

On Input-to-State Stability for Nonlinear Systems with Delayed Feedbacks



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OUTLINE

- Background and Motivation
- Definitions and Assumptions
- Main Theorem
- Extensions to Cascades
- Identification Theory Example
- Two Other Examples
- Conclusions and Summary

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BACKGROUND

Lyapunov Functions: Useful for building stabilizing feedbacks when given by explicit expressions. For example, Lyapunov functions for

$$\dot{x} = f(x)$$

yield input-to-state stability (ISS) stabilizers $K(x)$ for

$$\dot{x}(t) = f(x(t)) + g(x(t))[K(x(t)) + d(t)]$$

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STATEMENT of PROBLEM

Our Approach: Use a given Lyapunov function V for a UGAS system

$$\dot{x} = f(x, t) + g(x, t)u_s(x, t) \quad (\Sigma_{nd})$$

to explicitly construct an ISS Lyapunov-Krassovski functional for

$$\dot{x}(t) = f(x(t), t) + g(x(t), t)[u_s(\xi_\tau(t), t) + d(t)], \quad (\Sigma_d)$$

where $\xi_\tau(t) = (x_1(t - \tau_1), x_2(t - \tau_2), \dots, x_n(t - \tau_n))$ and $0 \leq \tau_i \leq \bar{\tau}$.

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Motivation: General delayed systems of this kind are useful in networks; see e.g. [Nesic-Teel \(2004\)](#).

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ASSUMPTIONS and DEFINITIONS

Assumption A: f , g , and u_s are locally Lipschitz. \exists constant $\bar{L} > 0$ s.t.
 $\forall x \in \mathbb{R}^n, t \geq 0$, **(A1)** $|f(x, t)| \leq \bar{L}|x|$, **(A2)** $|g(x, t)| \leq \bar{L}(|x| + 1)$, and
(A3) $|\partial u_s / \partial x(x, t)| \leq \bar{L}$.

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Consequences: **(I)** For all choices of $t_o \geq 0$, $\tau > 0$, $x_o \in \mathcal{C}_n([t_o - \tau, t_o])$,
and $d \in \mathcal{L}_m^\infty([0, \infty))$, the initial value problem

$$\begin{aligned} \dot{x}(t) &= f(x(t), t) + g(x(t), t)[u(x(t - \tau), t) + d(t)] \\ \forall t \geq t_o \text{ a.e. and } x(r) &= x_o(r) \quad \forall r \in [t_o - \tau, t_o]. \end{aligned} \tag{IP}$$

has a unique solution $t \mapsto x(t; t_o, x_o, d, \tau)$ defined on $[t_o - \tau, +\infty)$.

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(II) For each $\kappa \in \mathbb{N}$ and $\tau > 0$, we can construct $\bar{\gamma}_{\kappa, \tau} \in \mathcal{K}_\infty$ s.t.

$$|x(t; t_o, x_o, d, \tau)| \leq \bar{\gamma}_{\kappa, \tau}(|x_o|_{[t_o - \tau, t_o]}) + \bar{\gamma}_{\kappa, \tau}(|d|_{[t_o, t]}) \quad \forall t \in [t_o, t_o + \kappa\tau]$$

for all $t_o \geq 0$, $x_o \in \mathcal{C}_n([t_o - \tau, t_o])$, and $d \in \mathcal{L}_m^\infty([0, \infty))$.

INPUT-TO-STATE STABILITY

We say $\beta \in \mathcal{KL}$ provided $\beta(\cdot, t) \in \mathcal{K}_\infty \forall t \geq 0$, $\beta(s, \cdot)$ non-increasing $\forall s \geq 0$, and $\beta(s, t) \rightarrow 0$ as $t \rightarrow +\infty \forall s \geq 0$. Set $x_t(\theta) = x(t + \theta)$.

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ISS: We say that (Σ_d) is input-to-state stable (ISS) provided there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|x(t; t_o, x_o, d, \tau)| \leq \beta(|x_o|_{[t_o - \tau, t_o]}, t - t_o) + \gamma(|d|_{[t_o, t]})$$

for all $t_o \geq 0$, $\tau \in (0, \bar{\tau}]$, $x_o \in \mathcal{C}_n([t_o - \tau, t_o])$, and $d \in \mathcal{L}_m^\infty([0, \infty))$.

