Control and Robustness Analysis for Curve Tracking with Unknown Control Gains

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Sponsored by NSF/ECCS/EPAS Program

Summary of Forthcoming Paper in Automatica

2013 Joint Mathematics Meetings
Special Session on Theory and Interdisciplinary Applications of Dynamical Systems
January 9, 2013
What Are Perturbed Control Systems?

These are triply parameterized families of ODEs of the form

$$Y' (t) = F(t, Y(t), u(t, Y(t)), \Gamma, \delta(t)), \quad Y(t) \in \mathbb{Y}.$$

(1)

$$Y \subseteq \mathbb{R}^n.$$ We have freedom to choose the control function $$u.$$ The functions $$\delta: [0, \infty) \to \mathbb{D}$$ represent uncertainty. $$\mathbb{D} \subseteq \mathbb{R}^m.$$ The vector $$\Gamma$$ is constant but unknown. Specify $$u$$ to get a doubly parameterized closed loop family

$$Y' (t) = G(t, Y(t), \Gamma, \delta(t)), \quad Y(t) \in \mathbb{Y},$$

(2)

where $$G(t, Y, \Gamma, \delta) = F(t, Y, u(t, Y), \Gamma, \delta).$$ Typically we construct $$u$$ such that all trajectories of (2) for all possible choices of $$\delta$$ satisfy some control objective.
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Specify \( u \) to get a *doubly* parameterized closed loop family

\[ Y'(t) = \mathcal{G}(t, Y(t), \Gamma, \delta(t)), \quad Y(t) \in \mathcal{Y}, \quad (2) \]

where \( \mathcal{G}(t, Y, \Gamma, d) = \mathcal{F}(t, Y, u(t, Y), \Gamma, d) \).
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What is One Possible Control Objective?

\[
y'(t) = G(t, y(t), \Gamma), \quad y(t) \in Y(\Sigma) \\
|y(t)| \leq \gamma_1(e^{t_0 - t} \gamma_2(|y(t_0)|)) \\
\]

\[\text{ISS}\]

\[
Y'(t) = G(t, y(t), \Gamma, \delta(t)) , \quad y(t) \in Y(\Sigma_{\text{pert}}) \\
|y(t)| \leq \gamma_1(e^{t_0 - t} \gamma_2(|y(t_0)|) + \gamma_3(|\delta|)[t_0, t]) \\
\]

Find \(\gamma_i\)'s by building special strict LFs for \(y'(t) = G(t, y(t), \Gamma, 0)\).

Ex: \(\Sigma_{\text{pert}}\) is ISS iff it has an ISS Lyapunov function (Sontag-Wang)
Input-to-state stability generalizes global asymptotic stability.
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Ex : \( \Sigma_{\text{pert}} \) is ISS iff it has an ISS Lyapunov function (Sontag-Wang)
Consider a perturbed control system
\[
\dot{\xi} = J(t, \xi, \Gamma, u, \delta) \quad (3)
\]
with a smooth reference trajectory \( \xi_R \) for a reference control \( u_R \).
That means
\[
\dot{\xi}_R(t) = J(t, \xi_R(t), \Gamma, u_R(t), 0) \quad \forall t \geq 0.
\]

**Problem:**
Find a dynamic feedback and a parameter estimator \( u(t, \xi, \hat{\Gamma}) \) and
\[
\dot{\hat{\Gamma}} = \tau(t, \xi, \hat{\Gamma}) \quad (4)
\]
that makes the \( \bar{Y} = (\tilde{\xi}, \tilde{\Gamma}) = (\xi - \xi_R, \hat{\Gamma} - \Gamma) \) dynamics ISS.

Flight control, electrical and mechanical engineering, etc.
Persistent excitation.

We proved a general theorem about how this can be solved.

Michael Malisoff (LSU) and Fumin Zhang (Georgia Tech)
Control for Curve Tracking with Unknown Control Gains
Consider a perturbed control system

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Adaptive Tracking and Parameter Identification

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We proved a general theorem about how this can be solved.
Application: 2D Curve Tracking for Marine Robots

Motivation: Search for pollutants from Deepwater Horizon disaster.

\[ \rho = |r_2 - r_1|, \quad \phi = \text{angle between } x_1 \text{ and } x_2, \quad \cos(\phi) = x_1 \cdot x_2 \]

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Adaptive Robust Curve Tracking

\[ \dot{\rho} = -\sin(\phi) \]
\[ \dot{\phi} = \kappa \cos(\phi) \]
\[ \frac{1}{1 + \kappa \rho} \Delta u(\rho, \phi) \in \text{state space} \]

\[ h(\rho) = \alpha \{ \rho + \rho^2 - 2 \rho_0^2 \}, \quad \rho_0 = \text{desired value for} \rho \]

