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Control-Lyapunov Functions for Systems Satisfying the Conditions of the Jurdjevic-Quinn Theorem

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Abstract— **For a broad class of nonlinear systems satisfying the Jurdjevic-Quinn conditions, we construct a family of smooth control-Lyapunov functions whose derivatives along the trajectories of the systems can be made negative definite by smooth control laws that are arbitrarily small in norm. We also design state feedbacks of arbitrarily small norm that render our systems integral-input-to-state stable to actuator errors.**

Index Terms—**Control-Lyapunov functions, global asymptotic and integral-input-to-state stabilization**

I. INTRODUCTION

Lyapunov stability is of paramount importance in nonlinear control theory. In many important applications, it is very beneficial to have a continuously differentiable Lyapunov function whose derivative along the trajectories of the system can be made negative definite by an appropriate choice of feedback. Observe in particular that:

• Recent advances in the stabilization of nonlinear delay systems (e.g., [9], [10], [17], [19], [27]) are based on knowledge of continuously differentiable Lyapunov functions.

• Lyapunov functions are very efficient tools for robustness analysis. For example, many proofs of nonlinear disturbance-to-state L^p stability properties rely on Lyapunov functions; see [8, Chapter 13] and [2], [14], [20], [26]. Moreover, control-Lyapunov function based control designs guarantee robustness to different types of deterministic [6] and stochastic disturbances, and to unmodeled dynamics [21], [22].

• When a control-Lyapunov function (CLF) satisfying the *small control property* (as defined below) is available, the universal formula provides an explicit expression for an asymptotically stabilizing feedback that is also an optimal control for a suitable optimization problem whose value function is the CLF; see [24].

• Backstepping and forwarding require Lyapunov functions of class at least C^1 for the subsystems [22].

The converse Lyapunov theorem (see [7], [13]) ensures that, for any system that is globally asymptotically stabilizable by continuously differentiable feedback, a control-Lyapunov function exists. Unfortunately, for nonlinear control systems, determining *explicit expressions* for control-Lyapunov functions is in general difficult. Fortunately, for large classes of systems, one can determine functions whose derivatives along the trajectories can be rendered negative *semi*-definite. If the systems satisfy the conditions of the celebrated Jurdjevic-Quinn Theorem (from [11]), then globally asymptotically stabilizing feedbacks can be constructed. However, in this case, no formula for control-Lyapunov functions is available, and this makes robustness analysis more difficult. This motivates the following fundamental question: *When the Jurdjevic-Quinn Theorem applies, is it possible to design explicit control-Lyapunov functions?*

In [5], where this issue was addressed for the first time, a method was presented for designing explicit CLFs for affine homogeneous systems that satisfy the Jurdjevic-Quinn conditions. Our objective in the present note is to extend [5] by constructing control-Lyapunov functions for systems satisfying the Jurdjevic-Quinn conditions but having no homogeneity property. Our work also complements [18] where strong Lyapunov functions are constructed for a large family of systems satisfying the conditions of LaSalle. The main difference between the present work and [18] is that in [18], only systems without input are considered whereas here we consider systems with input.

This paper is organized as follows. In Section II, we introduce our notation, and in Section III, we present our main result on constructing control-Lyapunov functions, and corresponding smooth stabilizing state feedbacks of arbitrarily small norm, for systems satisfying the Jurdjevic-Quinn conditions. Section IV is devoted to a discussion of our main result, Section V to its proof, and Section VI to an illustrating example. Section VII constructs feedbacks for our systems that have arbitrarily small norm and that in addition achieve integral-input-to-state stability relative to actuator errors. Concluding remarks in Section VIII end our work.

II. PRELIMINARIES

1. We assume throughout the paper that the functions encountered are sufficiently smooth. In particular, if the vector fields defining our systems are C^∞, then our constructions lead to Lyapunov functions and stabilizing state feedbacks that are C^p for any prescribed $p \in \mathbb{N}$.

2. We let $\mathcal{K}_\infty$ denote the class of all continuous functions $\rho : [0, \infty) \to [0, \infty)$ for which
 i) $\rho(0) = 0$ and
 ii) ρ is increasing and unbounded.

3. We let $\mathcal{KL}$ denote the class of all continuous functions $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ for which
 i) $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \geq 0$,
 ii) $\beta(s, \cdot)$ is nonincreasing for each $s \geq 0$, and
 iii) $\beta(s, t) \to 0$ as $t \to +\infty$ for each $s \geq 0$.

