Stabilization and Robustness Analysis for a Chemostat Model with Two Species and Monod Growth Rates Via a Lyapunov Approach

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• Background and Objectives
• Main Stability Theorem
• Proof Ideas: Explicit Lyapunov Function
• Robustness to Disturbances
• Numerical Validation
• Conclusions and Further Research
OUTLINE

- Background and Objectives
- Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Robustness to Disturbances
- Numerical Validation
- Conclusions and Further Research
CHEMOSTAT SET-UP

Feed Vessel → Culture Vessel → Collecting Receptacle
Basic Model: The two-species chemostat with nutrient concentration $S(t)$ and organism concentrations $X_i(t)$ evolving on $\mathcal{X} := (0, \infty)^3$ is

$$\begin{align*}
\dot{S} &= D[S_0 - S] - \frac{\mu_1(S)}{\mathcal{Y}_1} X_1 - \frac{\mu_2(S)}{\mathcal{Y}_2} X_2, \\
\dot{X}_i &= [\mu_i(S) - D] X_i, \quad i = 1, 2
\end{align*}$$
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\end{align*}$$

$D(\cdot) = \text{dilution rate.} \ S_0(\cdot) = \text{input nutrient concentration.} \ Y_i = \text{yield.} \ \mu_i(S) = \frac{K_i S}{L_i + S} = \text{(Monod) uptake function, with} \ K_i, L_i > 0 \ \text{constants.}$
BACKGROUND and GOAL

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$D(\cdot) =$ dilution rate. $S_0(\cdot) =$ input nutrient concentration. $Y_i =$ yield. $\mu_i(S) = \frac{K_i S}{L_i + S} =$ (Monod) uptake function, with $K_i, L_i > 0$ constants.

Importance: bioengineering, ecology, population biology
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$D(\cdot) =$ dilution rate. $S_0(\cdot) =$ input nutrient concentration. $\mathcal{Y}_i =$ yield. $\mu_i(S) = \frac{K_i S}{L_i + S} =$ (Monod) uptake function, with $K_i, L_i > 0$ constants.

Goal: Given any $X_{i*} > 0$, design $S_0$ and $D(\cdot)$, depending only on $Y = X_1 + A X_2$ (where $A$ is a given positive constant), that render $(S_*, X_{1*}, X_{2*}) \in \mathcal{X}$ robustly GAS.
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**Competitive Exclusion:** When $S_0(\cdot)$ and $D$ are constant and the $\mu_i$’s are increasing, at most one species survives.
**OVERVIEW of LITERATURE**

**Coexistence:** In real ecological systems, \( n > 1 \) species can **coexist** on 1 substrate, so much of the literature aims at choosing \( S_0 \) and/or \( D \) to force coexistence.
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Feedback Controls: De Leenheer-Smith (JMB’03) generated a coexistence equilibrium for $n = 2, 3$. See Mazenc-M-Harmand (ACC’07, TCAS’08) for $n = 2$ with explicit Lyapunov functions and tracking of oscillations.
Feedback Linearization: Ballyk-Barany (ACC’07, CDC’07, ECOMOD’08), $n = 2$. 
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Input-to-State Stability: Mazenc-M-De Leenheer (CDC’06, MBE’07) – explicit strict Lyapunov functions, one species case, (i)ISS tracking to actuator errors.
Feedback Linearization: Ballyk-Barany (ACC’07, CDC’07, ECOMOD’08), $n = 2$.

Input-to-State Stability: Mazenc-M-De Leenheer (CDC’06, MBE’07) – explicit strict Lyapunov functions, one species case, (i)ISS tracking to actuator errors.

