Stabilization and Robustness Analysis for a Chemostat Model with Two Species and Monod Growth Rates Via a Lyapunov Approach

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Control of Biological Systems Session ThC01.6
46th IEEE Conference on Decision and Control – 12/13/07
Hilton New Orleans Riverside, The Big Easy, LA

OUTLINE

- Background and Objectives
- Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Robustness to Disturbances
- Numerical Validation
- Conclusions and Further Research

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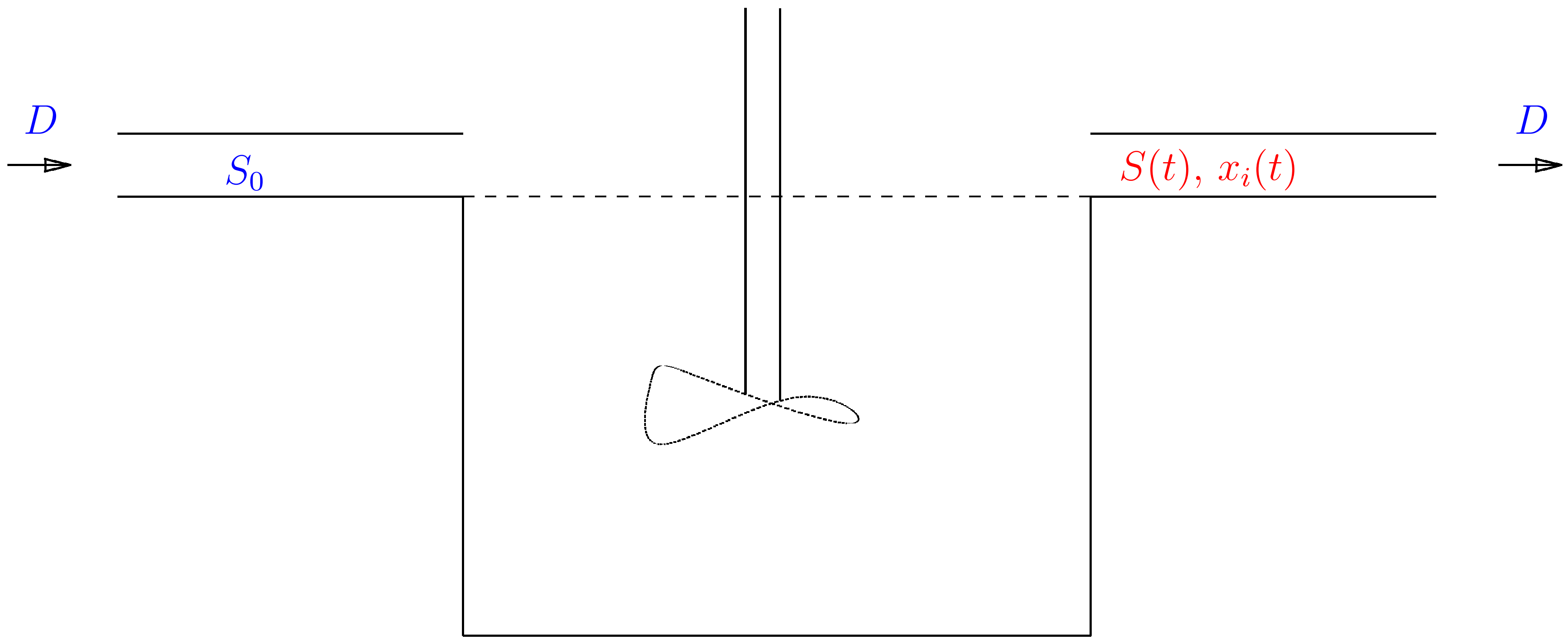
CHEMOSTAT SET-UP



Feed Vessel $\longrightarrow$ Culture Vessel $\longrightarrow$ Collecting Receptacle

BACKGROUND and GOAL

Basic Model: The two-species chemostat with nutrient concentration $S(t)$ and organism concentrations $X_i(t)$ evolving on $\mathcal{X} := (0, \infty)^3$ is

$$\begin{cases} \dot{S} &= D[S_0 - S] - \frac{\mu_1(S)}{\mathcal{Y}_1} X_1 - \frac{\mu_2(S)}{\mathcal{Y}_2} X_2 \,, \\ \dot{X}_i &= [\mu_i(S) - D] X_i \,, \quad i = 1, 2 \end{cases}$$

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$D(\cdot)$ = dilution rate. $S_0(\cdot)$ = input nutrient concentration. $\mathcal{Y}_i$ = yield. $\mu_i(S) = \frac{K_i S}{L_i + S}$ = (Monod) uptake function, with $K_i, L_i > 0$ constants.

