

# Asymptotic Stabilization for Feedforward Systems with Delayed Feedbacks

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$$Y'(t) = \mathcal{G}(t, Y_t, \delta(t)), \quad Y(t) \in \mathcal{Y}, \quad (2)$$

where  $\mathcal{G}(t, Y_t, d) = \mathcal{F}(t, Y(t), \mathbf{u}(t, Y(t - \tau)), d)$ .

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Typically we construct  $u$  such that all trajectories of (2) for all possible choices of  $\delta$  satisfy some control objective.

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Find  $\gamma_i$ 's by building certain LKFs for  $Y'(t) = \mathcal{G}(t, Y_t, 0)$ .

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Leveraging Undelayed Stabilizers :

- First we render the undelayed system UGAS and build a LF for the closed loop system.
- Then we see how long an input delay can be introduced in the controller without destroying the UGAS.
- The stabilization analysis typically adds integral terms to the LF to produce an LKF.

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Example: The function  $V(Y) = \frac{1}{2}|Y|^2$  is an ISS-LKF for  $Y'(t) = -Y(t) + \frac{1}{4}Y(t) + \delta(t)$  for any  $\mathcal{D}$ .

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$$V^\#(Y_t) = V(Y(t)) + \frac{1}{4} \int_{t-\tau}^t |Y(\ell)|^2 d\ell + \frac{1}{8\tau} \int_{t-\tau}^t \left[ \int_s^t |Y(r)|^2 dr \right] ds$$

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# Linear Feedforward Systems

Consider the set of all systems having the feedforward form

$$\begin{cases} \dot{x} &= h_1(z) + h_2(z)v(t - \tau) \\ \dot{z} &= f(z) + g(z)v(t - \tau). \end{cases} \quad (3)$$

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The state space is  $\mathbb{R}^n \times \mathbb{R}^p$ . Linearizing (3) around period  $\tau$  reference trajectories produces a system of the form

$$\begin{cases} \dot{x}(t) &= C(t)z(t) + D(t)u(t - \tau) \\ \dot{z}(t) &= A(t)z(t) + B(t)u(t - \tau), \end{cases} \quad (4)$$

where  $A$ ,  $B$ ,  $C$ , and  $D$  are  $C^1$  matrix valued functions of period  $\tau$ .



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We focus on (4), and cases where uncertainties  $\delta$  are added to  $u$ .

**Assumption 1.** *The system*

$$\dot{\theta}(t) = A(t)\theta(t) \quad (5)$$

*is UGAS. The matrices  $A$ ,  $B$ ,  $C$ , and  $D$  are  $C^1$  and have period  $\tau$ .*

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Hence, (5) admits a Lyapunov function  $V(t, \theta) = \theta^\top P(t)\theta$  such that  $\dot{V} \leq -|\theta|^2$  along all trajectories of (5) and  $P$  has period  $\tau$ .

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$$\begin{cases} \frac{\partial \psi_a}{\partial t}(t, m) &= -\psi_a(t, m)A(t) \\ \psi_a(m, m) &= \mathbf{I} \end{cases} \quad (6)$$

for all  $t \in \mathbb{R}$  and  $m \in \mathbb{R}$ .

## Lemma

Let Assumption 1 hold. Then the function  $I - \psi_a(l, l - \tau)$  is invertible for all  $l \in \mathbb{R}$ . Also, the function  $q : \mathbb{R} \rightarrow \mathbb{R}^{n \times p}$  defined by

$$q(t) = - \int_{t-\tau}^t C(l) [I - \psi_a(l, l - \tau)]^{-1} \psi_a(t, l) dl \quad (7)$$

has period  $\tau$ , and  $\dot{q}(t) + q(t)A(t) + C(t) = 0$  for all  $t \in \mathbb{R}$ .  $\square$

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**Assumption 2.** There exists a constant  $c > 0$  such that the matrix  $R(t) = q(t)B(t) + D(t)$  satisfies

$$\int_{t-\tau}^t R(m)R(m)^\top dm \geq cI \quad (8)$$

for all  $t \in \mathbb{R}$ . (That means  $I$  is the  $n \times n$  identity matrix.)