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4. A function $V(\cdot)$ on $\mathbb{R}^n$ is called *positive definite* (resp., *negative definite*) provided $V(x) > 0$ (resp., $V(x) < 0$) for all $x \neq 0$ and $V(0) = 0$; it is called *radially unbounded* provided $\lim_{|x|\to+\infty} V(x) = +\infty$.
5. Given a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, we use the notation $L_f V(x) := \nabla V(x) f(x)$, where ∇V is the gradient of V.
6. A positive definite function $V(\cdot)$ on $\mathbb{R}^n$ is called a *control-Lyapunov function (CLF)* for a system

$$\dot{\chi} \;=\; \varphi_1(\chi) + \varphi_2(\chi)u \tag{1}$$

with input u provided it is radially unbounded and satisfies

$$L_{\varphi_1} V(\chi) \;\geq\; 0 \;\Rightarrow\; [\; \chi = 0 \quad \text{or} \quad L_{\varphi_2} V(\chi) \;\neq\; 0 \;] \;. \tag{2}$$

We use $\dot{V}(x,u)$ to denote the derivative along trajectories of (1), i.e., $\dot{V}(x,u) = L_{\varphi_1} V(x) + L_{\varphi_2} V(x) u$. We often suppress the argument of $\dot{V}$ to simplify the notation. We say that a CLF $V(\cdot)$ for (1) satisfies the *small control property* provided for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $0 < |\chi| < \delta(\varepsilon)$, then there exists u (possibly depending on χ) such that $|u| < \varepsilon$ and $L_{\varphi_1} V(\chi) + L_{\varphi_2} V(\chi) u < 0$.

III. Main result

In this section, we state our main result. Consider the nonlinear system

$$\dot{x} \;=\; f(x) + g(x)u \tag{3}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ is the input and the function $f(\cdot)$ satisfies $f(0) = 0$. We make the following assumptions:

Assumption H1. A smooth function $V(x)$ that is radially unbounded and positive definite and such that

$$L_f V(x) \;\leq\; 0 \quad \forall x \in \mathbb{R}^n \tag{4}$$

is known.

Assumption H2. A vector field $G(x)$ such that when $L_g V(x) = 0$ and $x \neq 0$, then either $L_f L_G V(x) < 0$ or $L_f V(x) < 0$ is known.

Theorem 1: Assume that (3) satisfies Assumptions H1 and H2. Then one can determine a positive definite smooth function $\delta(\cdot)$ such that

$$U(x) \;=\; V(x) + \delta(V(x)) L_G V(x) \tag{5}$$

is a control-Lyapunov function for the system (3) that satisfies the small control property. In fact, for each real-valued positive function $\xi(\cdot)$, one can determine a function $\delta(\cdot)$ such that the corresponding function (5) is a smooth control-Lyapunov function for the system (3) that satisfies the small control property and whose derivative along the trajectories of (3) in closed-loop with the feedback

$$u \;=\; -\xi(V(x)) L_g V(x)^\top \tag{6}$$

is negative definite.

IV. Discussion of Theorem 1

1. Assumptions H1 and H2 are similar to the assumptions of the main result of [5]. In particular, [5] provides an explicit expression for a vector field $G(x)$ such that Assumption H2 holds whenever the so-called "weak Jurdjevic-Quinn conditions" are satisfied. This vector field is not continuous at the origin but it turns out that there exists an integer $N \geq 1$ such that the vector field $G_N(x) = V(x)^N G(x)$ is of class C^∞ for V satisfying our assumptions. The equality

$$\begin{aligned} L_f L_{G_N} V(x) \;&=\; N V(x)^{N-1} L_f V(x) L_G V(x) \\ &\quad + V(x)^N L_f L_G V(x) \end{aligned} \tag{7}$$

then implies that if $G(x)$ satisfies Assumption H2, and if Assumption H1 also holds, then $G_N(x)$ satisfies Assumption H2 as well. Consequently, one can take advantage of the formula in [5] to determine a C^∞ vector field for which Assumption H2 is satisfied.
2. No restriction on the size of the function $\xi(\cdot)$ in (6) is imposed. Therefore, the family of feedbacks (6) contains elements that are arbitrarily small in (sup) norm. By taking advantage of this fact, one can extend Theorem 1 to systems that are not affine in the input.
3. An important class of dynamics covered by Theorem 1 is described by the so-called *Euler-Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau \tag{8}$$

for the motion of mechanical systems, in which q represents the generalized configuration coordinates, $L = K - P$ is the difference between the kinetic energy K and potential energy P, and τ is the control [28]. In standard cases, $K(q,\dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q}$ where the inertia matrix $M(q)$ is smooth and everywhere symmetric and positive definite. Then the generalized momenta $\frac{\partial L}{\partial \dot{q}}$ are given by $p = M(q)\dot{q}$, so in terms of the state $x = (q,p)$, the equations (8) become

