Stabilization of a Chain of Exponential Integrators Using a Strict Lyapunov Function

Michael Malisoff

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Chain of Exponential Integrators

$$\left\{ \begin{array}{lcl} \dot{X} & = & (Y^* - Y)X \\ \dot{Y} & = & (D^* - D)Y, \quad (X, Y) \in (0, \infty)^2 \end{array} \right.$$

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Constant $D^* > 0$ and $Y^* > 0$

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Theorem: The closed loop system is GAS to (X_r, Y^*) on $(0, \infty)^2$.

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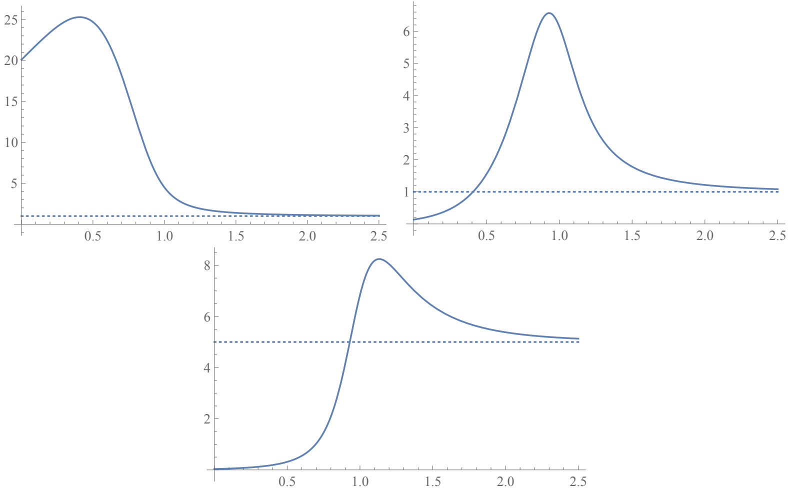

Fig. 1: X (Left), Y (Right), $D = 5Y/X$ (Bottom). $Y^* = X_r = 1$. $D^* = 5$.

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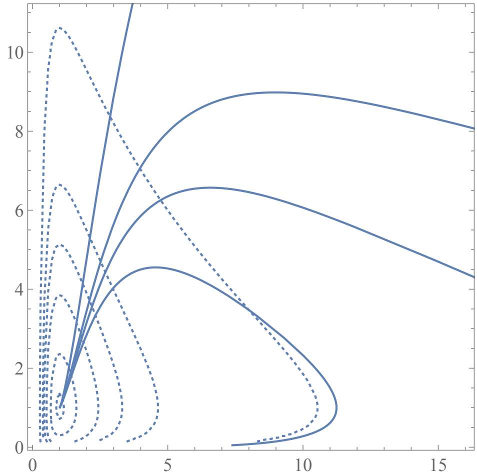

Fig. 2: $(X(t), Y(t))$ Converging to $(1, 1)$ through Level Curves of V.

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Malisoff, M., and F. Mazenc, *Constructions of Strict Lyapunov Functions*, Communications and Control Engineering Series, Springer-Verlag London Ltd., London, UK, 2009.

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Theorem: We can build a function $\mathcal{R} : [0, \infty) \to [1, \infty)$ such that

$$V_3(x, y) = \int_0^{V_1(x,y)} \mathcal{R}(s)\mathrm{d}s - xy, \quad \text{where}$$
$$V_1(x, y) = x + e^{-x} - 1 + \frac{Y^*}{D^*}(e^y - y - 1)$$

satisfies $V_3(x, y) \geq \frac{1}{2} V_1(x, y)$ for all $(x, y) \in \mathbb{R}^2$ and

$$\dot{V}_3(x, y) \leq -\frac{1}{2}\left\{\mathcal{R}(V_1(x, y))e^{-x} Y^*(1 - e^y)^2 + D^* x(1 - e^{-x})\right\}$$

along all solutions $(x(t), y(t))$ so V_3 is a strict Lyapunov function.

Extensions and Applications

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Thank you for your attention!