Outputs: De Leenheer-Smith and Gouzé-Robledo (IJRNC’06..) stabilized chemostats where only $X_1 + X_2$ or $S$ is known. Did not use ISS.
OUTLINE

- Background and Objectives
- **Main Stability Theorem**
- Proof Ideas: Explicit Lyapunov Function
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STANDING ASSUMPTION

$\exists S_* > 0$ such that (i) $\mu_1(S_*) = \mu_2(S_*)$, (ii) $\mu_2(S) < \mu_1(S')$ if $0 < S < S_*$, and (iii) $\mu_2(S) > \mu_1(S)$ if $S > S_*$. 
\[ \exists S_* > 0 \text{ such that (i) } \mu_1(S_*) = \mu_2(S_*) \text{, (ii) } \mu_2(S) < \mu_1(S') \text{ if } 0 < S < S_* \text{, and (iii) } \mu_2(S) > \mu_1(S) \text{ if } S > S_* . \]

I.e., \( \mu_2 \) crosses over \( \mu_1 \) exactly once and then stays higher.
STANDING ASSUMPTION

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Example: Take \( \mu_1(S') = \frac{0.5S}{0.05+S} \) and \( \mu_2(S) = \frac{S}{1+S} \).
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STANDING ASSUMPTION

\[ \exists S^* > 0 \text{ such that (i) } \mu_1(S^*) = \mu_2(S^*), \text{ (ii) } \mu_2(S) < \mu_1(S) \text{ if } 0 < S < S^*, \text{ and (iii) } \mu_2(S) > \mu_1(S) \text{ if } S > S^*. \]

Example: Take \( \mu_1(S) = \frac{0.5S}{0.05+S} \) and \( \mu_2(S) = \frac{S}{1+S} \). \( S^* = 0.9 \).
Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/Y_i$, $y = x_1 + ax_2$, $a = AY_2/Y_1$. 
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Fix any \( x_{i*} > 0 \). Errors: \( \xi_i = \ln(x_i/x_{i*}) \) and \( \Sigma = \ln(S/S_*) \).
Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = \frac{X_i}{Y_i}$, $y = x_1 + ax_2$, $a = A\frac{Y_2}{Y_1}$.

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**Theorem 1:** Assume $\varepsilon \in (0, \bar{\varepsilon}]$ and $a \neq 1$. Then $(S_*, x_{1*}, x_{2*})$ is a GAS equilibrium for the $(S, x_1, x_2)$ dynamics when

$$S_0 = S_* + x_{1*} + x_{2*}$$

$$D(y) = \mu_1(S_*) - \varepsilon(a - 1)\sigma(y - x_{1*} - ax_{2*}).$$
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**Theorem 1:** Assume \( \varepsilon \in (0, \bar{\varepsilon}] \) and \( a \neq 1 \). Then \( (S_*, x_{1*}, x_{2*}) \) is a GAS equilibrium for the \((S, x_1, x_2)\) dynamics when

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S_0 = S_* + x_{1*} + x_{2*} \\
D(y) = \mu_1(S_*) - \varepsilon(a - 1)\sigma (y - x_{1*} - ax_{2*}) .
\]

More precisely, we can construct a function \( \beta \in \mathcal{KL} \) such that \( |(\Sigma, \xi_1, \xi_2)(t)| \leq \beta(\|(\Sigma, \xi_1, \xi_2)(0)\|, t) \) for all \( t \geq 0 \) along all trajectories \((S, x_1, x_2)(t)\) of the closed loop dynamics.
Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i / Y_i$, $y = x_1 + ax_2$, $a = AY_2 / Y_1$. Fix any $x_{i*} > 0$. Errors: $\xi_i = \ln(x_i / x_{i*})$ and $\Sigma = \ln(S / S_*)$.

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See proceedings for the explicit construction of $\bar{\varepsilon} > 0$ and $\beta$. 
Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/Y_i$, $y = x_1 + ax_2$, $a = AY_2/Y_1$. Fix any $x_{i*} > 0$. Errors: $\xi_i = \ln(x_i/x_{i*})$ and $\Sigma = \ln(S/S_*)$.

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Remark: $KL$: Means (1) $\beta(\cdot, t) \in K_\infty \forall t \geq 0$ and (2) $\forall r \geq 0$, $\beta(r, \cdot)$ is non-increasing and $\beta(r, t) \to 0$ as $t \to +\infty$. 
Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/Y_i$, $y = x_1 + ax_2$, $a = AY_2/Y_1$.

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**Remark:** Cannot pick $\varepsilon = 0$. 
Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/Y_i$, $y = x_1 + ax_2$, $a = AY_2/Y_1$.

