

Stabilization of a Periodic Trajectory for a Chemostat with Two Species



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Joint with **Frédéric Mazenc** (Projet MERE INRIA-INRA)
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Biochemical Reactors and Reaction Networks Session
2007 American Control Conference – July 11-13, 2007
Marriott Marquis Hotel at Times Square, New York City

BACKGROUND and GOAL

Basic Model: The two-species chemostat with nutrient concentration $s(t)$ and organism concentrations $x_i(t)$ evolving on $[0, \infty) \times (0, \infty)^2$ is

$$\begin{cases} \dot{s} &= D[s_{in} - s] - \sum_{j=1}^2 \mu_j(s)x_j, \\ \dot{x}_i &= [\mu_i(s) - D]x_i, \quad i = 1, 2 \end{cases} \quad (\Sigma_c)$$

where $D(\cdot)$ is the dilution rate, $s_{in}(\cdot)$ is the concentration of the input nutrient, and $\mu_1, \mu_2 : [0, \infty) \rightarrow [0, \infty)$ are uptake functions.

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Competitive Exclusion: When $s_{in}(\cdot)$ and D are constant and the μ_i 's are increasing, **at most one species survives**. (There is a steady state with at most one nonzero species concentration, which attracts a.a. solutions.)

OVERVIEW of LITERATURE

Coexistence: In real ecological systems, **many species can coexist**, so much of the literature aims at choosing s_{in} and/or D to force coexistence. “**The Paradox of the plankton**,” Hutchinson, *American Naturalist*, 1961.

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Intra-Specific Competition: This can be modeled with growth rates $\mu_i(S, x_i)$ that decrease in x_i . See Mazenc-Lobry-Rapaport (*EJDE*'07), Grogard-Mazenc-Rapaport (*DCDS*'07).

LOCAL STABILIZATION RESULT

Assumption A1: $\mu_i(0) = 0$, $\mu_i \in C^1$, $\mu'_i > 0$ bounded for $i = 1, 2$.
 $\exists s_c > 0$ such that $\chi(s) := \mu_2(s) - \mu_1(s)$ satisfies $\chi(s_c) = 0$, $\chi(s) < 0$
when $0 < s < s_c$, $\chi(s) > 0$ when $s > s_c$, and $\chi'(s_c) > 0$. $\Gamma = \mu_1(s_c)$.

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Reference Trajectory: Given any constant $\alpha \in [0, \Gamma)$, we wish to track $(s_r, x_{1r}, x_{2r}) = (s_c, \exp(\cos(\alpha t)), \exp(\cos(\alpha t)))$, i.e., stabilize the error $(s_e, \xi_{1e}, \psi) := (s - s_c, \xi_1 - \cos(\alpha(t)), \xi_2 - \xi_1)$ to 0, where $\xi_i = \ln(x_i)$.
More general reference trajectories are tractable by analogous methods.

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Theorem 1: Under Assumption A1, the control laws

$$D(t, \xi_1) := \Gamma + \alpha \sin(\alpha t) + \frac{(\Gamma - \alpha)^2}{\Gamma} (\xi_1 - \cos(\alpha t))$$
$$s_{in}(t) := s_c + \frac{2\Gamma e^{\cos(\alpha t)}}{\Gamma + \alpha \sin(\alpha t)}$$

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render the error dynamics locally exponentially stable to 0. Hence, they locally exponentially stabilize the reference trajectory.