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Importance: bioengineering, ecology, population biology

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Goal: Given any $X_{i*} > 0$, design S_0 and $D(\cdot)$, depending only on $Y = X_1 + AX_2$ (where A is a given positive constant), that render $(S_*, X_{1*}, X_{2*}) \in \mathcal{X}$ robustly GAS.

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Competitive Exclusion: When $S_0(\cdot)$ and D are constant and the μ_i's are increasing, at most one species survives.

OVERVIEW of LITERATURE

Coexistence: In real ecological systems, $n > 1$ species can coexist on 1 substrate, so much of the literature aims at choosing S_0 and/or D to force coexistence.

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Feedback Controls: De Leenheer-Smith (*JMB*'03) generated a coexistence equilibrium for $n = 2, 3$. See Mazenc-M-Harmand (*ACC'07*, *TCAS'08*) for $n = 2$ with explicit Lyapunov functions and tracking of oscillations.

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Feedback Linearization: Ballyk-Barany (ACC'07, CDC'07, ECOMOD'08), $n = 2$.

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Outputs: De Leenheer-Smith and Gouzé-Robledo (*IJRNC*'06..) stabilized chemostats where only $X_1 + X_2$ or S is known. Did not use ISS.

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STANDING ASSUMPTION

$\exists\ S_* > 0$ such that (i) $\mu_1(S_*) = \mu_2(S_*)$, (ii) $\mu_2(S) < \mu_1(S)$ if $0 < S < S_*$, and (iii) $\mu_2(S) > \mu_1(S)$ if $S > S_*$.

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I.e., μ_2 crosses over μ_1 exactly once and then stays higher.

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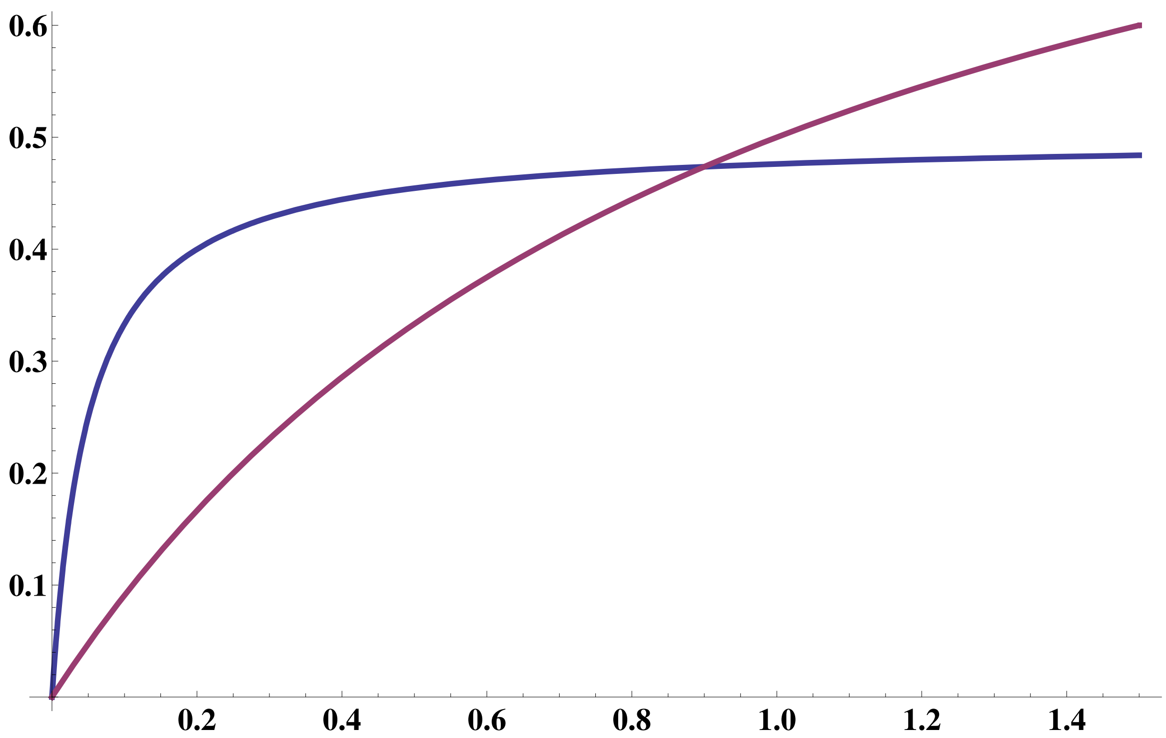

