Lyapunov Functions, Point Stabilization, and Strictification

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- Strict and nonstrict Lyapunov functions

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M. Malisoff and F. Mazenc. Constructions of Strict Lyapunov Functions. Communications and Control Engineering Series, Springer-Verlag London Ltd., London, UK, 2009.

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Using LaSalle Invariance, we can often use nonstrict Lyapunov functions to prove stability.

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For example, take $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2^3$. Use $V(x) = 0.5|x|^2$.
Then $\dot{V} = -x_2^4$.

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For example, take $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2^3$. Use $V(x) = 0.5|x|^2$.
Then $\dot{V} = -x_2^4$. The largest invariant set in $\{x : x_2 = 0\}$ is $\{0\}$.

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However, explicit strict Lyapunov function *constructions* are often needed in applications to certify robustness.

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This has led to significant research on explicitly constructing strict Lyapunov functions.

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We assume standard assumptions on the dynamics which hold under smooth forward completeness and time-periodicity.

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ISS is defined using comparison functions.

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A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is of class $\mathcal{KL}$ provided there exist $\theta_i \in \mathcal{K}_\infty$ such that $\beta(s, t) = \theta_1(\theta_2(s)e^{-t})$ everywhere.

ISS Motivation-Part 2/3

We say that (1) is ISS provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ and a modulus $\bar{\alpha}$ with respect to $\mathcal{X}$ s.t. for all initial conditions $x(t_0) = x_0 \in \mathcal{X}$ and all disturbances d, the corresponding trajectories $t \mapsto \zeta(t; t_0, x_0, d)$ satisfy

$$|\zeta(t; t_0, x_0, d)| \leq \beta\Big(\bar{\alpha}(x_0), t - t_0\Big) + \gamma(|d|_\infty) \;\; \forall t \geq t_0 \,. \qquad (2)$$

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The special case where γ and d are not present is UGAS. This corresponds to point stabilization but not just attractivity.

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$$\frac{1}{2} \; = \; \frac{1+t_0}{2+2t_0} \; \leq \; \beta(1, t_0 + 1) \; \to \; 0 \; \text{ as } \; t_0 \to +\infty. \tag{6}$$

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Multiply both sides by e^{kt} and integrate. That gives ISS since $V(x(t)) \leq e^{-kt}V(x(0)) + \gamma(|d|_\infty)/k$ along all trajectories.

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Assume in addition that there are positive constants c_i such that $c_1|x|^2 \le V(x) \le c_2|x|^2$ everywhere.

$$|x(t)| \le \sqrt{\frac{c_2}{c_1}}e^{-tk/2}|x(0)| + \sqrt{\frac{\gamma(|d|_\infty)}{kc_1}}.$$

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More general ISS decay:

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$$|x(t)| \le \sqrt{\frac{c_2}{c_1}} e^{-tk/2}|x(0)| + \sqrt{\frac{\gamma(|d|_\infty)}{kc_1}}.$$

More general ISS decay: $\dot{V} \le -\alpha_1(V) + \alpha_2(|d|)$, $\alpha_i \in \mathcal{K}_\infty$.

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Example:

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is UGAS to the origin.

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$$\dot{x} = f(t, x) + g(t, x)\Big[K(t, x) - D_x V(t, x) \cdot g(t, x) + d\Big] \tag{8}$$

is ISS with respect to actuator errors d.

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Need $K(t,x)$ and $D_xV(t,x)\cdot g(t,x)$.

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There may be virtual obstacles to time-invariant stabilization.

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 u_1 . \end{cases} \tag{9}$$

There is no C^1 feedback $k(x)$ stabilizing the origin of (9).

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Brockett's Stabilization Theorem: Let a system $\dot{x} = f(x, u)$ with $f \in C^1$ admit an equilibrium point x_* and a C^1 feedback $u_s(x)$ such that $\dot{x} = f\big(x, u_s(x)\big)$ has the LAS equilibrium point x_*.

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Proof: Use degree theory (functional analysis) and homotopy arguments (general topology). See Chapter 5 of Sontag's book *Mathematical Control Theory*.

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More generally, we cannot locally continuously stabilize

$$\dot{x} = u_1 g_1(x) + \ldots + u_m g_m(x) = G(x)u, \quad x \in \mathbb{R}^n$$

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$$|x(t, t_0, x_0)| \; \leq \; (4 + 10\sqrt{e})e^{-0.5(t-t_0)}|x_0|$$

Effects of Perturbations

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The system

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obtained by adding a perturbation to u_1 is ISS.

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$$V_s(t,x) = \frac{1}{2}x_1^2 + \left(4 + \frac{\pi}{2} - 2\sin(t)\cos(t)\right)[\cos(t)x_1 + x_2]^2 \tag{11}$$

is an ISS Lyapunov function for (10).