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Assumption A2: $\mu_1, \mu_2 \in C^2$. $\exists \theta_1, \theta_2 > 0$ such that

$$\sup\{|\mu_1''(l)| : l \geq 0\} \leq \theta_1, \sup\{|\mu_2''(l)| : l \geq 0\} \leq \theta_2.$$

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$$c_1 := \frac{\Gamma}{16} \min \left\{ \frac{s_c}{\mu_1'(s_c)}, \frac{1}{\theta_1} \right\}, \quad c_2 := \frac{\Gamma}{16} \min \left\{ \frac{s_c}{\mu_2'(s_c) - \mu_1'(s_c)}, \frac{1}{\theta_1 + \theta_2} \right\}$$

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Theorem 2: Let Assumptions A1-A2 hold and $\alpha \in [0, \Gamma/2)$. Then

$$\begin{aligned} D(t, \xi_{1e}) &= \Gamma + \alpha \sin(\alpha t) + \frac{\Gamma}{4} \langle \xi_{1e} \rangle \quad \text{and} \\ s_{in}(t, \xi_{1e}, \psi) &= s_c + \frac{1}{D(t, \xi_{1e})} \left\{ \Gamma e^{\cos(\alpha t)} (e^{\xi_{1e}} + e^{\psi + \xi_{1e}}) \right. \\ &\quad \left. - c_1 \mu_1'(s_c) \langle \xi_{1e} \rangle - c_2 [\mu_2'(s_c) - \mu_1'(s_c)] \langle \psi \rangle \right\} \end{aligned}$$

render the error dynamics GAS and locally exponentially stable to the origin.

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STRICT LYAPUNOV FUNCTION

Step 1: Show that the error dynamics

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has the nonstrict Lyapunov function

$$V(s_e, \xi_{1e}, \psi) := \frac{1}{2}s_e^2 + c_1 \left(\sqrt{1 + \xi_{1e}^2} - 1 \right) + c_2 \left(\sqrt{1 + \psi^2} - 1 \right)$$

i.e. $\dot{V} \leq -\frac{\Gamma}{8}s_e^2 - \frac{\Gamma}{4}c_1 \langle \xi_{1e} \rangle^2 \leq 0$ along the error dynamics.

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Step 2: Construct a positive increasing function κ so that

$$V_a(s_e, \xi_{1e}, \psi) := s_e \langle \psi \rangle + \int_0^{V(s_e, \xi_{1e}, \psi)} \kappa(r) dr$$

is a strict Lyapunov function for the error dynamics.

EXAMPLE

We track $(s_r, x_{1r}, x_{2r}) = (0.9, \exp(\cos(t/4)), \exp(\cos(t/4)))$ for

$$\left\{ \begin{array}{l} \dot{s} = D[s_{in} - s] - \frac{10s}{1+20s}x_1 - \frac{s}{1+s}x_2 \\ \dot{x}_1 = \left[\frac{10s}{1+20s} - D \right] x_1 \\ \dot{x}_2 = \left[\frac{s}{1+s} - D \right] x_2. \end{array} \right. \quad (\Sigma_e)$$

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Our assumptions are satisfied using

$$\mu_1(s) = \frac{10s}{1+20s}, \quad \mu_2(s) = \frac{s}{1+s}, \quad s_c = \frac{9}{10}, \quad \text{and} \quad \Gamma = \frac{9}{19}.$$

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Therefore, we get the locally uniformly stabilizing controllers

$$D(t, \xi_1) = \frac{9}{19} + \frac{1}{4} \sin(t/4) + \frac{19}{9} \left(\frac{9}{19} - \frac{1}{4} \right)^2 (\xi_1 - \cos(t/4))$$
$$s_{in}(t) = \frac{9}{10} + \frac{72e^{\cos(t/4)}}{36+19 \sin(t/4)}$$

which cause the trajectories of (Σ_e) to locally track the green trajectory.