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See proceedings for the explicit construction of $\bar{\varepsilon} > 0$ and β.

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Remark: $\mathcal{KL}$: Means (1) $\beta(\cdot, t) \in \mathcal{K}_\infty$ $\forall t \geq 0$ and (2) $\forall r \geq 0$, $\beta(r, \cdot)$ is non-increasing and $\beta(r, t) \to 0$ as $t \to +\infty$.

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Remark: *Cannot* pick $\varepsilon = 0$.

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Simpler than Mazenc-M-Harmand (*ACC'07*, *TCAS'08*), outputs, robust stability, explicit strict Lyapunov function.

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Using a suitable bound $\bar{\Delta}$ on $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2)$, we can design $\beta \in \mathcal{KL}$, $\alpha \in \mathcal{K}_\infty$ so that along the trajectories of

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the errors satisfy an iISS [Sontag, 1998] estimate of the form

$$\alpha(|(\Sigma, \xi_1, \xi_2)(t)|) \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t) + \int_0^t |\mathbf{d}(r)|\mathrm{d}r.$$

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In the special case where $\mathbf{d}_2 \equiv 0$, we get iISS if

$$\bar{\Delta} = \frac{0.16\mu_1(S_*)S_*}{\mu_1(S_*) + \varepsilon|a - 1|}.$$

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Further reducing $\bar{\Delta}$ gives usual ISS [Sontag, 1989] estimate

$$|(\Sigma, \xi_1, \xi_2)(t)| \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t) + \gamma(|\mathbf{d}|_\infty).$$

REMARKS on ROBUSTNESS ESTIMATES

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- The dilution rate D is proportional to the speed of the pump that supplies the fresh nutrient and so is prone to variability that can be modeled by $D(y) + \mathbf{d}_2$.
- Our (i)ISS proof uses the Lyapunov function $\mathcal{V}$ to explicitly construct β, γ, and α from the (i)ISS estimates. Hence, we can precisely quantify the overshoot from $\mathbf{d}$. New: (i)ISS for 2 species chemostat.

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- ISS implies persistence, since if e.g. $x_2(t) \to 0$ as $t \to t_*^-$, then $\xi_2(t) = \ln(x_2(t)) - \ln(x_{2*}) \to -\infty$, contrary to the ISS bound on $|(\Sigma, \xi_1, \xi_2)(t)|$.
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- Similarly, iISS implies persistence when $\mathbf{d}$ is integrable.

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EXAMPLE

$$\begin{cases} \dot{S} = (D(y)+\mathbf{d}_2)(S_0+\mathbf{d}_1-S)-\frac{0.05Sx_1}{20+S}-\frac{.052Sx_2}{25+S} \\ \dot{x}_1 = \left[\frac{.05S}{20+S} - D(y) - \mathbf{d}_2\right] x_1 \\ \dot{x}_2 = \left[\frac{.052S}{25+S} - D(y) - \mathbf{d}_2\right] x_2 \end{cases}$$

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Choose $y = x_1 + 0.8x_2$, and $x_{1*} = 0.05$ and $x_{2*} = 0.02$.