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More precisely, we can construct a function $\beta \in KL$ such that $|(\Sigma, \xi_1, \xi_2)(t)| \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t)$ for all $t \geq 0$ along all trajectories $(S, x_1, x_2)(t)$ of the closed loop dynamics.

**Simpler** than Mazenc-M-Harmand *(ACC’07, TCAS’08)*, outputs, robust stability, explicit strict Lyapunov function.
Background and Objectives

Main Stability Theorem

Proof Ideas: Explicit Lyapunov Function

Robustness to Disturbances

Numerical Validation

Conclusions and Further Research
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ROBUSTNESS

Using a suitable bound $\bar{\Delta}$ on $d = (d_1, d_2)$, we can design $\beta \in \mathcal{KL}$, $\alpha \in \mathcal{K}_\infty$ so that along the trajectories of

$$\dot{S} = [D(y)+d_2](S_0+d_1-S) - \mu_1(S)x_1 - \mu_2(S)x_2$$
$$\dot{x}_i = [\mu_i(S) - D(y) - d_2]x_i, \quad i = 1, 2$$

the errors satisfy an iISS [Sontag, 1998] estimate of the form

$$\alpha(||(\Sigma, \xi_1, \xi_2)(t)||) \leq \beta(||(\Sigma, \xi_1, \xi_2)(0)||, t) + \int_0^t |d(r)| dr.$$
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\[
\alpha(||(\Sigma, \xi_1, \xi_2)(t)||) \leq \beta(||(\Sigma, \xi_1, \xi_2)(0)|, t) + \int_0^t |d(r)| dr.
\]

In the special case where $d_2 \equiv 0$, we get iISS if

\[
\bar{\Delta} = \frac{0.16\mu_1(S_*)S_*}{\mu_1(S_*) + \varepsilon|a - 1|}.
\]
Using a suitable bound \( \tilde{\Delta} \) on \( d = (d_1, d_2) \), we can design \( \beta \in K_L, \alpha \in K_\infty \) so that along the trajectories of

\[
\begin{align*}
\dot{S} &= \left[D(y) + d_2\right](S_0 + d_1 - S) - \mu_1(S)x_1 - \mu_2(S)x_2 \\
\dot{x}_i &= [\mu_i(S) - D(y) - d_2]x_i, \quad i = 1, 2
\end{align*}
\]

the errors satisfy an iISS [Sontag, 1998] estimate of the form

\[
\alpha(\| (\Sigma, \xi_1, \xi_2)(t) \|) \leq \beta(\| (\Sigma, \xi_1, \xi_2)(0) \|, t) + \int_0^t |d(r)| \, dr.
\]

Further reducing \( \tilde{\Delta} \) gives usual ISS [Sontag, 1989] estimate

\[
| (\Sigma, \xi_1, \xi_2)(t) | \leq \beta(\| (\Sigma, \xi_1, \xi_2)(0) \|, t) + \gamma(\| d \|_\infty).
\]
REMARKS on ROBUSTNESS ESTIMATES

\[ \alpha(\|(\Sigma, \xi_1, \xi_2)(t)\|) \leq \beta(\|(\Sigma, \xi_1, \xi_2)(0)\|, t) + \int_0^t |d(r)| dr. \]

\[ |(\Sigma, \xi_1, \xi_2)(t)| \leq \beta(\|(\Sigma, \xi_1, \xi_2)(0)\|, t) + \gamma(|d|_\infty). \]

\cdot

\cdot
REMARKS on ROBUSTNESS ESTIMATES

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\alpha((\Sigma, \xi_1, \xi_2)(t)) \leq \beta((\Sigma, \xi_1, \xi_2)(0), t) + \int_0^t |d(r)|dr.
\]

\[
|\Sigma, \xi_1, \xi_2)(t)| \leq \beta((\Sigma, \xi_1, \xi_2)(0), t) + \gamma(\|d\|_\infty).
\]

- The dilution rate \(D\) is proportional to the speed of the pump that supplies the fresh nutrient and so is prone to variability that can be modeled by \(D(y) + d_2\).
\[ \alpha(||(\Sigma, \xi_1, \xi_2)(t)||) \leq \beta(||(\Sigma, \xi_1, \xi_2)(0)||, t) + \int_0^t |d(r)|dr. \]

\[ ||(\Sigma, \xi_1, \xi_2)(t)|| \leq \beta(||(\Sigma, \xi_1, \xi_2)(0)||, t) + \gamma(|d|_\infty). \]

- The dilution rate \( D \) is proportional to the speed of the pump that supplies the fresh nutrient and so is prone to variability that can be modeled by \( D(y) + d_2 \).