$$\dot{q} = \frac{\partial H}{\partial p}(q,p)^\top = M^{-1}(q)p, \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p)^\top + \tau, \tag{9}$$

where

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + P(q)$$

is the total energy in the dynamics. We make the following additional assumptions: (a) $P(q)$ is positive definite and radially unbounded and (b) $\nabla P(q) \neq 0$ when $q \neq 0$. (These assumptions are not too restrictive since one can often perform a modification of H and τ resulting in a new system that satisfies these assumptions.) Then H is positive definite and radially unbounded, so $V = H$ satisfies Assumption H1. The radial unboundedness follows from the continuity of the (positive) eigenvalues of the positive definite matrix $M^{-1}(q)$ as functions of q [25, Appendix A4], which implies that each compact set S of q values admits a constant $c_S > 0$ such that $p^\top M^{-1}(q) p \geq c_S |p|^2$ for all $q \in S$ and all p. In fact, in our general notation with $x = (q,p)$, we get $L_f V(x) \equiv 0$ and $L_g V(x) = H_p(x) = p^\top M^{-1}(q)$. Choosing

$$G(x) = \begin{pmatrix} 0 \\ \nabla P(q)^\top \end{pmatrix} \tag{10}$$

gives $L_G V(x) = H_p(x)\nabla P(q)^\top$. Therefore, if $L_g V(x) = p^\top M^{-1}(q) = 0$ and $x \neq 0$, then $p = 0$ and therefore also $L_f L_G V(x) = -\nabla P(q) M^{-1}(q) \nabla P(q)^\top$ and $q \neq 0$. Since M^{-1} is everywhere positive definite, Assumption H2 reduces to our assumption (b) and therefore is satisfied as well. We study a special case of (9) in Section VI below where we explicitly compute the corresponding CLF (5) and stabilizing feedback.

V. Proof of Theorem 1

We fix a positive function $\xi : [0, \infty) \to (0, \infty)$.

First step. We exhibit a family of functions $\delta(\cdot)$ for which the function $U(x)$ in (5) is positive definite and radially unbounded. One can determine $\alpha_i(\cdot)$ of class $\mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)\ ,\ |L_G V(x)| \le \alpha_3(|x|) \tag{11}$$

for all $x \in \mathbb{R}^n$. It follows that

$$\begin{array}{rcl} U(x) & \ge & \alpha_1(|x|) - \delta(V(x))\alpha_3(|x|) \\ & \ge & \alpha_1(\alpha_2^{-1}(V(x))) \\ & & -\delta(V(x))\alpha_3(\alpha_1^{-1}(V(x))) \quad \forall x \in \mathbb{R}^n\ . \end{array} \tag{12}$$

For each $p \in \mathbb{N}$, we can use standard results to find a function $\delta : [0, \infty) \to [0, \infty)$, of class C^p, such that

$$\delta(v) \le \frac{\alpha_1(\alpha_2^{-1}(v))}{1 + 2\alpha_3(\alpha_1^{-1}(v))} \quad \forall v \ge 0\ . \tag{13}$$

With such a function $\delta(\cdot)$, the inequality

$$U(x)\ \ge\ \frac{1}{2}\alpha_1(\alpha_2^{-1}(V(x)))\ \ \forall x \in \mathbb{R}^n \tag{14}$$

is satisfied. Since $V(x)$ is positive definite and radially unbounded and $\frac{1}{2}\alpha_1(\alpha_2^{-1}(\cdot))$ is of class $\mathcal{K}_\infty$, (14) implies that $U(x)$ is positive definite and radially unbounded as well. In the next steps, we impose further restrictions on δ.

Second step. Along the trajectories $x(t)$ of (3), the derivative $\dot{U}$ of $U(x)$ satisfies

$$\begin{array}{rcl} \dot{U} & = & \dot{V}\left[1 + \delta'(V(x))L_G V(x)\right] \\ & & +\delta(V(x))\dfrac{d}{dt}L_G V(x) \\ & = & \left[L_f V(x) + L_g V(x) u\right]\left[1 + \delta'(V(x))L_G V(x)\right] \\ & & +\delta(V(x))\left[L_f L_G V(x) + L_g L_G V(x) u\right] \end{array} \tag{15}$$

When the system (3) is in closed-loop with the feedback (6), (15) reads

$$\begin{array}{rcl} \dot{U} & = & \left[L_f V(x) - \xi(V(x))|L_g V(x)|^2\right] \\ & & \times\left[1 + \delta'(V(x))L_G V(x)\right] \\ & & +\delta(V(x))L_f L_G V(x) \\ & & -\xi(V(x))\delta(V(x))L_g L_G V(x) L_g V(x)^\top. \end{array} \tag{16}$$