# Main Result

Our coordinate change  $\xi(t) = x(t) + q(t)z(t)$  gave the system

$$\begin{cases} \dot{\xi}(t) &= R(t)u(t - \tau) \\ \dot{z}(t) &= A(t)z(t) + B(t)u(t - \tau) \end{cases} \quad (9)$$

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## Theorem

Let Assumptions 1 and 2 hold. Then for all constants  $\tau > 0$  and  $\epsilon \in (0, \frac{1}{1+4\tau\|R\|^2})$ , the controller

$$u(t-\tau) = -\epsilon \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \quad (10)$$

renders (9) UGAS.

# Proof of Theorem

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Show that the closed loop system (9) admits the LKF

$$V^\#(t, \xi_t, z(t)) = z^\top(t)P(t)z(t) + 21\beta_1 W_3(t, \xi_t), \quad \text{where}$$

$$W_3(t, \xi_t) = W_2(t, \xi_t) + k \left[ (1 + 2U(\xi_t))^{3/2} - 1 \right],$$

$$W_2(t, \xi_t) = W_1(t, \xi_t) + \beta_0 \int_{t-\tau}^t \left| \frac{R(m)^\top \xi(m)}{\sqrt{1+|\xi(m)|^2}} \right|^2 dm,$$

$$W_1(t, \xi_t) = \xi(t)^\top \left[ \int_{t-\tau}^t \int_m^t R(\ell)R(\ell)^\top d\ell dm \right] \xi(t),$$

$$U(\xi_t) = \frac{1}{2}|\xi|^2 + \frac{1}{4\tau} \int_{t-2\tau}^t \int_m^t \frac{\epsilon |R(\ell)^\top \xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}} d\ell dm,$$

$$\beta_0 = \frac{1}{2c} \|R\|^6 \tau^4 \epsilon^2, \quad k = \frac{4\sqrt{2}}{3\epsilon} (\tau + \beta_0),$$

$$\beta_1 = \max\{v_1, v_2\}, \quad v_1 = \frac{2}{c} [4\|P\|^2 \|B\|^2 \|R\|^2 + 1],$$

$$\text{and } v_2 = \frac{16\sqrt{2}\tau}{3\epsilon k} (1 + 8\tau\|P\|^2 \|B\|^2 \|R\|^4).$$

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The  $\xi$ -subsystem  $\dot{\xi}(t) = R(t)u(t - \tau)$  is

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since  $R$  has period  $\tau$ .

# Key Ideas of Proof

The  $\xi$ -subsystem  $\dot{\xi}(t) = R(t)\mathbf{u}(t - \tau)$  is

$$\dot{\xi}(t) = -\epsilon \frac{R(t)R(t)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}}, \quad (11)$$

since  $R$  has period  $\tau$ . For all  $t \geq \tau$ , we have

$$\xi(t - \tau) = \xi(t) + \epsilon \int_{t-\tau}^t \frac{R(m)R(m)^\top \xi(m - \tau)}{\sqrt{1 + |\xi(m - \tau)|^2}} dm. \quad (12)$$

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Hence, for all  $t \geq \tau$ , we have

$$\begin{aligned} \dot{\xi}(t) &= -\epsilon \frac{R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}} \xi(t) \\ &\quad - \epsilon^2 \frac{R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}} \int_{t-\tau}^t \frac{R(m)R(m)^\top \xi(m - \tau)}{\sqrt{1 + |\xi(m - \tau)|^2}} dm. \end{aligned} \quad (13)$$

If the orange term were not present, the  $\xi$  subsystem would be

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That admits the strict Lyapunov function

$$U(\xi_t) = \frac{1}{2} |\xi(t)|^2 + \frac{1}{4\tau} \int_{t-2\tau}^t \int_m^t \frac{\epsilon |R(\ell)^\top \xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}} d\ell dm.$$

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When the asymptotic stability of the  $\xi$ -subsystem is established, we can (more) easily prove the UGAS result with the additional component  $\dot{z}(t) = A(t)z(t) + B(t)u(t - \tau)$  from the dynamics.