SIMULATION for (Σ_e) with $(s, x_1, x_2)(0) = (.001, 10, .001)$

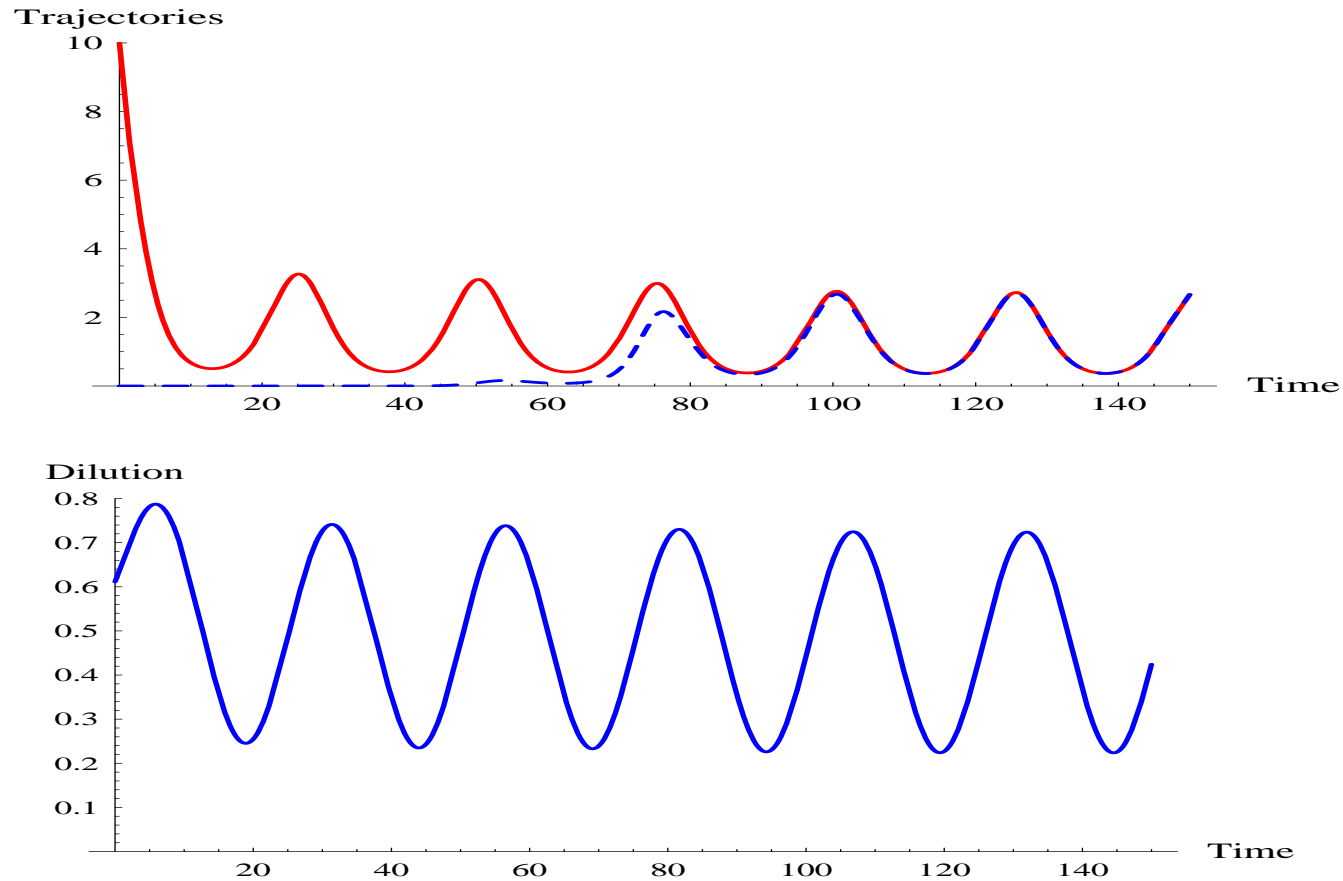


Figure 1: x_1 (solid red line). x_2 (dashed blue line). D (solid blue line).