- Our assumptions hold with $S_* = 105$, $\varepsilon \in (0, .00753]$, $S_0 = 105.07$, and $D(y) = .042 + 0.001506\sigma(y - 0.066)$. Hence, all closed loop trajectories converge to $(105, 0.05, 0.02)$ when $\mathbf{d} = 0$.

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- When $\mathbf{d}_1 \equiv 0$, we get iISS to disturbances $\mathbf{d}_2(t)$ bounded by $\bar{\Delta} \approx 0.20\mu_1(S_*)$ i.e. about 20% of D.

EXAMPLE

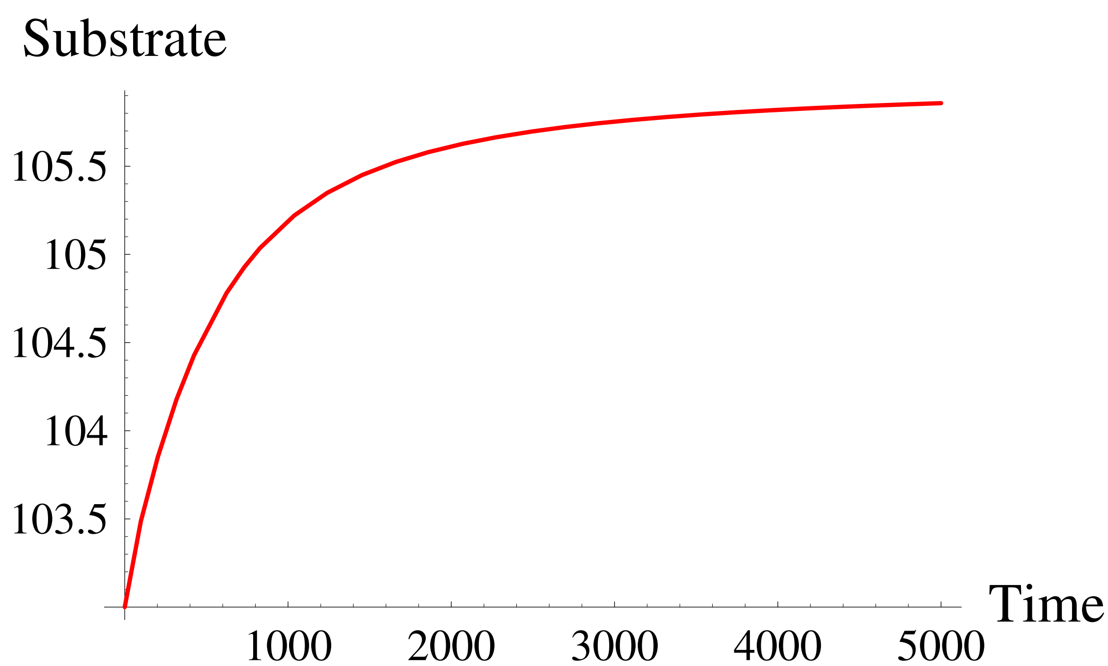
$$\begin{cases} \dot{S} = (D(y)+\mathbf{d}_2)(S_0+\mathbf{d}_1-S) - \frac{0.05Sx_1}{20+S} - \frac{.052Sx_2}{25+S} \\ \dot{x}_1 = \left[\frac{.05S}{20+S} - D(y) - \mathbf{d}_2\right] x_1 \\ \dot{x}_2 = \left[\frac{.052S}{25+S} - D(y) - \mathbf{d}_2\right] x_2 \end{cases}$$

Choose $y = x_1 + 0.8x_2$, and $x_{1*} = 0.05$ and $x_{2*} = 0.02$.

- Our assumptions hold with $S_* = 105$, $\varepsilon \in (0, .00753]$, $S_0 = 105.07$, and $D(y) = .042 + 0.001506\sigma(y - 0.066)$. Hence, all closed loop trajectories converge to $(105, 0.05, 0.02)$ when $\mathbf{d} = 0$.
- If instead $\mathbf{d}_2 \equiv 0$, then we have iISS to disturbances $\mathbf{d}_1(t)$ bounded by $\bar{\Delta} \approx 16$, or about 15% of $S_0 = 105.07$.