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Benefit of LKF: Leads to **robustness** to actuator errors, using ISS.



Allowing additive uncertainties on the control gives

$$\begin{cases} \dot{\xi}(t) &= R(t)[u(t - \tau) + \delta(t)] \\ \dot{z}(t) &= A(t)z(t) + B(t)[u(t - \tau) + \delta(t)] . \end{cases} \quad (15)$$

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$$\bar{\delta} = \frac{c}{9k\|R\|(1+2\bar{u})^{1/2}}, \quad \text{where } k = \frac{4\sqrt{2}}{3\epsilon} \left( \tau + \frac{1}{2c}\|R\|^6\tau^4\epsilon^2 \right) \quad (16)$$

$$\text{and } \bar{u} = \max \left\{ \frac{1}{2} + \frac{\epsilon\|R\|^2\tau}{4\sqrt{2}}, \frac{\epsilon\|R\|^2\tau}{4\sqrt{2}} (1 + 2\epsilon\|R\|^2\tau) \right\} .$$

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### Theorem

*Under the preceding assumptions, (15) in closed loop with*

$$\mathbf{u}(t - \tau) = -\epsilon \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \quad (17)$$

*is ISS with respect to the set of all disturbances  $\delta$  bounded by  $\bar{\delta}$ .*



# Application to UAV Dynamics

We study the UAV with standard autopilots which is first order for heading and Mach hold and second order for the altitude hold.

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Key Model : Underactuated kino-dynamic representation that is justifiable for high-level formation flight control.

See e.g. 2004 IEEE-TCST paper by Ren and Beard.

We are given a  $C^1$  reference trajectory  $(x_r, y_r, \theta_r, v_r) : \mathbb{R} \rightarrow \mathbb{R}^4$ , so there is a reference input  $(\theta_{cr}, v_{cr}) : \mathbb{R} \rightarrow \mathbb{R}^2$  such that

$$\begin{cases} \dot{x}_r(t) &= v_r(t) \cos(\theta_r(t)) \\ \dot{y}_r(t) &= v_r(t) \sin(\theta_r(t)) \\ \dot{\theta}_r(t) &= \alpha_\theta(\theta_{cr}(t) - \theta_r(t)) \\ \dot{v}_r(t) &= \alpha_v(v_{cr}(t) - v_r(t)) \end{cases} \quad (19)$$

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**Assumption 3 :** The functions  $\cos(\theta_r(t))$  and  $\sin(\theta_r(t))$  have period  $\tau$ , there exists a constant  $t_c \in [0, \tau]$  such that  $\dot{\theta}_r(t_c) \neq 0$ , and  $v_r$  is bounded.

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Tracking Error :  $(\bar{x}, \bar{y}, \bar{\theta}, \bar{v}) = (x - x_r, y - y_r, \theta - \theta_r, v - v_r)$ .

# Application to UAV Tracking Dynamics

$$\left\{ \begin{array}{l} \dot{\bar{x}} = \cos(\theta_r(t))\bar{v} \\ \quad + [\bar{v} + v_r(t)][\cos(\bar{\theta} + \theta_r(t)) - \cos(\theta_r(t))] \\ \dot{\bar{y}} = \sin(\theta_r(t))\bar{v} \\ \quad + [\bar{v} + v_r(t)][\sin(\bar{\theta} + \theta_r(t)) - \sin(\theta_r(t))] \\ \dot{\bar{v}} = -\alpha_v \bar{v} + \mathbf{u}(t - \tau) \\ \dot{\bar{\theta}} = -\alpha_\theta \bar{\theta} . \end{array} \right. \quad (20)$$



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We apply our theory to the  $(\bar{x}, \bar{y}, \bar{v})$  dynamics obtained by setting  $\bar{\theta} = 0$ , and then we reincorporate the  $\bar{\theta}$  dynamics to get  $\theta_c$  and  $v_c$ .