Control:
\[ u(\rho, \phi, \hat{\Gamma}) = -\frac{1}{\hat{\Gamma}} \hat{\Gamma} \left( \kappa \cos(\phi) \right) \frac{h'(\rho)}{\cos(\phi)} + \mu \sin(\phi) \]

Estimator:
\[ \dot{\hat{\Gamma}} = (\hat{\Gamma} - c_{\min})(c_{\max} - \hat{\Gamma}) \]

\[ V^{\sharp}(\rho, \phi) = -h'(\rho) \sin(\phi) + \int_{0}^{\gamma(\mu)} V(\rho, \phi) \]

\[ \gamma(\mu) = \frac{1}{\mu} \left( 2 \alpha^2 \rho_0^4 \left( q + 2 \alpha \rho_0 \right)^3 + 1 \right) + 18 \alpha \rho_0 + 576 \rho_4^3 (9) \]

\[ V(\rho, \phi) = -\ln(\cos(\phi)) + h(\rho) (10) \]
Adaptive Robust Curve Tracking

\[
\begin{align*}
\dot{\rho} &= -\sin(\phi) \\
\dot{\phi} &= \frac{\kappa \cos(\phi)}{1+\kappa \rho} + \Gamma[u + \delta]
\end{align*}
\]

\( (\rho, \phi) \in (0, \infty) \times (-\pi/2, \pi/2) \) (\( \Sigma_c \))

\[h(\rho) = \alpha \{\rho + \rho^2 \rho - \rho^2\}, \rho_0 = \text{desired value for } \rho (5)\]

Control :

\[u(\rho, \phi, \hat{\Gamma}) = -\frac{1}{\hat{\Gamma}} \left(\kappa \cos(\phi) \frac{1}{1+\kappa \rho} - h'(\rho) \cos(\phi) + \mu \sin(\phi)\right) (6)\]

Estimator :

\[\dot{\hat{\Gamma}} = (\hat{\Gamma} - \gamma_{\min}) (\gamma_{\max} - \hat{\Gamma}) \frac{\partial V}{\partial \phi} (\rho, \phi) \]

\[V(\rho, \phi) = -h'(\rho) \sin(\phi) + \int_{\gamma(\rho)} V(\rho, \phi) d\mu (8)\]

\[\gamma(\rho) = \frac{1}{\mu} \left(\frac{2}{\alpha^2 \rho_0^4} (\rho + 2\alpha \rho_0) + 1 + \frac{1}{\rho_0^4} (2 + \frac{1}{18}) \alpha \rho_0 + \frac{576}{\rho_0^4} \alpha^2 \rho_0^3\right) (9)\]

\[V(\rho, \phi) = -\ln(\cos(\phi)) + h(\rho) (10)\]
Adaptive Robust Curve Tracking

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\begin{cases}
\dot{\rho} = -\sin(\phi) \\
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\end{cases}
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\( (\rho, \phi) \in (0, \infty) \times (-\pi/2, \pi/2) \) \( (\Sigma_c) \)

\[ h(\rho) = \alpha \left\{ \rho + \frac{\rho_0^2}{\rho} - 2\rho_0 \right\}, \quad \rho_0 = \text{desired value for } \rho \] (5)
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Control: \[u(\rho, \phi, \hat{\Gamma}) = -\frac{1}{\hat{\Gamma}} \left( \frac{\kappa \cos(\phi)}{1+\kappa \rho} - h'(\rho) \cos(\phi) + \mu \sin(\phi) \right) \quad (6)\]

Estimator: \[\dot{\hat{\Gamma}} = (\hat{\Gamma} - c_{\min})(c_{\max} - \hat{\Gamma}) \frac{\partial V^{\#}(\rho, \phi)}{\partial \phi} u(\rho, \phi, \hat{\Gamma}) \quad (7)\]
Adaptive Robust Curve Tracking

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\begin{aligned}
\dot{\rho} &= -\sin(\phi), \\
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\end{aligned}
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h(\rho) = \alpha \left\{ \rho + \frac{\rho_0^2}{\rho} - 2\rho_0 \right\}, \quad \rho_0 = \text{desired value for } \rho \tag{5}
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\[
\gamma(q) = \frac{1}{\mu} \left( \frac{2}{\alpha^2 \rho_0^4} (q + 2\alpha \rho_0)^3 + 1 \right) + \frac{\mu}{2} + 2 + \frac{18\alpha}{\rho_0} + \frac{576}{\rho_0 \alpha^2} q^3 \tag{9}\]

\[
V(\rho, \phi) = -\ln(\cos(\phi)) + h(\rho) \tag{10}\]
Robustly Forwardly Invariant Hexagonal Regions

We must restrict the suprema of the perturbations $\delta(t)$ to keep $(\rho, \phi)$ from exiting the required state space $(0, \infty) \times (-\pi/2, \pi/2)$.