SIMULATION of EXAMPLE

We used $\mathbf{d}(t) \equiv (1,0)$ and $(S, x_1, x_2)(0) = (103, 2, 1)$.



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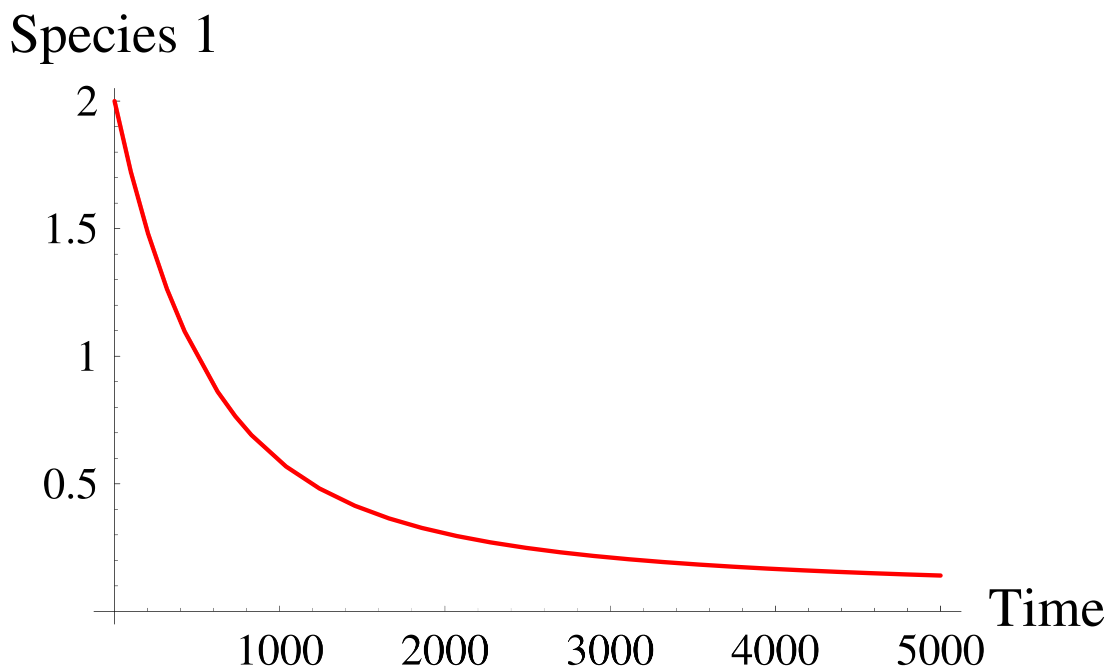

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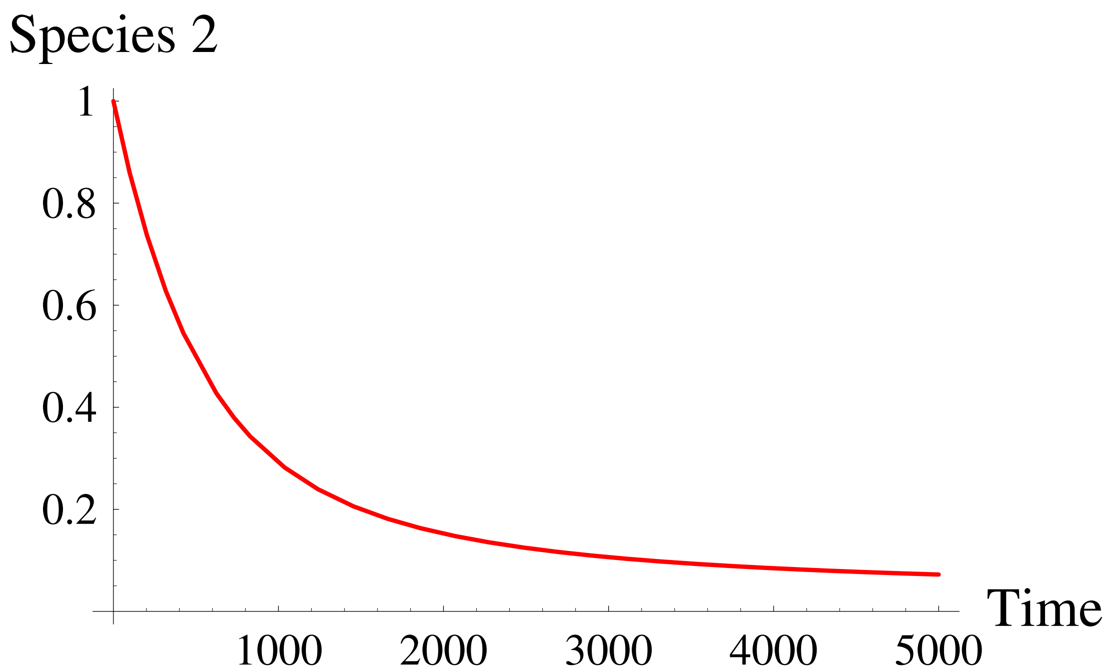