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$$\begin{aligned} x &= (\bar{x}, \bar{y}), \quad z = \bar{v}, \quad A(t) = -\alpha_v, \quad \psi_a(t, \ell) = e^{\alpha_v(t-\ell)}, \quad B(t) = 1, \\ C(t) &= (\cos(\theta_r(t)), \sin(\theta_r(t)))^\top, \quad D(t) = 0, \\ \xi &= (\xi_1, \xi_2)^\top = (\bar{x}, \bar{y})^\top + q(t)\bar{v}, \quad q(t) = R(t), \end{aligned}$$

$$R(t) = \frac{1}{e^{\alpha_v \tau} - 1} \int_{t-\tau}^t \begin{pmatrix} \cos(\theta_r(\ell)) \\ \sin(\theta_r(\ell)) \end{pmatrix} e^{\alpha_v(t-\ell)} d\ell. \quad (21)$$

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To illustrate our work, we simulated the UAV dynamics

$$\begin{cases} \dot{x} &= v \cos(\theta) \\ \dot{y} &= v \sin(\theta) \\ \dot{\theta} &= \alpha_{\theta}(\theta_c(t - \tau) - \theta) \\ \dot{v} &= \alpha_v(v_c(t - \tau) + \delta(t) - v), \end{cases} \quad (22)$$

with our controllers

$$\begin{aligned} \theta_c(t - \tau) &= \theta_{cr}(t) \quad \text{and} \\ v_c(t - \tau) &= v_{cr}(t) - \frac{\epsilon}{\alpha_v} \frac{R(t - \tau)^{\top} \xi(t - \tau)}{\sqrt{1 + |\xi(t - \tau)|^2}}. \end{aligned} \quad (23)$$

Autopilot constants:  $\alpha_v = 0.192$  and  $\alpha_\theta = 0.55$ .

Delay:  $\tau = 2$

Reference trajectory:  $(20 + 10 \sin(\pi t)/\pi, 20 - 10 \cos(\pi t)/\pi, \pi t, 10)$ .

Reference control:  $(\theta_{cr}(t), v_{cr}(t)) = (\pi(t + 1/\alpha_\theta), 10)$ .

Controller parameter:  $\epsilon = 0.257732$ .

Disturbance:  $\delta(t) = 0.1 \sin(t)$  added to  $v_c$ .

Initial function:  $(x_0, y_0, \theta_0, v_0) = (17, 22, -0.5, 8)$ .

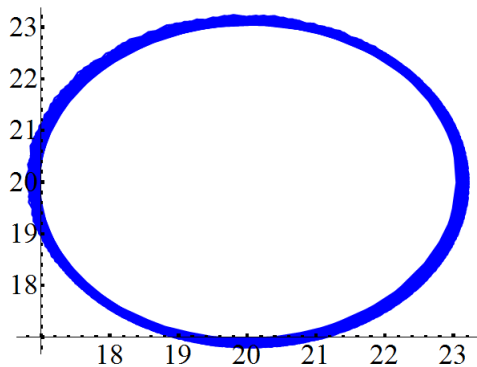


FIGURE:  $(x(t), y(t))$  for Times [480, 1000]

