Extension of Razumikhin's Theorem for Time-Varying Systems with Delay

Frederic Mazenc

Michael Malisoff

Proving GAS of Nonlinear Time Delay Systems

Delay: arise from latencies, communication time lags, and time consuming information gathering, often too long to ignore,...

Indirect methods: flow maps usually not expressible in explicit closed form, hard to measure decay of solutions,...

Lyapunov-Krasovskii functionals: infinite dimensional domains, sum of a Lyapunov function for undelayed system + integral,....

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Assumption 1: There exist a function $V : [0, \infty) \times \mathbb{R}^n \to [0, \infty)$ that is C^1 on $([0, \infty) \times \mathbb{R}^n) \setminus \{0\}$, and α_1 and α_2 in $\mathcal{K}_\infty$, such that

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \text{ on } [0, \infty) \times \mathbb{R}^n \tag{2}$$

and such that there are bounded piecewise continuous functions $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to [0, \infty)$ such that

$$\tfrac{d}{dt} V(t, x(t)) \leq a(t) V(t, x(t)) + b(t) \sup_{\ell \in [t-\tau, t]} V(\ell, x(\ell)) \tag{3}$$

holds along all trajectories of (1). □

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Assumption 2: There exist constants $\beta > 0$ and $\varpi > 0$ and a bounded piecewise continuous function ϵ such that the function

$$\kappa(t) = \sup_{\ell \in [t-\tau, t]} \int_{\ell}^{t} (-\epsilon(s) - a(s) - b(s)) \mathrm{d}s \quad (4)$$

is such that

$$\begin{aligned} \left| \int_0^t (\epsilon(\ell) + a(\ell) + b(\ell)) \mathrm{d}\ell \right| &\leq \beta \\ \text{and } \left(e^{\kappa(t)} - 1 \right) b(t) - \epsilon(t) &\leq -\varpi \end{aligned} \quad (5)$$

hold for all $t \geq 0$. □

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Theorem: Under the preceding assumptions, (1) is UGAS to 0. □

Ideas of Proof of Theorem

$$\left|\int_0^t(\epsilon(\ell)+a(\ell)+b(\ell))\mathrm{d}\ell\right| \leq \beta$$
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$$U(t,x) = \exp\left(-\int_0^t(\epsilon(\ell) + a(\ell) + b(\ell))\mathrm{d}\ell\right) V(t,x)$$

$$e^{-\beta}\alpha_1(|x|) \leq U(t,x) \leq e^{\beta}\alpha_2(|x|) \text{ on } [0,\infty) \times \mathbb{R}^n \tag{R1}$$

$$U(t,x(t)) \geq \frac{1}{r+1} \sup_{\ell\in[t-\tau,t]} U(\ell,x(\ell))$$
$$\Longrightarrow \quad \frac{d}{dt}\big(U(t,x(t))\big) \leq -\frac{\varpi}{2} U(t,x(t)) \tag{R2}$$

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Applications and Extensions

Conditions hold if a and b have a period $\mathcal{T}$ with $\epsilon(t) = \epsilon_*$, where

$$\epsilon_* = -\frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} (a(\ell) + b(\ell))\mathrm{d}\ell, \quad \text{if} \quad \tau < \frac{1}{s_\mu} \ln\left(1 + \frac{\epsilon_*}{\overline{b}}\right) \tag{6}$$

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$$\dot{x}(t) = -[1+2\cos(t)]x(t-\tau), \quad \text{if} \quad \sqrt{e^{12\tau} - 1} < \frac{\sqrt{2}\pi}{\sqrt{6\tau}(\pi/3+2\sqrt{3})} \tag{7}$$

$$\dot{x}(t) = -m(t)m^\top(t)x(t - \Delta(t)) \text{ in any dimension.} \tag{8}$$

Constructions of LKF's, discrete time analogs,...

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Mazenc, F., and M. Malisoff, "Extensions of Razumikhin's theorem and Lyapunov-Krasovskii functional constructions for time-varying systems with delay," *Automatica*, regular paper.

More about Identification Example

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Arises when identifying coefficients of a stable plant transfer function; see Anderson's 1977 adaptive identification TAC paper.

Aeyels-Sepulchre (94), Peuteman-Aeyels (02),...

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Δ piecewise continuous, upper bounded by a constant $\tau > 0$.

$|m(t)|$ admits some Lipschitz constant $l_m > 0$ and a period ω.

$$M = \frac{1}{\omega}\int_0^\omega m(s)m^\top(s)\mathrm{d}s \ \text{ positive definite} \tag{9}$$

Δ bound in terms of preceding coefficients....

Conclusions

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Thank you for your attention!