Minorizing $1/(1 + 4|L_G V(x)|)$ by a positive function of the form $x \mapsto \mathcal{P}(V(x))$ using, e.g.,

$$\mathcal{P}(s) = \inf\{1/(1 + 4|L_G V(x)|) : x \in \mathbb{R}^n, V(x) = s\}$$

and arguing as above provides a C^1 function $\delta(\cdot)$ such that

$$\delta'(V(x))L_G V(x) \ge -\frac{1}{4}\ \ \forall x \in \mathbb{R}^n \tag{17}$$

so recalling (4) gives the inequality

$$\begin{array}{rcl} \dot{U} & \le & \frac{3}{4}\left[L_f V(x) - \xi(V(x))|L_g V(x)|^2\right] \\ & & +\delta(V(x))L_f L_G V(x) \\ & & +\xi(V(x))\delta(V(x))|L_g L_G V(x)||L_g V(x)|. \end{array} \tag{18}$$

The inequality $(\delta(V(x))|L_g L_G V(x)| - \frac{1}{2}|L_g V(x)|)^2 \ge 0$ multiplied by $\xi(V(x))$, with (4) and (18), gives

$$\begin{array}{rcl} \dot{U} & \le & \frac{1}{2}\left[L_f V(x) - \xi(V(x))|L_g V(x)|^2\right] \\ & & +\delta(V(x))L_f L_G V(x) \\ & & +\xi(V(x))\delta^2(V(x))|L_g L_G V(x)|^2\ . \end{array} \tag{19}$$

Third step. The remaining part of the proof relies extensively on the following consequence of Assumption H2:

Lemma 2: Assume that the system (3) satisfies Assumption H2. Then, there exist continuous positive definite functions Γ and N such that

$$\begin{array}{l} |L_g V(x)| \le \Gamma(|x|) \quad \text{implies} \\ \left[\ L_f V(x) \le -N(|x|)\ \text{ or }\ L_f L_G V(x) \le -N(|x|)\ \right]\ . \end{array}$$

Proof: We first show that the continuous function

$$\begin{array}{rcl} S(x) & = & \min\{0, L_f L_G V(x)\} \\ & & +\min\{0, L_f V(x)\} - |L_g V(x)| \end{array} \tag{20}$$

is negative definite. Observe first that $S(0) = 0$ and $S(x) \le 0$ for all x. Assume now that $S(x) = 0$. Each term of $S(x)$ is nonpositive. It follows that

$$\min\{0, L_f L_G V(x)\} = \min\{0, L_f V(x)\} = |L_g V(x)| = 0.$$

By Assumption H2, it follows that $x = 0$, which gives the negative definiteness. Therefore $-S(x)$ is positive definite. It follows that one can determine a continuous positive definite real-valued function ρ such that

$$\rho(|x|)\ \le\ -S(x) \tag{21}$$

Let us now prove that $|L_g V(x)| \le \frac{1}{2}\rho(|x|)$ implies that either $L_f L_G V(x) \le -\frac{1}{4}\rho(|x|)$ or $L_f V(x) \le -\frac{1}{4}\rho(|x|)$.

Consider x such that $|L_g V(x)| \le \frac{1}{2}\rho(|x|)$. Then (20) and (21) imply

$$\begin{array}{rcl} \rho(|x|) & \le & -\min\{0, L_f L_G V(x)\} - \min\{0, L_f V(x)\} \\ & & +|L_g V(x)| \\ & \le & -\min\{0, L_f L_G V(x)\} - \min\{0, L_f V(x)\} \\ & & +\frac{1}{2}\rho(|x|) \end{array}$$

We deduce that

$$\frac{1}{2}\rho(|x|)\ \le\ -\min\{0, L_f L_G V(x)\} - \min\{0, L_f V(x)\}$$

i.e., $\min\{0, L_f L_G V(x)\} + \min\{0, L_f V(x)\}\ \le\ -\frac{1}{2}\rho(|x|)$. It follows that either $\min\{0, L_f L_G V(x)\} \le -\frac{1}{4}\rho(|x|)$ or $\min\{0, L_f V(x)\} \le -\frac{1}{4}\rho(|x|)$. Therefore one can conclude that $|L_g V(x)| \le \frac{1}{2}\rho(|x|)$ implies $L_f L_G V(x) \le -\frac{1}{4}\rho(|x|)$

or $L_f V(x) \leq -\frac{1}{4}\rho(|x|)$, so we can take $\Gamma(s) = \frac{1}{2}\rho(s)$ and $N(s) = \frac{1}{4}\rho(s)$. ■

Fourth step. We prove that the right hand side of (19) is negative definite when the smooth positive definite function $\delta(\cdot)$ is suitably chosen, by considering all cases in Lemma 2.

