)) \leq -\frac{\varpi}{2} U(t, x(t)) \]
Applications and Extensions

Conditions hold if \( a \) and \( b \) have a period \( \mathcal{T} \) with \( \epsilon(t) = \epsilon_* \), where

\[
\epsilon_* = -\frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} (a(\ell) + b(\ell)) d\ell, \quad \text{if} \quad \tau < \frac{1}{s_\mu} \ln \left( 1 + \frac{\epsilon_*}{b} \right)
\]  

(6)

and if \( s_\mu = \sup_{s \in [0, \mathcal{T}]} (-\epsilon_* - a(s) - b(s)) > 0 \).
Applications and Extensions

Conditions hold if $a$ and $b$ have a period $\mathcal{T}$ with $\epsilon(t) = \epsilon_*$, where

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$$
\dot{x}(t) = -[1 + 2 \cos(t)]x(t - \tau), \quad \text{if } \sqrt{e^{12\tau}} - 1 < \frac{\sqrt{2\pi}}{\sqrt{6\pi (\pi/3 + 2\sqrt{3})}}
$$

(7)

$$
\dot{x}(t) = -m(t)m^\top(t)x(t - \Delta(t)) \text{ in any dimension.}
$$

(8)

Constructions of LKF’s, discrete time analogs,...
Applications and Extensions

Conditions hold if $a$ and $b$ have a period $T$ with $\epsilon(t) = \epsilon_*$, where

$$\epsilon_* = -\frac{1}{T} \int_0^T (a(\ell) + b(\ell)) d\ell, \quad \text{if} \quad \tau < \frac{1}{s_\mu} \ln \left(1 + \frac{\epsilon_*}{b}\right)$$

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and if $s_\mu = \sup_{s \in [0,T]} (-\epsilon_* - a(s) - b(s)) > 0$.

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Constructions of LKF's, discrete time analogs,...

More about Identification Example

\[ \dot{x} = -m(t)m^\top(t)x(t - \Delta(t)) \text{ in any dimension.} \quad (8) \]

Arises when identifying coefficients of a stable plant transfer function; see Anderson’s 1977 adaptive identification TAC paper.

Aeyels-Sepulchre (94), Peuteman-Aeyels (02),...

Constant Delays using LKFs: Mazenc-M-Lin (08),...
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\( \Delta \) piecewise continuous, upper bounded by a constant \( \tau > 0 \).

\( |m(t)| \) admits some Lipschitz constant \( l_m > 0 \) and a period \( \omega \).

\[ M = \frac{1}{\omega} \int_0^\omega m(s)m^\top(s)ds \text{ positive definite} \quad (9) \]

\( \Delta \) bound in terms of preceding coefficients....
Conclusions

New way to relax decay condition on Razumikhin functions
Uses new strictification that produces Razumikhin functions
Allows nonlinear systems with time-varying and distributed delay
Covers identification theory and other interesting examples
Has discrete time analogs for discretized nonlinear systems
Our related results construct Lyapunov-Krasovskii functionals
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Thank you for your attention!