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is an ISS Lyapunov function for (10). In fact,

$$\dot{V}_s(t,x) \leq -\tfrac{1}{204}V_s(t,x) + 3 \times 102^2\delta_1^2(t). \tag{12}$$

along all trajectories of (10), so we have exponential ISS.

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Integral ISS (Sontag, SCL, 1998)

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This is typically verified by finding iISS Lyapunov functions, which are defined the same way as ISS Lyapunov functions except the decay condition is $\exists$ a positive definite function α and $\gamma \in \mathcal{K}_\infty$ such that $\dot{V} \leq -\alpha(|x|) + \gamma(|d|)$ along all trajectories.

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For example, $\Pi(V_s)$ is an iISS Lyapunov function for the previous system for a suitable Π. Also, $\dot{x} = -\arctan(x) + u$.

Effect of Exponentially Decaying Disturbances

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Consider the following example of Teel-Hespanha, T-AC'04:

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Hence, there is no strict Lyapunov function for the $d = 0$ case that has a gradient bound C. In fact, if one existed, then $V(x(t)) \leq V(x(0)) + C|x_2(0)|$, by taking $d(t) = x_2(0) e^{-t}$.

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We can build strict Lyapunov functions under generalized Jurdjevic-Quinn conditions for much more general systems.

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We can extend to $\dot{x} = \mathcal{F}(x, u)$ by assuming its first order expansion in u satisfies Assumption J.

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We assume the Weak Jurdjevic Quinn Conditions: There exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying:

1. V is positive definite and radially unbounded;
2. for all $x \in \mathbb{R}^n$, $L_{f_0} V(x) \leq 0$; and
3. there exists an integer $l \geq 2$ such that the set

   $$W(V) = \left\{ \begin{array}{l} x \in \mathbb{R}^n : \forall k \in \{1, \ldots, m\} \text{ and } \forall i \in \{0, \ldots, l\}, \\ L_{f_0} V(x) = L_{ad^i_{f_0}(f_k)} V(x) = 0 \end{array} \right\}$$

   equals $\{0\}$.

Constructing the Auxiliary Function ψ (MM-FM)

Proposition: If $\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2 + \ldots + f_m(x)u_m$ satisfies the Weak Jurdjevic Quinn Conditions for some integer l and some storage function V, and if we define G by

$$G = \sum_{i=0}^{l-1} \sum_{k=1}^{m} \lambda_{i,k} \mathrm{ad}_{f_0}^{i}(f_k), \tag{24}$$

where

$$\lambda_{i,k} = \sum_{j=i}^{l-1} (-1)^{j-i+1} L_{\mathrm{ad}_{f_0}^{(2j-i+1)}(f_k)} V \quad \forall i, k, \tag{25}$$

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then $\psi(x) = L_G V(x)$ satisfies: If $x \in \mathbb{R}^n \setminus \{0\}$, and if $L_{f_i} V(x) = 0$ for $i = 0, 1, \ldots, m$, then $L_{f_0} \psi(x) < 0$.

LaSalle Invariance

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Assume $\dot{x} = f(x)$ has a nonstrict Lyapunov function V so that:

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Objective:

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Objective: Find a simpler construction that also applies to t-v systems, and that has a much less restrictive NDC on V.

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Let $V \in C^\infty$ be a nonstrict Lyapunov function for $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, with f and V having period T in t.

First Construction (MM-FM)

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Theorem 1

Assume $\exists$ constants $\tau \in (0, T]$ and $\ell \in \mathbb{N}$ and a positive definite continuous function ρ such that for all $x \in \mathbb{R}^n$ and all $t \in [0, \tau]$,

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Then we can explicitly determine functions $\mathcal{F}_j$ and $\mathcal{G}$ such that

$$V^\sharp(t,x) = \sum_{j=1}^{\ell-1} \mathcal{F}_j\big(V(t,x)\big)A_j(t,x) + \mathcal{G}\big(t, V(t,x)\big) \tag{27}$$

is a strict Lyapunov function, giving UGAS of the dynamics.

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Hence, (5) holds with $\tau = \frac{\pi}{4}$ and $\rho(r) = r^2/\{200(r+1)\}$.

Idea of Proof of Thm 1, Part 1/3

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Let $\Gamma \in C^1$ be any everywhere positive increasing function s.t.

$$\Gamma(V(t,x)) \geq (\ell + 2)|a_m(t,x)| + 1$$

for all $m \in \{1, ..., \ell + 1\}$ and all $(t,x) \in [0,\infty) \times \mathbb{R}^n$.