First Case. $|L_g V(x)| \leq \Gamma(|x|)$ and $L_f V(x) \leq -N(|x|)$. Then the inequality (19) implies that

$$\begin{array}{rcl} \dot{U} & \leq & -\frac{1}{2}N(|x|) + \delta(V(x))L_f L_G V(x) \\ & & +\xi(V(x))\delta^2(V(x))|L_g L_G V(x)|^2 \ . \end{array} \quad (22)$$

Arguing as before provides $\delta(\cdot)$ such that

$$\begin{array}{rcl} \delta(V(x))L_f L_G V(x) & \leq & \frac{1}{8}N(|x|) \\ \xi(V(x))\delta^2(V(x))|L_g L_G V(x)|^2 & \leq & \frac{1}{8}N(|x|) \end{array} \quad (23)$$

for all $x \in \mathbb{R}^n$. We obtain $\dot{U} \leq -\frac{1}{4}N(|x|) < 0$ for all $x \neq 0$.

Second Case. $|L_g V(x)| \leq \Gamma(|x|)$, $L_f L_G V(x) \leq -N(|x|)$. Then the inequalities (4) and (19) imply

$$\begin{array}{rcl} \dot{U} & \leq & -\delta(V(x))N(|x|) \\ & & +\xi(V(x))\delta^2(V(x))|L_g L_G V(x)|^2 \ . \end{array} \quad (24)$$

Choosing $\delta(\cdot)$ such that

$$\xi(V(x))\delta(V(x))|L_g L_G V(x)|^2 \ \leq \ \frac{1}{2}N(|x|) \quad (25)$$

we obtain $\dot{U} \leq -\frac{1}{2}\delta(V(x))N(|x|) < 0$ for all $x \neq 0$.

Third Case. $|L_g V(x)| \geq \Gamma(|x|)$. Then the inequality (19) implies that

$$\begin{array}{rcl} \dot{U} & \leq & -\frac{1}{2}\xi(V(x))\Gamma^2(|x|) \ + \ \delta(V(x))L_f L_G V(x) \\ & & +\xi(V(x))\delta^2(V(x))|L_g L_G V(x)|^2 \ . \end{array}$$

Choosing $\delta(\cdot)$ such that

$$\begin{array}{rcl} \delta(V(x))L_f L_G V(x) & \leq & \frac{1}{8}\Gamma^2(|x|)\xi(V(x)) \\ \delta^2(V(x)|L_g L_G V(x)|^2 & \leq & \frac{1}{8}\Gamma^2(|x|) \ , \end{array} \quad (26)$$

we obtain $\dot{U} \leq -\frac{1}{4}\xi(V(x))\Gamma^2(|x|) < 0$ for all $x \neq 0$.

Fifth step. To conclude the proof, one has to prove that one can determine continuously differentiable and positive definite functions $\delta(\cdot)$ simultaneously satisfying the requirements (13), (17), (23), (25), (26). This can be done by first finding a C^1 positive definite function δ satisfying the requirements (13), (23), (25), (26) that is increasing on $[0, 1]$ and non-increasing on $[1, \infty)$, and then minorizing the result so that it also satisfies (17). The lengthy but easy proof of this fact is left to the reader. △

VI. Example

We illustrate Theorem 1 by applying it to the two-dimensional nonlinear system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \ , \\ \dot{x}_2 & = & -x_1^3 + u \ . \end{array} \right. \quad (27)$$

For this system, with our general notation, we have

$$f(x_1, x_2) = \begin{pmatrix} x_2 \\ -x_1^3 \end{pmatrix} \ , \ g(x_1, x_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \quad (28)$$

Let us check that (27) satisfies Assumptions H1 and H2.

i) The positive definite radially unbounded function

$$V(x_1, x_2) \ = \ \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \, , \quad (29)$$

while not a CLF for (27), satisfies $L_f V(x) = 0$ on $\mathbb{R}^2$. Therefore Assumption H1 is satisfied.

ii) Consider the vector field

$$G(x_1, x_2) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \ . \quad (30)$$

One can check readily that

$$\begin{array}{c} L_g V(x_1, x_2) = x_2 \ , \ L_G V(x_1, x_2) = x_1 x_2 \\ L_f L_G V(x_1, x_2) = x_2^2 - x_1^4 \ . \end{array} \quad (31)$$

If $L_g V(x_1, x_2) = 0$ and $(x_1, x_2) \neq (0, 0)$, then $x_2 = 0$ and $x_1 \neq 0$, so $L_f L_G V(x_1, 0) = -x_1^4 < 0$. Therefore Assumption H2 is satisfied. It follows that Theorem 1 applies to the system (27).