View the state space $(0, \infty) \times (-\pi/2, \pi/2)$ as a union of compact hexagonally shaped regions $H_1 \subseteq H_2 \subseteq ... \subseteq H_i \subseteq ...$. For each $i$, all trajectories of $(\Sigma c)$ starting in $H_i$ stay in $H_i$. The tilted legs have slope $c_{\min} \mu/c_{\max}$.

For each index $i$, we take $\delta^*_i$ to be the largest allowable disturbance bound to maintain forward invariance of $H_i$. Then we can prove ISS of the tracking and parameter identification dynamics for each set $H_i$ for the disturbance set $D = [-\delta^*_i, \delta^*_i]$. 
We must restrict the suprema of the perturbations \( \delta(t) \) to keep \((\rho, \phi)\) from exiting the required state space \((0, \infty) \times (-\pi/2, \pi/2)\).
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View the state space $(0, \infty) \times (-\pi/2, \pi/2)$ as a union of compact hexagonally shaped regions $H_1 \subseteq H_2 \subseteq \ldots \subseteq H_i \subseteq \ldots$. For each $i$, all trajectories of $(\Sigma_c)$ starting in $H_i$ for all $\delta : [0, \infty) \rightarrow [-\delta^*_i, \delta^*_i]$ stay in $H_i$. The tilted legs have slope $c_{\min} \mu / c_{\max}$. 
Robustly Forwardly Invariant Hexagonal Regions

We must restrict the suprema of the perturbations $\delta(t)$ to keep $(\rho, \phi)$ from exiting the required state space $(0, \infty) \times (-\pi/2, \pi/2)$.

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For each index $i$, we take $\delta_{*i}$ to be the largest allowable disturbance bound to maintain forward invariance of $H_i$. 
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For each index $i$, we take $\delta_{*i}$ to be the largest allowable disturbance bound to maintain forward invariance of $H_i$.

Then we can prove ISS of the tracking and parameter identification dynamics for each set $H_i$ for the disturbance set $\mathcal{D} = [-\delta_{*i}, \delta_{*i}]$. 
Theorem

Let $\kappa > 0$ and $\rho > 0$ be any constants. Let $c_{\min}$ and $c_{\max}$ be any positive constants such that $c_{\min} < \Gamma < c_{\max}$. Let $i \in \mathbb{N}$, and let $H_i$ and $\delta^* i$ satisfy the above requirements. Then the augmented perturbed 2D tracking and parameter identification dynamics

\[
\begin{align*}
\dot{Y}_1 &= -\sin(Y_2) \\
\dot{Y}_2 &= \kappa \cos(Y_2) + 1 + \kappa (Y_1 + \rho_0) + \Gamma u(Y_1 + \rho_0, Y_2, \hat{\Gamma}) + \Gamma \delta^* \\
\dot{\tilde{\Gamma}} &= u(Y_1 + \rho_0, Y_2, \hat{\Gamma})(\hat{\Gamma} - c_{\min})(c_{\max} - \hat{\Gamma})
\end{align*}
\]

for $Y = (Y_1, Y_2, \tilde{\Gamma}) = (\rho - \rho_0, \phi, \hat{\Gamma} - \Gamma)$ is ISS on the state space $Y = (H_i - \{(\rho_0, 0)\}) \times (c_{\min} - \Gamma, c_{\max} - \Gamma)$ for $D = [-\delta^* i, \delta^* i]$. 

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Control for Curve Tracking with Unknown Control Gains
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\dot{\hat{\Gamma}} &= u(Y_1 + \rho_0, Y_2, \hat{\Gamma}) (\hat{\Gamma} - c_{\min}) (c_{\max} - \hat{\Gamma}) / \partial V^#(Y_1 + \rho_0, Y_2)
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Control for Curve Tracking with Unknown Control Gains
Conclusions

It is important but complicated to design controllers when there are unknown parameters that we must identify. We overcame this challenge for an interesting large class of dynamics, including 2D curve tracking dynamics. Our robust forward invariance approach leads to input-to-state stability under maximal perturbation bounds. We can also cover time delayed perturbed systems which model intermittent communication in marine environments. We can generalize our work to 3D curve tracking and we plan extensions to cases with other obstacles.
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