# Tracking Control for Neuromuscular Electrical Stimulation

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SPONSORED BY NSF/ECCS/EPCN PROGRAM

Applied and Computational Mathematics Seminar
Georgia Tech – Skiles 005 – April 20, 2015

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Our new control only needs sampled observations, allows any delay, and tracks position and velocity under a state constraint.

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$$Y'(t) = \mathcal{G}(t, Y_t, \delta(t)), \quad Y(t) \in \mathcal{Y}, \qquad (2)$$

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Typically we construct $u$ such that all trajectories of (2) for all possible choices of $\delta$ satisfy some control objective.

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Find $\gamma_i$'s by building certain LKFs for $Y'(t) = \mathcal{G}(t, Y_t, 0)$.

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$E_3$ Step $E_2$ is often done by converting $V$ into an ISS-LKF $V^\sharp$.

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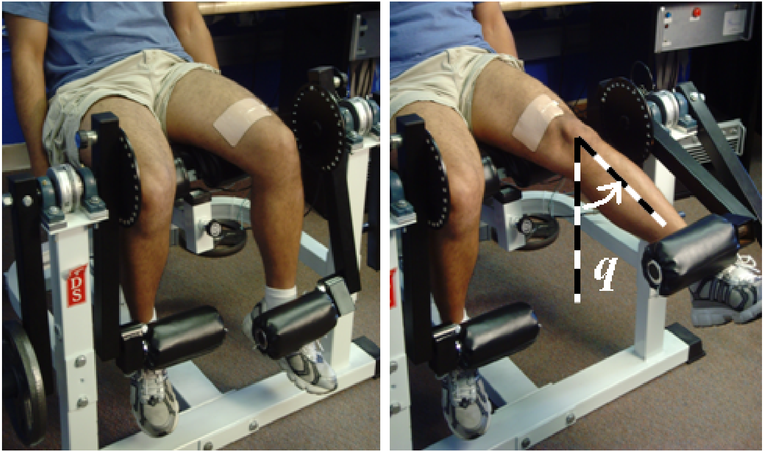


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$M_e(q) = k_1 q e^{-k_2 q} + k_3\tan(q)$: elastic effects due to joint stiffness with constants $k_i > 0$. We introduce the tan term to accommodate our state constraint $q \in (-\pi/2, \pi/2)$.

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$F = \xi(q, \dot{q})v(t - \tau)$: $v$ = voltage across quadriceps. $\tau$ = latency between applying voltage and force production.

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$$+ \underbrace{\mathcal{M}gl\sin(q)}_{M_g(q)} = \mathcal{A}(q,\dot{q})\, v(t-\tau), \quad q \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \qquad (3)$$

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Our Requirements:

- $F : (-\pi/2, \pi/2) \to [0,\infty)$ is $C^2$ and $\lim_{q\to\pm\pi/2} F(q) = \infty$.
- $G : (-\pi/2, \pi/2) \times \mathbb{R} \to (0,\infty)$ is $C^1$ and bounded.
- $H : \mathbb{R} \to \mathbb{R}$ is $C^1$ and $\inf_{x\in\mathbb{R}} xH(x) \geq 0$.

# Tracking Problem

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$$\begin{aligned} F(q) &= \frac{k_1 \exp(-k_2 q)}{J k_2^2}\left(\exp(k_2 q) - 1 - k_2 q\right) \\ &\quad + \frac{mgl}{J}(1 - \cos(q)) + \frac{k_3}{J}\ln\left(\frac{1}{\cos(q)}\right), \\ G(q, \dot{q}) &= \frac{1}{J}\mathcal{A}(q, \dot{q}), \text{ and} \\ H(\dot{q}) &= \frac{b_2}{J}\tanh(b_3 \dot{q}) + \frac{b_1}{J}\dot{q}. \end{aligned} \qquad (5)$$

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We want $(q - q_d, \dot{q} - \dot{q}_d) \to 0$ in a UGAS exponential way.

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Error variables:

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Applying the predictor feedback $v(t)$, i.e., the nominal control with the state variables replaced by their predicted values.

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$$v(t) = \frac{g_2(\zeta_d(t+\tau))v_d(t) - g_1(\zeta_d(t+\tau)+\xi(t)) + g_1(\zeta_d(t+\tau)) - (1+\mu^2)\xi_1(t) - 2\mu\xi_2(t)}{g_2(\zeta_d(t+\tau)+\xi(t))}$$

for all $t \in [T_i, T_{i+1})$ and each $i$

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for all $t \in [T_i, T_{i+1})$ and each $i$, where

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$$\zeta_d(t) = (\zeta_{1,d}(t), \zeta_{2,d}(t)) = \left(\tan(q_d(t)), \frac{\dot{q}_d(t)}{\cos^2(q_d(t))}\right),$$

$$\xi_1(t) = e^{-\mu(t-T_i)}\{(\xi_2(T_i) + \mu\xi_1(T_i))\sin(t-T_i) + \xi_1(T_i)\cos(t-T_i)\},$$

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and $\xi(T_i) = z_{N_i}$.

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and $\xi(T_i) = z_{N_i}$. The time-varying Euler iterations $\{z_k\}$ at each time $T_i$ use measurements $(q(T_i), \dot{q}(T_i))$.