We apply our CLF construction using

$$\Gamma(s) \ = \ \frac{1}{4}\frac{s^2}{1+s^2} \ , \ N(s) \ = \ \frac{1}{4}s^4 \ . \quad (32)$$

In this case, it suffices to prove that

$$\begin{array}{c} |L_g V(x_1, x_2)| \leq \Gamma(|(x_1, x_2)|) \\ \Rightarrow L_f L_G V(x_1, x_2) \ \leq \ -N(|(x_1, x_2)|) \ . \end{array} \quad (33)$$

According to (31) and (32), property (33) is satisfied if

$$|x_2| \leq \frac{x_1^2 + x_2^2}{4(1 + x_1^2 + x_2^2)} \Rightarrow x_2^2 - x_1^4 \leq -\frac{1}{4}(x_1^2 + x_2^2)^2. \quad (34)$$

The inequality $|x_2| \ \leq \ \frac{x_1^2+x_2^2}{4(1+x_1^2+x_2^2)}$ implies

$$\begin{array}{rcl} x_2^4 + x_2^2 + \frac{1}{4}(x_1^2 + x_2^2)^2 & \leq & \frac{(x_1^2 + x_2^2)^4}{[4(1 + x_1^2 + x_2^2)]^4} \\ & & +\frac{(x_1^2 + x_2^2)^2}{[4(1 + x_1^2 + x_2^2)]^2} \\ & & +\frac{1}{4}(x_1^2 + x_2^2)^2 \end{array}$$

which implies $x_2^4 + x_2^2 + \frac{1}{4}(x_1^2 + x_2^2)^2 \leq x_1^4 + x_2^4$ and therefore (34) holds.

Next we find a positive definite function $\delta(\cdot)$ such that

$$\begin{array}{rcl} U(x) & = & V(x) + \delta(V(x))L_G V(x) \\ & = & \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \delta\left(\frac{1}{4}x_1^4 + \frac{1}{2}x_2^2\right) x_1 x_2 \end{array} \quad (35)$$

is a CLF for the system (27) that admits a negative definite derivative along the trajectories of (27) in closed-loop with

$$u \ = \ -L_g V(x)^\top \ = \ -x_2 \ . \quad (36)$$

Let us prove that these properties are satisfied in particular with the function

$$\delta(v) \ = \ \frac{v^2}{8(1+v)^2} \ . \quad (37)$$

First observe that

$$\frac{1}{2}x_1^2 \le 1 + \frac{1}{4}x_1^4, \text{ so } |x_1x_2| \le 1 + V(x) \ \ \forall x \in \mathbb{R}^2 \quad (38)$$

and therefore choosing (37) in (35) gives

$$\begin{aligned} U(x) &\ge \tfrac{1}{4}x_1^4 + \tfrac{1}{2}x_2^2 \\ &\quad -\frac{\left(\frac{1}{4}x_1^4+\frac{1}{2}x_2^2\right)^2}{8\left(1+\frac{1}{4}x_1^4+\frac{1}{2}x_2^2\right)^2}\left(1+\frac{1}{4}x_1^4+\frac{1}{2}x_2^2\right) \\ &= \tfrac{1}{4}x_1^4 + \tfrac{1}{2}x_2^2 - \frac{\left(\frac{1}{4}x_1^4+\frac{1}{2}x_2^2\right)^2}{8\left(1+\frac{1}{4}x_1^4+\frac{1}{2}x_2^2\right)} \\ &\ge \tfrac{1}{8}x_1^4 + \tfrac{1}{4}x_2^2 \ . \end{aligned}$$

The derivative of $U(x)$ along the trajectories of (27) in closed-loop with the feedback (36) is

$$\begin{aligned} \dot{U} &= -x_2^2\left[1+\frac{V(x)}{4(1+V(x))^3}x_1x_2\right] \\ &\quad +\delta(V(x))[-x_1^4 - x_1x_2 + x_2^2] \\ &\le -\frac{5}{8}x_2^2 - \delta(V(x))x_1^4 - \delta(V(x))x_1x_2 \\ &\le -\frac{3}{8}x_2^2 - \delta(V(x))x_1^4 + \delta^2(V(x))x_1^2 \\ &\le -\frac{3}{8}x_2^2 - \delta(V(x))x_1^4 + \delta(V(x))\frac{\left(\frac{1}{4}x_1^4+\frac{1}{2}x_2^2\right)x_1^2}{8\left(1+\frac{1}{4}x_1^4+\frac{1}{2}x_2^2\right)} \\ &\le -\frac{1}{4}x_2^2 - \frac{1}{4}\delta(V(x))x_1^4, \end{aligned} \quad (39)$$

where we used (38) to get the first and last inequalities, and the second inequality used $(\delta(V(x))x_1 + \frac{1}{2}x_2)^2 \ge 0$. The right hand side of this inequality is negative definite. This proves our assertion. △