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ISS-LKF: A continuous $U : \mathcal{C}_n(\mathbb{R}) \times [0, \infty) \rightarrow [0, \infty)$ is called an *ISS Lyapunov-Krasovskii functional (ISS-LKF)* for (Σ_d) provided $\forall \tau \in (0, \bar{\tau}]$ and $x(t) := x(t; t_o, x_o, d, \tau)$, the function $t \mapsto U(x_t, t)$ is locally AC & $\exists \alpha_i \in \mathcal{K}_\infty$ and $\kappa \in \mathbb{N}$ s.t. $\forall x(t), \phi \in \mathcal{C}_n([- \kappa \bar{\tau}, 0])$, and $t \geq t_o + \kappa \bar{\tau}$,

- (i) $\alpha_1(|\phi(0)|) \leq U(\phi, t) \leq \alpha_2(|\phi|_{[- \kappa \bar{\tau}, 0]})$ and
- (ii) $D_t U(x_t, t) \leq -\alpha_3(U(x_t, t)) + \alpha_4(|d|_{[t_o, t]})$ a.e..

MORE DEFINITIONS and ASSUMPTIONS

Assumption H: The feedback $u_s \in C^1$. Also, there are $\sigma \in \mathcal{K}_\infty$ such that $\sigma(r) \leq r$ for all $r \geq 0$; constants $K_1 \geq 1$ and $K_i \geq 0$ ($i = 2, 3, 4$); and a C^1 uniformly proper and positive definite $V : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty)$ such that for all $x \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, $l \geq 0$, and $t \geq 0$, we have

$$\mathbf{H1} \quad V_t(x, t) + V_x(x, t)[f(x, t) + g(x, t)u_s(x, t)] \leq -\sigma(|x|)^2;$$

$$\mathbf{H2} \quad |V_x(x, t)g(x, t)| \leq K_1\sigma(|x|), \quad \left| \frac{\partial u_s}{\partial x}(x, t)f(x, l) \right|^2 \leq K_2\sigma(|x|)^2;$$

$$\mathbf{H3} \quad \left| \frac{\partial u_s}{\partial x}(x, t)g(x, l) \right|^2 \leq K_3(\sigma(|x|) + 1); \text{ and}$$

$$\mathbf{H4} \quad \left[\left| \frac{\partial u_s}{\partial x}(x, t)g(x, l) \right| |u_s(q, l)| \right]^2 \leq K_4[\sigma^2(|x|) + \sigma^2(|q|)].$$

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Allows stable linear system with bounded g and quadratic V , plus cases where the system is not exponentially stable or g is unbounded.

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Allows stable linear system with bounded g and quadratic V , plus cases where the system is not exponentially stable or g is unbounded. Set

$$\bar{\tau} := \frac{1}{4K_1\sqrt{3K_2 + 3K_4 + 1}}$$

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THEOREM 1

Theorem 1: Under the above assumptions, the feedback delayed system

$$\dot{x}(t) = f(x(t), t) + g(x(t), t)[u_s(x(t - \tau), t) + d(t)] \quad (\Sigma_d)$$

with any constant feedback delay satisfying

$$0 < \tau \leq \bar{\tau} := \frac{1}{4K_1\sqrt{3K_2 + 3K_4 + 1}}$$

admits the ISS-LKF

$$U(x_t, t) = V(x(t), t) + \frac{1}{4\bar{\tau}} \int_{t-2\bar{\tau}}^t \left(\int_r^t \sigma^2(|x(l)|) dl \right) dr$$

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Remark: When $V_t \equiv 0$ and the drift $f \equiv 0$, we can make the delay bound $\bar{\tau}$ arbitrarily large by taking $K_2 = 0$ and scaling u_s :

$$\sigma \rightarrow \sqrt{\eta}\sigma, \quad u_s \rightarrow \eta u_s, \quad K_1 \rightarrow K_1/\sqrt{\eta}, \quad K_4 \rightarrow \eta^3 K_4, \quad \eta \downarrow 0.$$

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EXTENSION to CASCADES

Cascades: Under the above assumptions on the subsystem

$$\dot{x}(t) = f(x(t)) + g(x(t))z(t)$$

with fictitious input z and time-invariant functions, we design u to render

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))z(t), \\ \dot{z}(t) = u(x(t - 2\tau), x(t - \tau), z(t - \tau)) + d(t) \end{cases} \quad (\Sigma_{\text{cas}})$$

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ISS Estimate: We use the generalization

$$|(x, z)(t)| \leq \beta(|(x, z)|_{[t_o-\tau, t_o]}, t-t_o) + \gamma(|d|_{[t_o, t]})$$

of our earlier ISS estimate. Initial functions constant on $(-\infty, t_o - \tau]$.

