# Stability and Control Design for Time-Varying Systems with Time-Varying Delays using a Trajectory-Based Approach

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Joint with Frederic Mazenc and Silviu-Iulian Niculescu

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Z. Artstein, I. Karafyllis, M. Krstic, S. Niculescu, P. Pepe, ...

# Our Contractiveness Lemma

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Lemma: Let $T^* > 0$ be a constant, $w : [-T^*, \infty) \to [0, \infty)$ admit a sequence $\{v_i\}$ and positive constants $\overline{v}_a$ and $\overline{v}_b$ such that $v_0 = 0$, $v_{i+i} - v_i \in [\overline{v}_a, \overline{v}_b]$ for all $i \geq 0$, $w$ be continuous on $[v_i, v_{i+1})$ for all $i \geq 0$ with finite left limits at each $v_i$, and $d : [0, \infty) \to [0, \infty)$ be piecewise continuous.

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Assume there is a constant $\rho \in (0, 1)$ such that

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Then

$$w(t) \leq |w|_{[-T^*,0]} e^{\frac{\ln(\rho)}{T^*} t} + \frac{1}{(1-\rho)^2} |d|_{[0,t]} \text{ for all } t \geq 0. \tag{2}$$

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Corollary: Let $X : [0,\infty) \to [0,\infty)$ be piecewise $C^1$ and admit constants $g \geq 0$, $a_s \geq 0$, $T > 0$, and $\delta \in (0,1)$ and piecewise continuous functions $a : [0,\infty) \to [-a_s,\infty)$, $b : [0,\infty) \to [0,\infty)$, and $\lambda : [0,\infty) \to [0,\infty)$ such that

$$\dot{X}(t) \leq -a(t)X(t) + b(t) \sup_{s\in[t-g,t]} X(s) + \lambda(t) \;\; \forall t \geq g \qquad (3)$$

and

$$e^{-\int_{t-T}^{t} a(m)\mathrm{d}m} + \int_{t-T}^{t} b(m) e^{-\int_{m}^{t} a(\ell)\mathrm{d}\ell} \mathrm{d}m \leq \delta \;\; \forall t \geq T+g. \qquad (4)$$

Assume that $\lim_{t\to p^-} X(t) \in \mathbb{R}$ at each $p \geq 0$. Then

$$X(t) \;\; \leq \;\; |X|_{[0,T+g]} e^{\frac{\ln(\delta)}{T+g}(t-T-g)} + \frac{Te^{Ta_s}|\lambda|_{[g,t]}}{(1-\delta)^2} \;\; \forall t \geq T+g. \qquad (5)$$

# Assumptions and Theorem

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Assumption 1: The functions $A$, $B$, and $h$ are bounded and piecewise continuous. Also, there exist a bounded piecewise continuous $p_1 : [0, \infty) \to \mathbb{R}$, constants $p_2 > 0$ and $p_3 > 0$, and a $C^1$ function $P : [0, \infty) \to \mathbb{R}^{n \times n}$ such that $P(t)$ is symmetric and positive definite for all $t$, such that $V(t, z) = z^\top P(t)z$ satisfies

$$p_2|z|^2 \leq V(t, z) \leq p_3|z|^2 \tag{7a}$$

$$V_t(t, z) + V_z(t, z)[A(t) + B(t)]z \leq -p_1(t)V(t, z) \tag{7b}$$

for all $t \geq 0$ and $z \in \mathbb{R}^n$.

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$$\dot{x}(t) = A(t)x(t) + B(t)x(t - h(t)) \tag{3}$$

Assumption 2: There exist $T > 0$ and $\delta \in (0, 1)$ such that

$$r(t) = \frac{2|P(t)B(t)|}{p_2} \int_{t-h(t)}^{t} [|A(m)| + |B(m)|] \, \mathrm{d}m, \tag{8}$$

is such that we have

$$e^{-\int_{t-T}^{t} p_1(m)\mathrm{d}m} + \int_{t-T}^{t} r(m) e^{-\int_m^t p_1(\ell)\mathrm{d}\ell} \mathrm{d}m \leq \delta \tag{GA}$$

for all $t \geq 2(T + |h|_\infty)$.

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- Decay rates $p_1(t)$ can take positive and negative values.
- Takes into account the case where $B(t)x(t - h(t))$ has a stabilizing effect.
- If $p_1$ is a nonnegative constant and $P$, $A$ and $B$ are constant, then (GA) simplifies to this averaging condition:

$$\sup_{t \geq 2(T+|h|_\infty)} \frac{1}{T} \int_{t-T}^{t} h(m)\mathrm{d}m \leq \frac{p_2}{2|PB|\left(|A| + |B|\right)} \frac{\delta - e^{-Tp_1}}{T} \tag{9}$$

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# Backup Slides to Use if Time Allows or Questions Warrant

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Two Examples From:
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# Example 1

$$\dot{x}(t) = -x(t) + b(t)x(t-1), \tag{10}$$

with $x \in \mathbb{R}$, and $b$ periodic of period 1 defined by

(i) $b(t) = 0$ when $t \in [0, c)$ and

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Our theorem applies with $V(x) = \frac{1}{2}x^2$ and $u = 0$.

# Example 2

Let $\ell > 0$ be arbitrary and $k \in \mathbb{N}$ be an odd integer and consider

$$\dot{x}(t) = -x(t - h(t)) \ , \tag{12}$$

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For each $T > 0$, there is a $k$ such that Assumption 2 is satisfied.

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