VII. Robustness to Actuator Errors

Theorem 1 provided a smooth stabilizing feedback law $u = K_1(x)$ for (3) such that $\dot{x} = f(x) + g(x)K_1(x)$ is globally asymptotically stable (GAS) to $x = 0$, meaning, there exists $\beta \in \mathcal{KL}$ such that $|x(t)| \le \beta(|x(0)|, t)$ for all trajectories $x(t)$ of the closed loop system and all $t \ge 0$. Moreover, for each $\varepsilon > 0$, we can choose K_1 to satisfy $|K_1(x)| \le \varepsilon$ for all $x \in \mathbb{R}^n$.

One natural and widely used generalization of the GAS condition is the so-called input-to-state stable (ISS) property, as introduced by Sontag in his seminal paper [23]. For a general nonlinear system $\dot{x} = F(x, d)$ evolving on $\mathbb{R}^n \times \mathbb{R}^m$ (where d represents the disturbance), the ISS property is the requirement that there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the following holds for all measurable essentially bounded functions $\mathbf{d} : [0, \infty) \to \mathbb{R}^m$ and corresponding trajectories $x(t)$ for $\dot{x}(t) = F(x(t), \mathbf{d}(t))$:

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(|\mathbf{d}|_\infty) \ \ \forall t \ge 0. \quad \text{(ISS)}$$

Here $|\cdot|_\infty$ is the essential supremum norm.

The ISS property reduces to GAS to 0 for systems with no controls, in which case the overshoot term $\gamma(|\mathbf{d}|_\infty)$ in the ISS decay condition is 0; see also [15], [16] for the relationship between the ISS property and asymptotic controllability. It is therefore natural to look for a feedback $K : \mathbb{R}^n \to \mathbb{R}^m$ for (3) (which could in principle differ from the feedback K_1 constructed above) for which

$$\dot{x} = F(x, d) := f(x) + g(x)[K(x) + d] \quad (40)$$

is ISS, and for which $|K(x)| \le \varepsilon$ for all $x \in \mathbb{R}^n$, where ε is any prescribed positive constant. In other words, we would want an arbitrarily small feedback K that renders (3) GAS to $x = 0$ and that has the additional property that (40) is also ISS with respect to actuator errors d.

However, it is clear that this objective cannot be met, since one can easily check that one cannot find a *bounded* feedback $K(x)$ such that the one-dimensional system $\dot{x} = K(x) + d$ is ISS. On the other hand, if we add

Assumption H3. There exists a positive nondecreasing smooth function h such that (i) $\int_0^{+\infty} 1/h(s)\,\mathrm{d}s = +\infty$ and (ii) $|L_gV(x)| \le h(V(x))$ for all $x \in \mathbb{R}^n$.

where V satisfies our continuing Assumptions H1-H2, then any feedback $K := -\xi(V(x))L_gV(x)^\top$ obtained from Theorem 1 and chosen such that $|\xi(V(x))L_gV(x)| \le \varepsilon$ for all $x \in \mathbb{R}^n$ also renders (40) *integral-input-to-state stable (iISS)*. For a general nonlinear system $\dot{x} = F(x, d)$ evolving on $\mathbb{R}^n \times \mathbb{R}^m$, the iISS condition is the following: There exist $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for all measurable locally essentially bounded functions $\mathbf{d} : [0, \infty) \to \mathbb{R}^m$ and corresponding trajectories $x(t)$ for $\dot{x}(t) = F(x(t), \mathbf{d}(t))$,

$$\alpha(|x(t)|) \le \beta(|x(0)|, t) + \int_0^t \gamma(|\mathbf{d}(s)|)\mathrm{d}s \ \ \forall t \ge 0. \quad \text{(iISS)}$$

The iISS condition reflects the qualitative property of having small overshoots when the disturbances have finite energy. It provides a nonlinear analog of "finite H^2 norm" for linear systems, and thus has obvious physical relevance and significance [1], [2], [3], [4]. The one-dimensional dynamic $F(x, d) = -\arctan(x) + d$ is an example that is iISS but not ISS [2]. Assumptions H1-H3 hold for our example in the previous section, since in that case, $|L_gV(x)| \le 2(V(x)+2)$ for all $x \in \mathbb{R}^n$, so we can take $h(s) = 2(s + 2)$. In fact, our assumptions hold for a broader class of Hamiltonian systems as well; see Remark 4 below.

