Controlling Human Heart Rate Response During Treadmill Exercise

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Hence, the existing linear models e.g. (Brodan, Hajek, & Kuhn, 1971) and (Cooper, Fletcher-Shaw, & Robertson, 1998) based on PID control may not be accurate.

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Model has been validated with human subjects. Unlike conventional linear models, it captures peripheral effects and is suitable for long duration exercise.

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$$\dot{\tilde{x}}_1 = -a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + a_2[u^2 - u_r(t)^2] \quad (3a)$$

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is globally exponentially stable to zero, i.e., there are constants $c_i > 0$ so that $|\tilde{x}(t)| \leq c_1 e^{-c_2 t} |\tilde{x}(0)|$ for all trajectories of (1).

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Knowing x_{1r} allows us to solve for x_{2r} from (2b).

$$x_{2r}(t) = x_{2r}(0)e^{-a_3 t} + a_4 \int_0^t e^{a_3(\tau - t)} \frac{x_{1r}(\tau)}{1 + b e^{-x_{1r}(\tau)}} \mathrm{d}\tau. \qquad (4)$$

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However, we need another condition to ensure trackability of x_r.

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Cheng et al. use the Levenberg-Marquardt method to estimate the a_i's.

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$$u_c(x,t) = \sqrt{\max\left\{0, u_r(t)^2 - \left(1 + \frac{R(\tilde{x}_1,t)}{P_\varepsilon}\right)\tilde{x}_2\right\}},$$
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Then $\dot{V} \leq -\sigma V$ along all trajectories of the closed loop system, where $\sigma = 2c_0/\max\{k,1\}$, $c_0 = \min\{-\lambda_{\max}, a_1, ka_3\}$, and $\lambda_{\max} < 0$ is the maximal eigenvalue of

$$M = \begin{bmatrix} -a_1 & a_2 \\ a_2 & -ka_3 \end{bmatrix}.$$

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Proposition. The $(\tilde{x}, \overline{x})$ dynamics in closed loop with (7) is globally exponentially stable to the origin.

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Proposition. The $(\tilde{x}, \overline{x})$ dynamics in closed loop with (7) is globally exponentially stable to the origin.

Proof:

Observer Design

Assume that $x_2(0)$ is unknown. Use

$$u_c(x_1,\hat{x}_2,t) = \sqrt{\max\left\{0, u_r(t)^2 - \left(1+\frac{R(\tilde{x}_1,t)}{P_\varepsilon}\right)\hat{x}_2\right\}}. \qquad (7)$$

The estimate $\hat{x}_2$ of $\tilde{x}_2$ is from the observer

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Simulations

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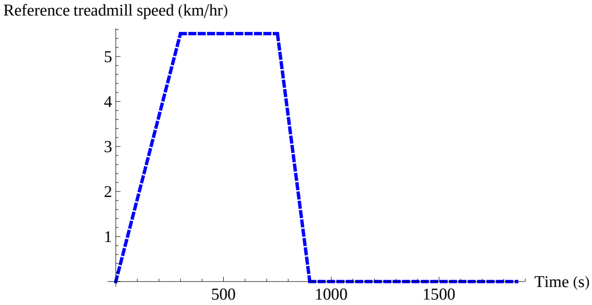

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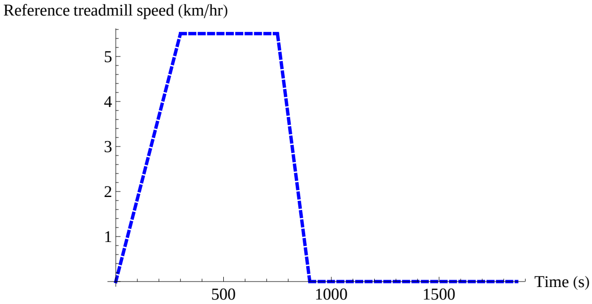

The resulting x_{1r} satisfies (SA) with $\varepsilon = 0.5$ so our results apply.

Tracking using State Feedback Control $u_c(x, t)$

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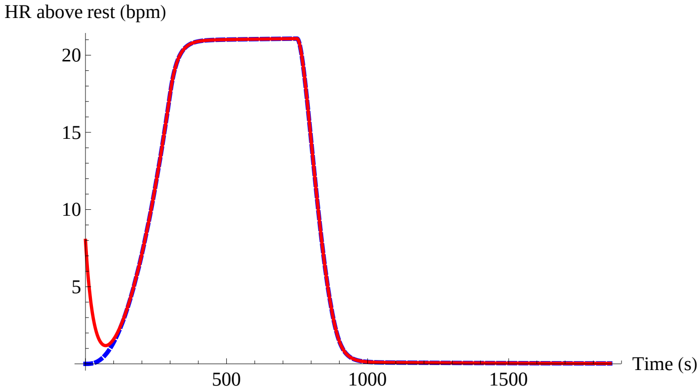