SIMULATION for (Σ_e) with $(s, x_1, x_2)(0) = (10, .015, 10)$

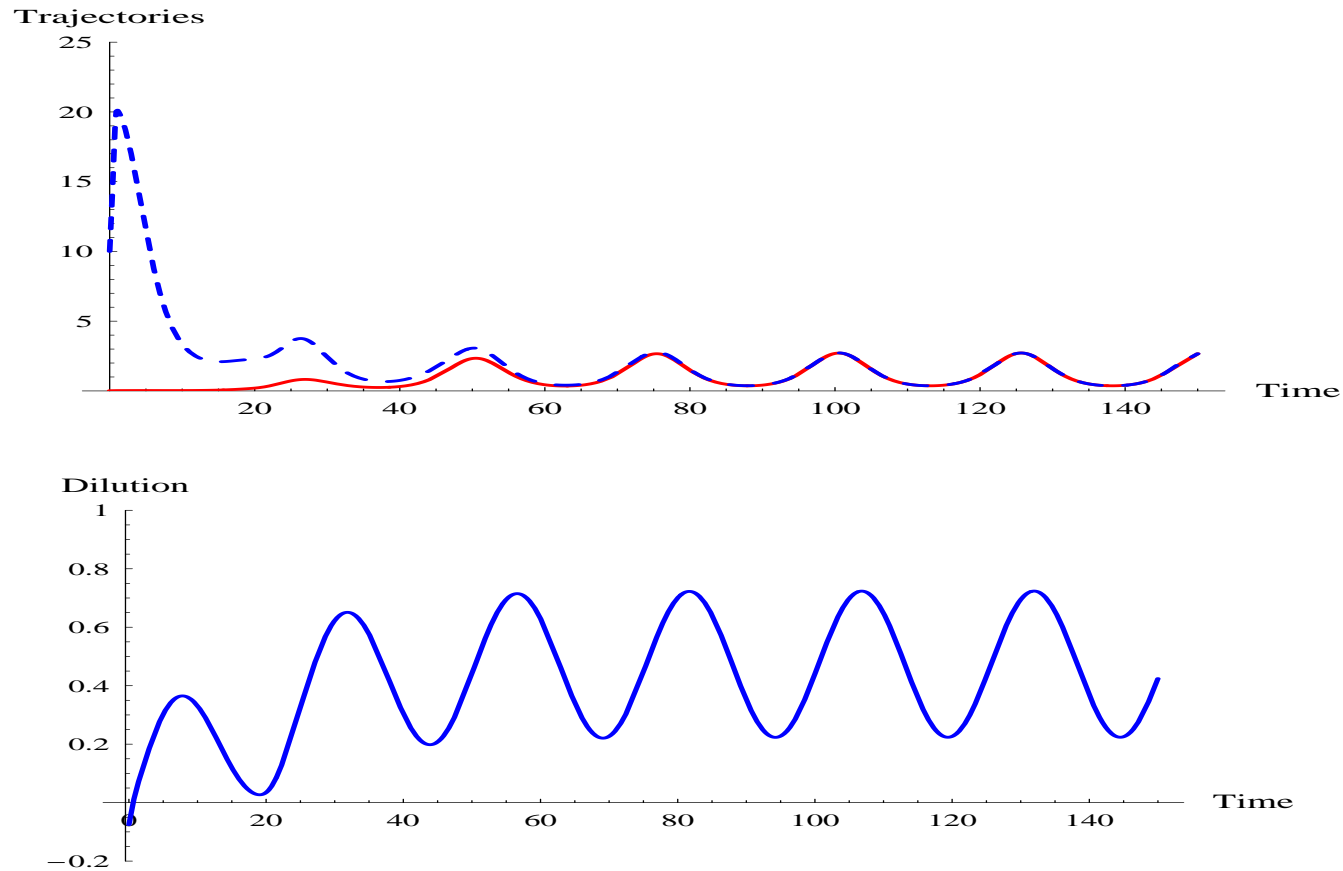


Figure 2: x_1 (solid red line). x_2 (dashed blue line). D (solid blue line).