To verify that the Theorem 1 feedback also renders (40) iISS, we begin by fixing $\varepsilon > 0$ and V satisfying our Assumptions H1-H3 for (3), and using the theorem to construct a CLF U for (3) corresponding to a positive function ξ that satisfies

$$|\xi(V(x))L_gV(x)| \le \varepsilon \ \ \forall x \in \mathbb{R}^n. \quad (41)$$

By reducing δ and δ' from the proof of the theorem, and replacing $h(p)$ with $p \mapsto h(2p) + 1$, we can assume that

$$|L_gU(x)| \le h(U(x)) \ \ \forall x \in \mathbb{R}^n. \quad (42)$$

It is then easy to check that

$$\begin{aligned} \tilde{U}(x) &= \textstyle\int_0^{U(x)} \frac{\mathrm{d}p}{h(p)}, \text{ where} \\ U(x) &= V(x) + \delta(V(x))L_GV(x) \end{aligned} \quad (43)$$

is again a CLF for our dynamic (3), since our choice of h gives $\tilde{U}(x) \to +\infty$ as $|x| \to \infty$ because U is radially unbounded, and because $\nabla\tilde{U}(x) \equiv \nabla U(x)/h(U(x))$ (which gives the Lyapunov decay condition). The smoothness of $\tilde{U}$ follows because U and h are both smooth. Finally, (42) gives

$$|L_g\tilde{U}(x)| = |L_gU(x)/h(U(x))| \leq 1 \quad \forall x. \tag{44}$$

Next choose $K_1(x) = -\xi(V(x))L_gV(x)^\top$, where ξ is a smooth positive function satisfying (41), so K_1 renders (3) GAS to $x = 0$, by Theorem 1. To check that $K(x) := K_1(x)$ also renders (40) iISS, notice that our choice of K_1 gives

$$\begin{aligned} &\nabla\tilde{U}(x)F(x,d) \\ &= \nabla\tilde{U}(x)[f(x) + g(x)K_1(x)] + L_g\tilde{U}(x)d \\ &\leq -\alpha_3(|x|) + |L_g\tilde{U}(x)|\,|d| \ \leq\ -\alpha_3(|x|) + |d| \end{aligned} \tag{45}$$

for all x and d for some continuous positive definite function α_3. This decay estimate differs from the usual ISS decay condition because α_3 need not be radially unbounded and in fact is *not* sufficient for ISS of (40) [2]. However, (45) says (see [2]) that the positive definite radially unbounded smooth function $\tilde{U}$ is an iISS-CLF for (40). Therefore, the fact that (40) is iISS now follows from the iISS Lyapunov characterization [2, Theorem 1]. We conclude as follows:

Corollary 3: Let (3) satisfy Assumptions H1-H3 for some vector field $G : \mathbb{R}^n \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$, and let $\varepsilon > 0$ be given. Then there exist two smooth functions $\delta, \xi : [0,\infty) \to [0,\infty)$ such that (i) the system (40) with the feedback $K(x) := -\xi(V(x))L_gV(x)$ is iISS and has a smooth iISS-CLF of the form (43) and (ii) $|K(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^n$.

Remark 4: Assume the Hamiltonian system (9) satisfies the conditions (a)-(b) we introduced in Section IV as well as the following additional condition: (c) There exist $\underline{\lambda}, \bar{\lambda} > 0$ such that $\text{spectrum}\{M^{-1}(q)\} \subseteq [\underline{\lambda}, \bar{\lambda}]$ for all q. (Assumption (c) means there are *positive* constants $\underline{c}$ and $\bar{c}$ such that $\underline{c}|p|^2 \leq p^\top M(q)p \leq \bar{c}|p|^2$ for all q and p.) Then (9) satisfies our Assumptions H1-H3 and so is covered by the preceding corollary. In fact, we saw in Section IV that (a)-(b) imply that Assumptions H1-H2 hold with $V = H$, and then Assumption H3 follows from (c) because $|L_gV(x)|^2 = |H_p(q,p)|^2 = |p^\top M^{-1}(q)|^2 \leq \bar{\lambda}^2|p|^2 \leq (\bar{\lambda}^2/\underline{\lambda})p^\top M^{-1}(q)p \leq 2(\bar{\lambda}^2/\underline{\lambda})V(x)$ for all states $x = (q,p)$, so we can satisfy the requirements of Assumption H3 using $h(s) = \sqrt{2(\bar{\lambda}^2/\underline{\lambda})(s+1)}$.

VIII. Conclusion

We showed how to construct control-Lyapunov functions for systems satisfying the Jurdjevic-Quinn conditions. We also constructed feedbacks of arbitrarily small norm that render our systems integral-input-to-state stable to actuator errors. Our constructions apply to important families of nonlinear systems, and in particular to systems described by Euler-Lagrange equations. Redesign and further robustness analysis for these systems via our construction of control-Lyapunov functions will be subjects of future work.

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