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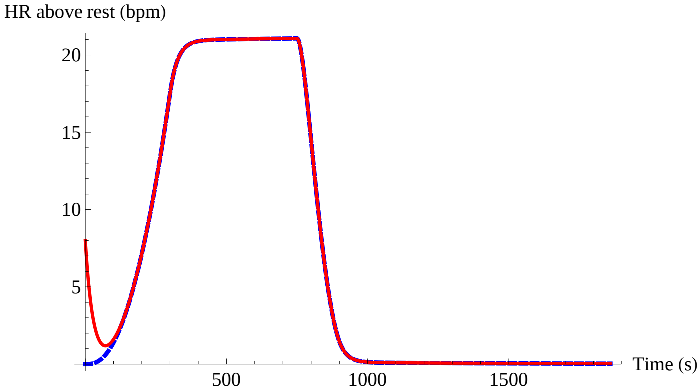

x_{1r} (blue and dashed) and state x_1 (red and solid).

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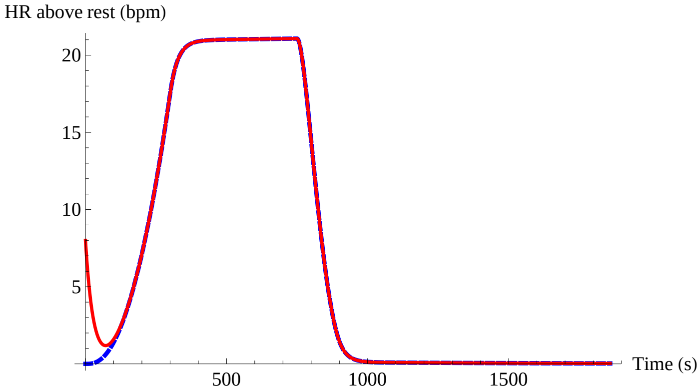

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Initial state: $x(0) = (2, 0)$.

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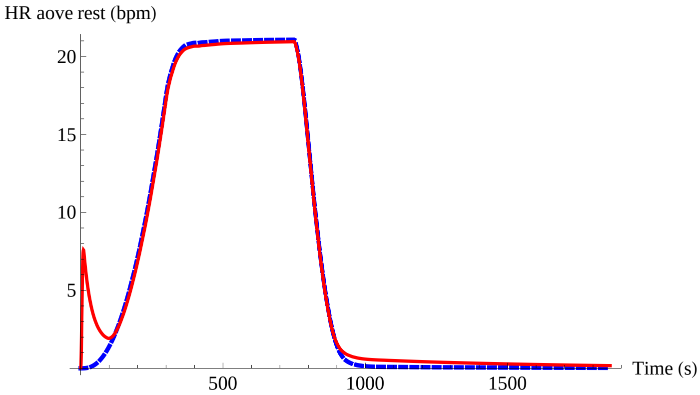

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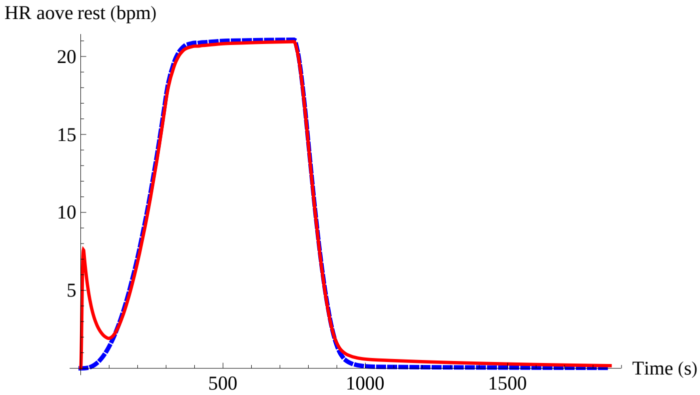

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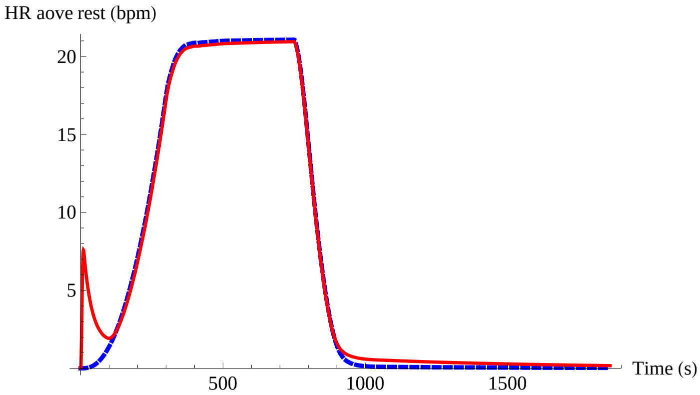

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Theorem: For each constant $\bar{\delta} > 0$, we can find constants $\bar{c}_i > 0$ depending on $\bar{\delta}$ so that along all trajectories of (9) for all measurable functions $\mathbf{d} : [0, \infty) \to [-\bar{\delta}, \bar{\delta}]$, we have $|(\tilde{x}(t), \bar{x}(t))| \leq \bar{c}_1 |(\tilde{x}(0), \bar{x}(0))| e^{-\bar{c}_2 t} + \bar{c}_3 |\mathbf{d}|_{[0,t]}$ for all $t \geq 0$.