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Pick $\omega \in \mathcal{K}_\infty \cap C^1$ and the strictly increasing everywhere positive function $K \in C^1$ such that

$$\rho(r) \geq \frac{\omega(r)}{K(r)} \quad \forall r \geq 0 \; . \tag{29}$$

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and

$$M_p(t,x) = \sum_{m=1}^{p} a_{m+1}(t,x)a_m(t,x) + \int_0^{V(t,x)} \Gamma(r)\mathrm{d}r. \tag{31}$$

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Let k_0 be any C^1 increasing function such that

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and $q : \mathbb{R} \to [0,1]$ be any continuous function with period T s.t. $q(t) = 0$ for all $t \in [\tau, T]$ and $q(t) = 1$ for all $t \in [\frac{\tau}{3}, \frac{2\tau}{3}]$.

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Let G be any C^1 function such that

$$G'(v) \;\geq\; T\left|k_{\ell-1}(v)\frac{\omega'(v)K(v)-\omega(v)K'(v)}{K^2(v)}+k'_{\ell-1}(v)\frac{\omega(v)}{K(v)}\right|$$

for all $v \geq 0$.

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and $\kappa \in C^1$ is any increasing function such that $\kappa(V(t,x)) \geq |S_3(t,x)| + 1$ everywhere.

Second Construction for $\dot{x} = f(x)$, $x \in \mathcal{X}$

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Assumptions 1

There exist a storage function $V_1 : \mathcal{X} \to [0, \infty)$*; functions* $h_1, \ldots, h_m$ *such that* $h_j(0) = 0$ *for all* j*; everywhere positive functions* $r_1, \ldots, r_m$ *and* ρ*; and an integer* $N > 0$ *for which*

$$\nabla V_1(x) f(x) \ \leq \ -r_1(x) h_1^2(x) - \ldots - r_m(x) h_m^2(x) \ \ \forall x \in \mathcal{X} \qquad (33)$$

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$$\text{and} \quad \sum_{k=0}^{N-1} \sum_{j=1}^{m} \left[L_f^k h_j(x) \right]^2 \ \geq \ \rho(V_1(x)) V_1(x) \ \ \forall x \in \mathcal{X}. \tag{34}$$

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Also, $f \in C^\infty(\mathbb{R}^n)$, and V_1 has a positive definite quadratic lower bound in some neighborhood of $0 \in \mathbb{R}^n$.

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One can determine explicit functions $k_\ell, \Omega_\ell \in \mathcal{K}_\infty \cap C^1$ such that

$$S(x) = \sum_{\ell=1}^{N} \Omega_\ell\Big(k_\ell(V_1(x)) + V_\ell(x)\Big) \tag{36}$$

is a strict Lyapunov function on $\mathcal{X}$ satisfying $S(x) \geq V_1(x)$ on $\mathcal{X}$.

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$$S(x) = \sum_{\ell=1}^{N} \Omega_\ell \Big(k_\ell(V_1(x)) + V_\ell(x) \Big) \qquad (36)$$

is a strict Lyapunov function on $\mathcal{X}$ satisfying $S(x) \geq V_1(x)$ on $\mathcal{X}$.

Significance:

Second Construction for $\dot{x} = f(x)$, $x \in \mathcal{X}$ (MM-FM)

Theorem 2

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Significance: New theorem says which functions V_i to pick.

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Significance: Allows any open state space $\mathcal{X}$ containing $0 \in \mathbb{R}^n$.

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Significance: Readily extends to time periodic t-v systems.

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Find everywhere positive C^1 increasing ϕ_1 and p_1 s.t.

$$\nabla V_i(x) f(x) \ \leq \ -\mathcal{N}_i(x) + \phi_1\big(V_1(x)\big)\sqrt{\mathcal{N}_{i-1}(x)}\sqrt{V_1(x)} \qquad (37)$$

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Finally, find a continuous everywhere positive $\tilde{\rho}$ so that

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$\Omega_N(r) = r$, and $\{\Omega_i\}_{i=1}^{N-1}$ satisfy

$$\Omega_i'(U_i) \geq (N-1)^2 \frac{8\phi_1^2(V_1)}{\tilde{\rho}(V_1)} \sum_{r=1+i}^{N} \Omega_r'(U_r)^2, \tag{42}$$

with $\Omega_i' : [0, \infty) \to [1, \infty)$ continuous and increasing for each i.

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$$p_1(r) \;=\; 1536(x_*+1)(\alpha+1)(1+x_*+y_*)^4(1+r)^3.$$

Along the trajectories of the L-V error dynamics,

$$\dot{S} \leq -\frac{1}{4}\left[\tilde{x}^2 + \left\{(\tilde{x}+\alpha\tilde{y})(\tilde{x}+x_*)\right\}^2\right]. \tag{52}$$

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- Extensions exist for multiple time scales and unknown parameters, e.g., adaptive, delayed, and hybrid systems.

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