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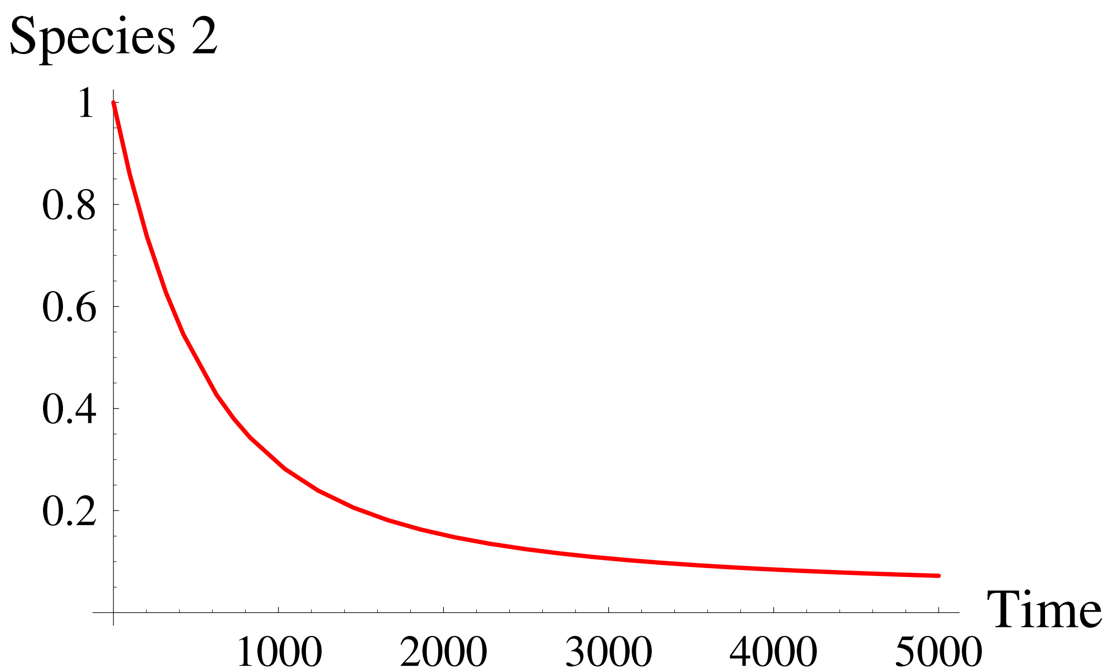

Persistence. $(S(t), x_1(t), x_2(t)) \to (105, 0.05, 0.02)$, but with an overshoot determined by iISS and the magnitude of $\mathbf{d}_1$.

OUTLINE

- Background and Objectives
- Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Robustness to Disturbances
- Numerical Validation
- Conclusions and Further Research

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- We can extend our Lyapunov function and robustness analysis to allow uncertain uptake functions and measurement noise in D.
- The novelty is in our explicit strict Lyapunov function, which made it possible to precisely quantify the effects of actuator errors using ISS.

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- The authors thank Patrick De Leenheer for illuminating discussions and the NSF for support for this work under DMS grant 0424011.