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This is input-to-state stability with exponential transient decay.

Ideas for Input-to-State Stability Proof

Step 1: Pick a positive definite quadratic function $W(\overline{x})$ such that

$$\dot{W} \;\leq\; -\frac{4a_2^2(1+8\bar{\delta}^2)}{c_0}|\overline{x}|^2 \tag{10}$$

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This means that $V^\sharp$ is an ISS Lyapunov function for (9) with disturbances $\mathbf{d}$ bounded by $\bar{\delta}$ in the sup norm.

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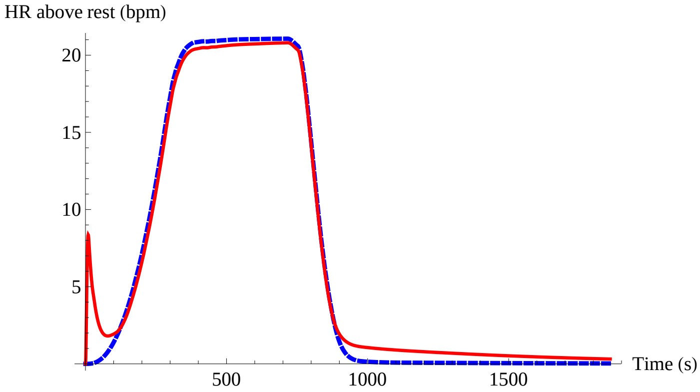

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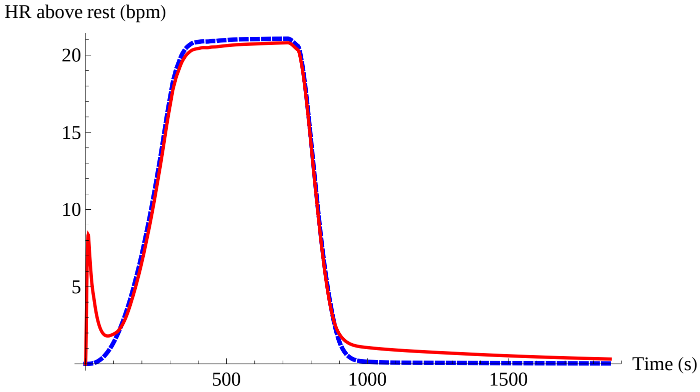

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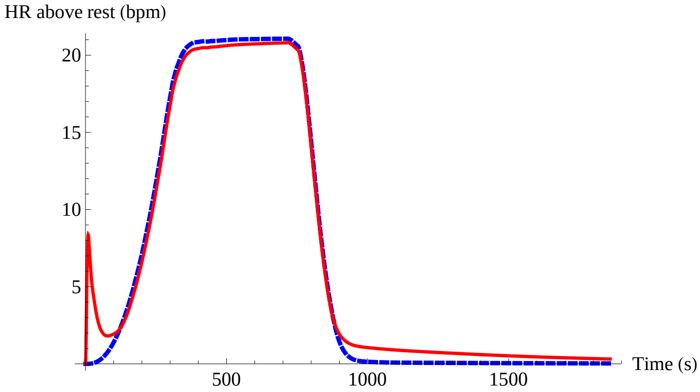

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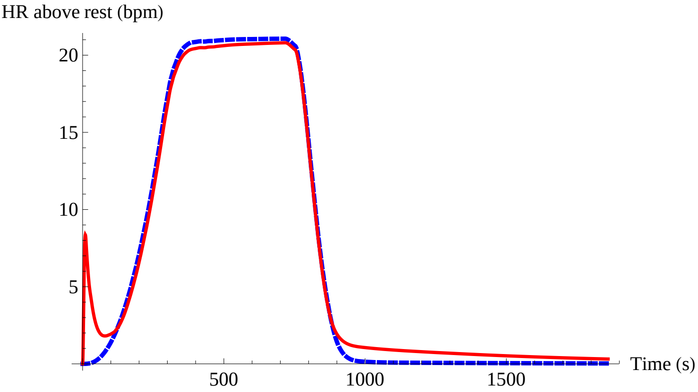

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- For complete proofs, see [FM, MM, and MdQ, “Tracking control and robustness analysis for a nonlinear model of human heart rate during exercise," Automatica, accepted.]