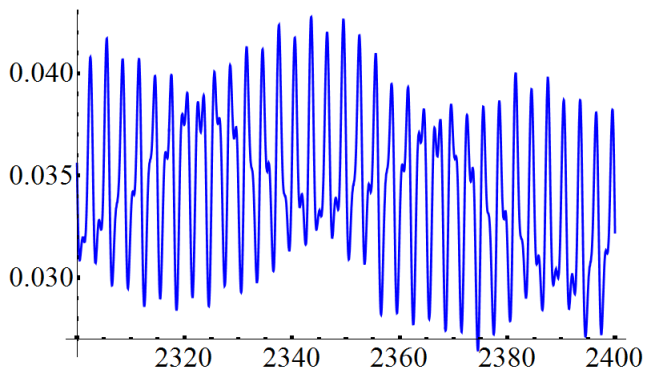


FIGURE:  $x(t) - x_r(t)$

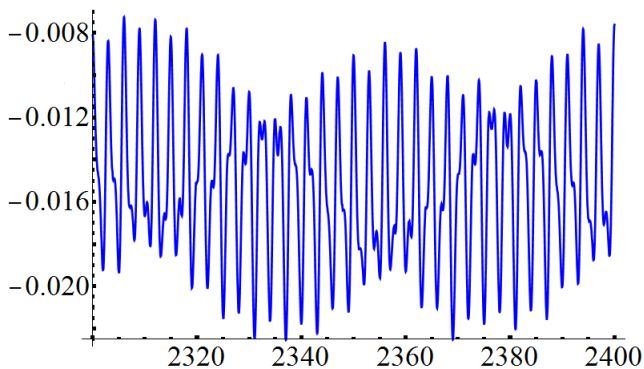


FIGURE:  $y(t) - y_r(t)$



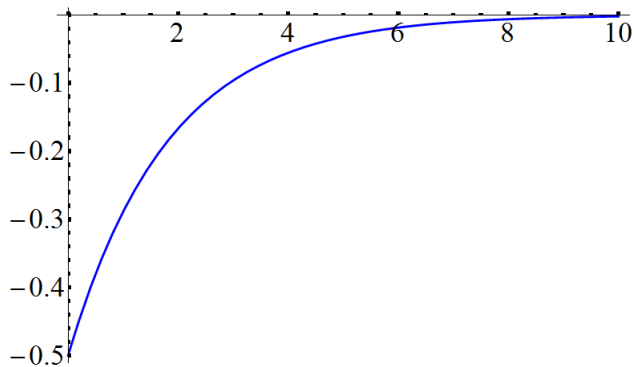


FIGURE:  $\theta(t) - \theta_r(t)$

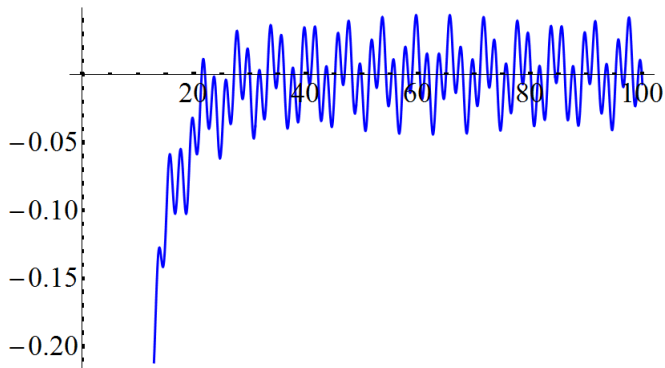


FIGURE:  $v(t) - v_r(t)$

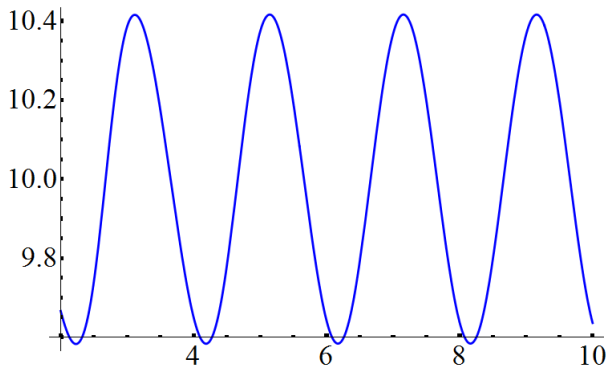


FIGURE:  $v_c(t - \tau)$

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Nonlinear analogs involving PDEs would also be interesting.