- Our (i)ISS proof uses the Lyapunov function \( \mathcal{V} \) to explicitly construct \( \beta, \gamma, \) and \( \alpha \) from the (i)ISS estimates. Hence, we can precisely quantify the overshoot from \( d \).

\textbf{New}: (i)ISS for 2 species chemostat.
\[ \alpha((\Sigma, \xi_1, \xi_2)(t))) \leq \beta(||(\Sigma, \xi_1, \xi_2)(0)||, t) + \int_0^t |d(r)|dr. \]

\[ ||(\Sigma, \xi_1, \xi_2)(t)|| \leq \beta(||(\Sigma, \xi_1, \xi_2)(0)||, t) + \gamma(|d|_\infty). \]

- ISS implies **persistence**, since if e.g. \( x_2(t) \to 0 \) as \( t \to t^- \), then \( \xi_2(t) = \ln(x_2(t)) - \ln(x_{2*}) \to -\infty \), contrary to the ISS bound on \( ||(\Sigma, \xi_1, \xi_2)(t)|| \).
REMARKS on ROBUSTNESS ESTIMATES

\[ \alpha(\|\Sigma, \xi_1, \xi_2\|(t)) \leq \beta(\|\Sigma, \xi_1, \xi_2\|(0), t) + \int_0^t |d(r)|dr. \]

\[ |\Sigma, \xi_1, \xi_2\|(t) \leq \beta(\|\Sigma, \xi_1, \xi_2\|(0), t) + \gamma(|d|_\infty). \]

- ISS implies persistence, since if e.g. \( x_2(t) \to 0 \) as \( t \to t^- \), then \( \xi_2(t) = \ln(x_2(t)) - \ln(x_{2*}) \to -\infty \), contrary to the ISS bound on \( |\Sigma, \xi_1, \xi_2\|(t) | \).

- Similarly, iISS implies persistence when \( d \) is integrable.
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EXAMPLE

\[
\begin{align*}
\dot{S} &= (D(y) + d_2)(S_0 + d_1 - S) - \frac{0.05Sx_1}{20+S} - \frac{.052Sx_2}{25+S} \\
\dot{x}_1 &= \left[\frac{0.05S}{20+S} - D(y) - d_2\right] x_1 \\
\dot{x}_2 &= \left[\frac{.052S}{25+S} - D(y) - d_2\right] x_2
\end{align*}
\]

Choose \( y = x_1 + 0.8x_2 \), and \( x_{1*} = 0.05 \) and \( x_{2*} = 0.02 \).
\[
\begin{aligned}
\dot{S} &= (D(y) + d_2)(S_0 + d_1 - S) - \frac{0.05Sx_1}{20+S} - \frac{0.052Sx_2}{25+S} \\
\dot{x}_1 &= \left[\frac{0.05S}{20+S} - D(y) - d_2\right] x_1 \\
\dot{x}_2 &= \left[\frac{0.052S}{25+S} - D(y) - d_2\right] x_2
\end{aligned}
\]

Choose \( y = x_1 + 0.8x_2 \), and \( x_{1*} = 0.05 \) and \( x_{2*} = 0.02 \).

- Our assumptions hold with \( S_* = 105 \), \( \varepsilon \in (0, 0.00753] \), \( S_0 = 105.07 \), and \( D(y) = 0.042 + 0.001506\sigma(y - 0.066) \).