EXTENSION to CASCADES (cont'd)

Theorem 2: Set

$$\bar{\tau}_c = \min\{1/\sqrt{8}, \bar{\tau}\} \quad \text{and} \quad Z(t) = z(t) - u_s(x(t - \tau)).$$

Then for each constant $\tau \in (0, \bar{\tau}_c]$, the dynamics (Σ_{cas}) is ISS when

$$\begin{aligned} &u(x(t - 2\tau), x(t - \tau), z(t - \tau)) := \\ &-Z(t - \tau) + \frac{\partial u_s}{\partial x}(x(t - \tau)) [f(x(t - \tau)) + g(x(t - \tau))z(t - \tau)]. \end{aligned}$$

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Strategy of Proof: First build an ISS-LKF for the $q = (x, Z)$ dynamics $\dot{q}(t) = F(q(t)) + G(q(t))[U_s(q(t - \tau)) + D(t)]$ with

$$F(q) = \begin{pmatrix} f(x) + g(x)Z \\ 0 \end{pmatrix}, \quad G(q) = \begin{pmatrix} g(x) & 0 \\ 0 & -1 \end{pmatrix},$$

the feedback $U_s(q) = (u_s(x), Z)^T$ and $D(t) = (0, -d(t))^T$.

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EXAMPLE from IDENTIFICATION THEORY

When $m : \mathbb{R} \rightarrow \mathbb{R}^n$ is continuous, we build an ISS-LKF for

$$\dot{x}(t) = -m(t)m^T(t)[x(t - \tau) + d(t)]. \quad (\Sigma_{\text{id}})$$

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Assume $|m(t)| = 1 \forall t \in \mathbb{R}$ and $\exists \alpha' \in (0, 1)$ and $\beta', \tilde{c} > 0$ such that

$$\alpha' I_{n \times n} \leq \int_t^{t+\tilde{c}} m(\tau)m^T(\tau)d\tau \leq \beta' I_{n \times n} \quad \forall t \in \mathbb{R}.$$

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Lemma: The function $V(x, t) := x^T P(t)x$ satisfies $\dot{V} \leq -\alpha'|x|^2/2$ along all trajectories of $\dot{x}(t) = -m(t)m^T(t)x(t)$ when

$$P(t) = \kappa I + \int_{t-\tilde{c}}^t \int_s^t m(l)m^T(l) dl ds$$

and $\kappa = 1 + \frac{\tilde{c}}{2} + \frac{1}{4\alpha'}\tilde{c}^4$. Moreover, $|P(t)| \leq \kappa + \tilde{c}^2$ everywhere.

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$$P(t) = \kappa I + \int_{t-\tilde{c}}^t \int_s^t m(l)m^T(l) dl ds$$

and $\kappa = 1 + \frac{\tilde{c}}{2} + \frac{1}{4\alpha'} \tilde{c}^4$. Moreover, $|P(t)| \leq \kappa + \tilde{c}^2$ everywhere.

Corollary: Let $\tau \in (0, \bar{\tau}]$. Then (Σ_{id}) has this ISS-LKF and so is ISS:

$$U(x_t, t) = x^T(t)P(t)x(t) + \frac{\alpha'}{16\bar{\tau}} \int_{t-2\tau}^t \left(\int_r^t |x(l)|^2 dl \right) dr.$$

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- Definitions and Assumptions
- Main Theorem
- Extensions to Cascades
- Identification Theory Example
- **Two Other Examples**
- Conclusions and Summary

MORE EXAMPLES

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$$\dot{x}(t) = \sqrt{1+x^2(t)} \left[-\int_0^{x(t-\tau)} \frac{1}{\sqrt{1+l^2}} dl + d(t) \right]$$

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