# Voltage Potential Controller (continued)

Euler iterations used for control:

$$z_{k+1} = \Omega(T_i + kh_i, h_i, z_k; v) \ \text{for} \ k = 0, ..., N_i - 1 \ , \ \text{where}$$

$$z_0 = \begin{pmatrix} \tan(q(T_i)) - \tan(q_d(T_i)) \\ \frac{\dot{q}(T_i)}{\cos^2(q(T_i))} - \frac{\dot{q}_d(T_i)}{\cos^2(q_d(T_i))} \end{pmatrix}, \quad h_i = \frac{\tau}{N_i} \ ,$$

and $\Omega : [0, +\infty)^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\Omega(T, h, x; v) = \begin{bmatrix} \Omega_1(T, h, x; v) \\ \Omega_2(T, h, x; v) \end{bmatrix} \tag{8}$$

and the formulas

$$\begin{aligned} \Omega_1(T, h, x; v) &= x_1 + hx_2 \ \text{ and} \\ \Omega_2(T, h, x; v) &= x_2 + \zeta_{2,d}(T) + \int_T^{T+h} g_1(\zeta_d(s)+x)\mathrm{d}s \\ &\quad + \int_T^{T+h} g_2(\zeta_d(s)+x)\, v(s-\tau)\mathrm{d}s - \zeta_{2,d}(T+h). \end{aligned}$$

# NMES Theorem (IK, MM, MK, Ruzhou, IJRNC)

For each pair $(\tau, r) \in (0, \infty)^2$, there exist a locally bounded function $N$, a constant $\omega \in (0, \mu/2)$ and a locally Lipschitz function $C$ satisfying $C(0) = 0$ such that: For all sample times $\{T_i\}$ in $[0, \infty)$ such that $\sup_{i \geq 0} (T_{i+1} - T_i) \leq r$ and each initial condition, the solution $(q(t), \dot{q}(t), v(t))$ with

$$N_i = N\left(\left|\left(\tan(q(T_i)), \frac{\dot{q}(T_i)}{\cos^2(q(T_i))}\right) - \zeta_d(T_i)\right| + ||v - v_d||_{[T_i - \tau, T_i]}\right) \quad (9)$$

satisfies

$$\begin{aligned} &|q(t) - q_d(t)| + |\dot{q}(t) - \dot{q}_d(t)| + ||v - v_d||_{[t-\tau, t]} \\ \leq \; & e^{-\omega t} C\left(\frac{|q(0) - q_d(0)| + |\dot{q}(0) - \dot{q}_d(0)|}{\cos^2(q(0))} + ||v_0 - v_d||_{[-\tau, 0]}\right) \end{aligned}$$

for all $t \geq 0$.

# MATLAB Simulation

$$J\ddot{q} + b_1\dot{q} + b_2 \tanh(b_3\dot{q}) + k_1 q e^{-k_2 q} + k_3 \tan(q) + \mathcal{M} g l \sin(q) = \mathcal{A}(q,\dot{q})\, v(t-\tau), \quad q \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \tag{10}$$

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$\tau = 0.07\text{s}$, $\mathcal{A}(q,\dot{q}) = \bar{a}e^{-2q^2}\sin(q) + \bar{b}$

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&J = 0.39\,\mathrm{kg\text{-}m^2/rad},\ b_1 = 0.6\,\mathrm{kg\text{-}m^2/(rad\text{-}s)},\ \bar{a} = 0.058, \\
&b_2 = 0.1\,\mathrm{kg\text{-}m^2/(rad\text{-}s)},\ b_3 = 50\,\mathrm{s/rad},\ \bar{b} = 0.0284, \\
&k_1 = 7.9\,\mathrm{kg\text{-}m^2/(rad\text{-}s^2)},\ k_2 = 1.681/\mathrm{rad}, \\
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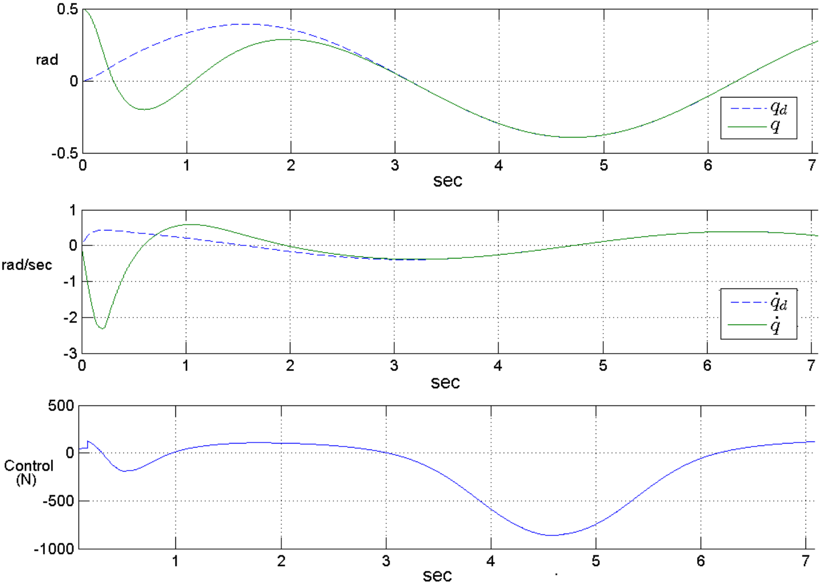
$q(0) = 0.5\,\text{rad}$, $\dot{q}(0) = 0\,\text{rad/s}$, $v(t) = 0$ on $[-0.07, 0)$, $N_i = N = 10$, and $T_{i+1} - T_i = 0.014\text{s}$, and $\mu = 2$.

# MATLAB Simulation



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In future work, we hope to apply input-to-state stability to better understand the effects of uncertainties under state constraints.