Hence, all closed loop trajectories converge to \((105, 0.05, 0.02)\) when \( d = 0 \).
\[
\begin{aligned}
\dot{S} &= (D(y) + \mathbf{d}_2)(S_0 + \mathbf{d}_1 - S) - \frac{0.05Sx_1}{20+S} - \frac{0.052Sx_2}{25+S} \\
\dot{x}_1 &= \left[\frac{0.05S}{20+S} - D(y) - \mathbf{d}_2\right] x_1 \\
\dot{x}_2 &= \left[\frac{0.052S}{25+S} - D(y) - \mathbf{d}_2\right] x_2
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Choose \( y = x_1 + 0.8x_2 \), and \( x_{1*} = 0.05 \) and \( x_{2*} = 0.02 \).

- Our assumptions hold with \( S_* = 105, \varepsilon \in (0, .00753], S_0 = 105.07 \), and \( D(y) = .042 + 0.001506\sigma(y - 0.066) \). Hence, all closed loop trajectories converge to \((105, 0.05, 0.02)\) when \( \mathbf{d} = 0 \).

- When \( \mathbf{d}_1 \equiv 0 \), we get iISS to disturbances \( \mathbf{d}_2(t) \) bounded by \( \bar{\Delta} \approx 0.20\mu_1(S_*) \) i.e. about 20\% of \( D \).
\[
\begin{align*}
\dot{S} &= (D(y) + d_2)(S_0 + d_1 - S) - \frac{0.05Sx_1}{20+S} - \frac{0.052Sx_2}{25+S} \\
\dot{x}_1 &= \left[\frac{0.05S}{20+S} - D(y) - d_2\right]x_1 \\
\dot{x}_2 &= \left[\frac{0.052S}{25+S} - D(y) - d_2\right]x_2
\end{align*}
\]

Choose \(y = x_1 + 0.8x_2\), and \(x_{1*} = 0.05\) and \(x_{2*} = 0.02\).

- Our assumptions hold with \(S_* = 105\), \(\varepsilon \in (0, .00753]\), \(S_0 = 105.07\), and \(D(y) = .042 + 0.001506\sigma(y - 0.066)\). Hence, all closed loop trajectories converge to \((105, 0.05, 0.02)\) when \(d = 0\).

- If instead \(d_2 \equiv 0\), then we have iISS to disturbances \(d_1(t)\) bounded by \(\bar{\Delta} \approx 16\), or about \(15\%\) of \(S_0 = 105.07\).
SIMULATION of EXAMPLE

We used \( d(t) \equiv (1, 0) \) and \((S, x_1, x_2)(0) = (103, 2, 1)\).

![Graph showing substrate over time]
We used $d(t) \equiv (1, 0)$ and $(S, x_1, x_2)(0) = (103, 2, 1)$. 

![Graph showing the decline of Species 1 over time, with time values from 1000 to 5000 on the x-axis and species values on the y-axis, starting at 2 and decreasing to below 0.5.]
We used $d(t) \equiv (1, 0)$ and $(S, x_1, x_2)(0) = (103, 2, 1)$. 
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**Persistence.** $(S(t), x_1(t), x_2(t)) \rightarrow (105, 0.05, 0.02)$, but with an overshoot determined by iISS and the magnitude of $d_1$. 
• Background and Objectives
• Main Stability Theorem
• Proof Ideas: Explicit Lyapunov Function
• Robustness to Disturbances
• Numerical Validation
• Conclusions and Further Research
SUMMARY and SUGGESTIONS

- We provided output feedback for robustly stabilizing equilibria with arbitrary prescribed species concentrations in two species chemostats.
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We can extend our Lyapunov function and robustness analysis to allow uncertain uptake functions and measurement noise in $D$. 
SUMMARY and SUGGESTIONS

- We provided output feedback for robustly stabilizing equilibria with arbitrary prescribed species concentrations in two species chemostats.

- We can extend our Lyapunov function and robustness analysis to allow uncertain uptake functions and measurement noise in $D$.

- The novelty is in our explicit strict Lyapunov function, which made it possible to precisely quantify the effects of actuator errors using ISS.
SUMMARY and SUGGESTIONS

- It would be of interest to extend our work to tracking of prescribed oscillations. This would explain oscillatory behaviors observed in nature and suggest feedback mechanisms for achieving them.
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