# Further Results on Robust Output Feedback Control for the Chemostat Dynamics

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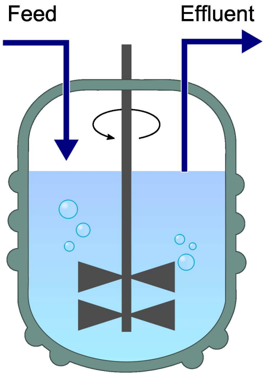


Figure 1: Chemostat

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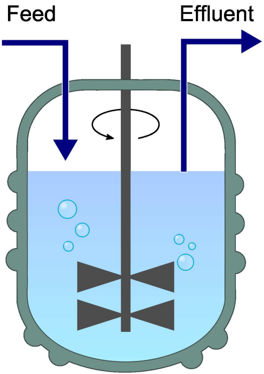


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Bioreactor.

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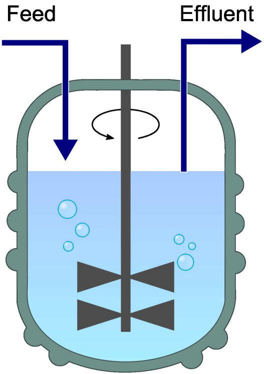


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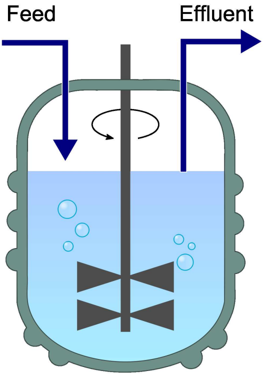


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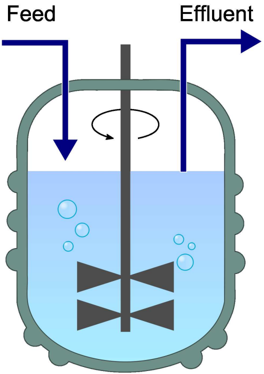


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Bioreactor. Fresh medium continuously added. Culture liquid continuously removed. Culture volume constant.

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- By competitive exclusion, $D$ cannot be constant.

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Theorem: For each constant $K \in (0, K_0)$ and each constant $a > 1$, (NV) in closed loop with the bounded positive controller

$$D(y) = \mu_1(s_*) - 2(1 + a)s_{\rm in}K\sigma\left(\frac{1}{2(1+a)s_{\rm in}}[y - x_{1*} - ax_{2*}]\right) \qquad \text{(C)}$$

is GAS to the equilibrium $(s_*, x_{1*}, x_{2*})$ on $(0, \infty)^3$.

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Let $\sigma : \mathbb{R} \to [-1, 1]$ denote the usual saturation.

Theorem: For each constant $K \in (0, K_0)$ and each constant $a > 1$, (NV) in closed loop with the bounded positive controller

$$D(y) = \mu_1(s_*) - 2(1+a)s_{\rm in}K\sigma\left(\frac{1}{2(1+a)s_{\rm in}}[y - x_{1*} - ax_{2*}]\right) \qquad \text{(C)}$$

is GAS to the equilibrium $(s_*, x_{1*}, x_{2*})$ on $(0, \infty)^3$.

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Corollary: We can choose $K$ and a constant $\varepsilon > 0$ such that if $\mathcal{T}(\mu, \nu) = \max\{|\mu_i{}'(s) - \nu_i{}'(s)| : i = 1, 2; s \in [0, s_{\rm in}]\} < \varepsilon$, then

$$\begin{cases} \dot{s} = D(y)[s_{\rm in} - s] - \nu_1(s)x_1 - \nu_2(s)x_2 \\ \dot{x}_i = [\nu_i(s) - D(y)]\, x_i, \;\; i = 1, 2 \end{cases} \qquad \text{(RC)}$$

is GAS to some point $(s_v, x_{1v}, x_{2v}) \in (0, \infty)^3$.

# Simulations

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$$D(y) = 0.042 - 0.00924616\sigma\left(\frac{y-0.074}{462.308}\right) \tag{C}$$

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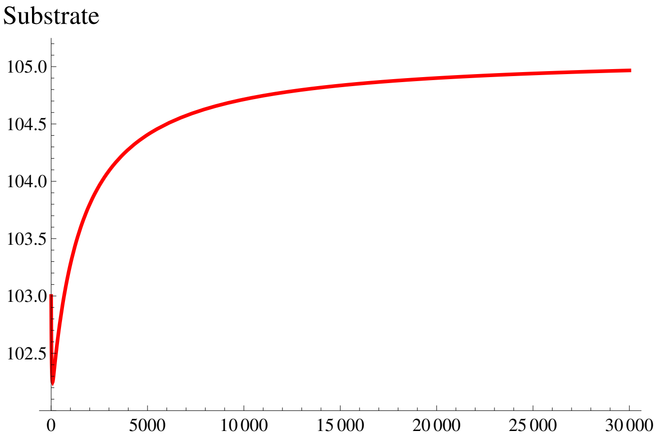


Figure 2: Substrate $s(t)$ from (RC) with $(d_1, d_2) = (0.1, 0.15)$

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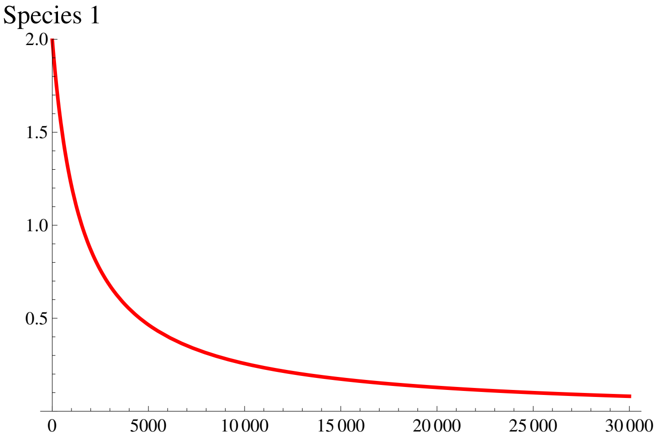


Figure 3: Species $x_1(t)$ from (RC) with $(d_1, d_2) = (0.1, 0.15)$

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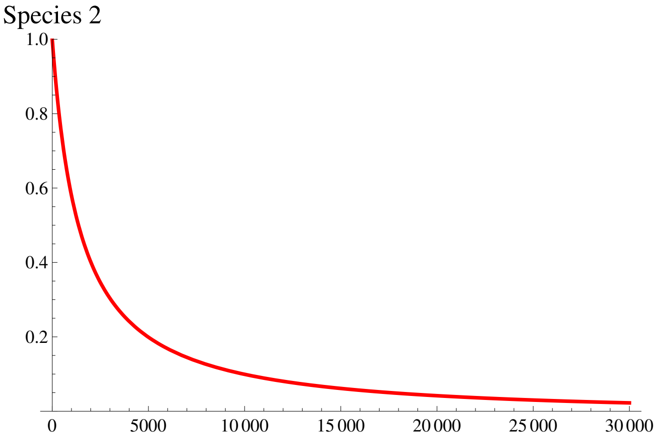


Figure 4: Species $x_2(t)$ from (RC) with $(d_1, d_2) = (0.1, 0.15)$

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- We also allowed uncertain monotone asymmetric uptake functions that are not necessary concave.
- Unlike the standard GAS treatments, our output feedback is a decreasing function of the output.
- Desirable extensions would allow nonmonotone $\mu_i$'s, more than two species, or multiple limiting substrates.