SIMULATION for (Σ_e) with $(s, x_1, x_2)(0) = (10, .001, .001)$

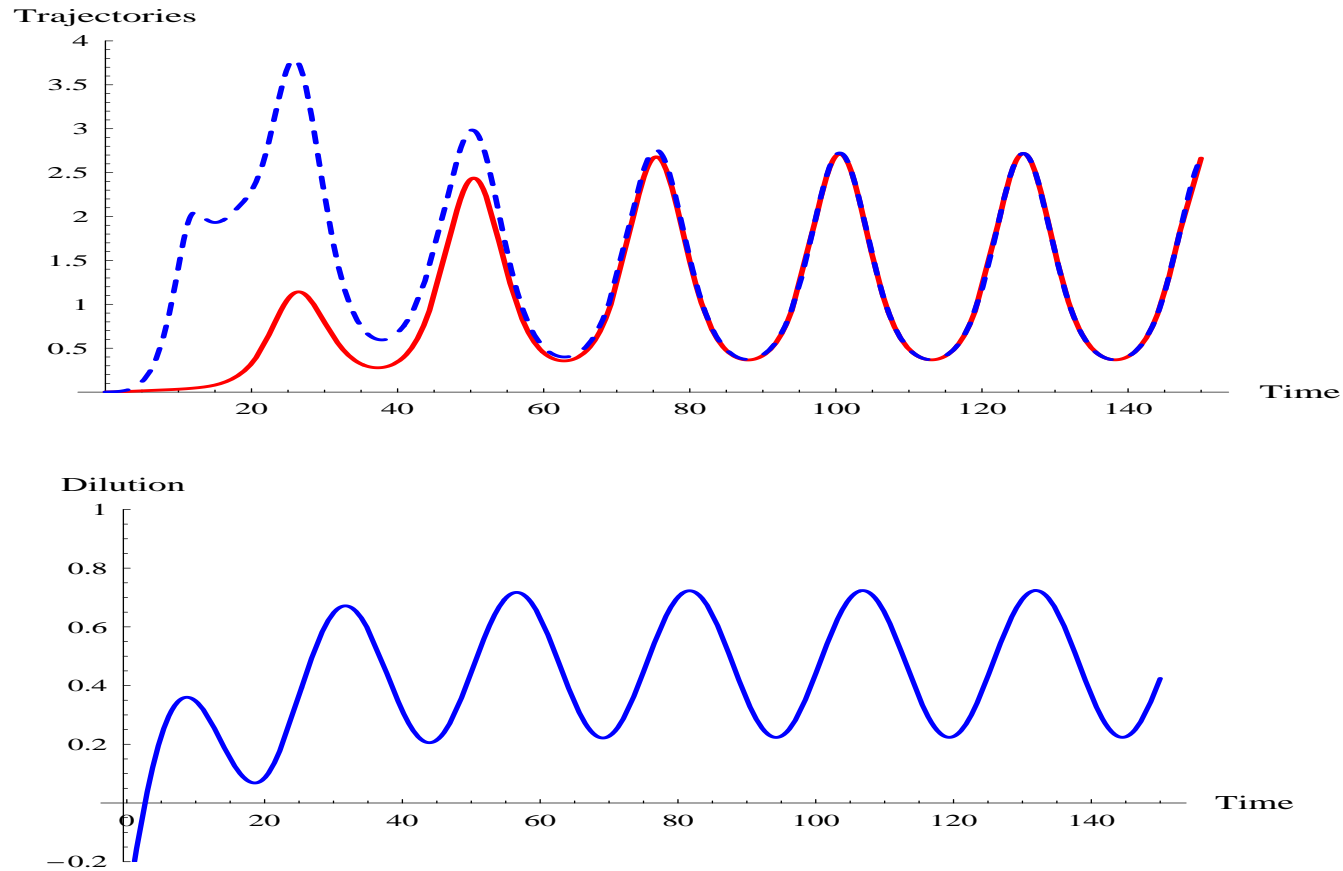


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