Bounded Backstepping for Nonlinear Control Systems

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- ▷ R. Freeman, H. Khalil, M. Krstic, F. Mazenc, J. Tsinias, ...

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$$\begin{cases} \dot{x}(t) &= \mathcal{F}(t, x(t), z_1(t)) \\ \dot{z}_i(t) &= z_{i+1}(t), \quad i \in \{1, \ldots, k-1\} \\ \dot{z}_k(t) &= u(t) + \sum_{j=1}^{k} v_j z_j(t) \end{cases} \tag{1}$$

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Assumption 1: There is a bounded locally Lipschitz ϑ such that

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Assumption 2: The function $\mathcal{F}$ in (1) is continuous in t and globally Lipschitz in (x, z_1) and $\mathcal{F}(t, 0, 0) = 0$ for all $t \geq 0$.

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Assumption 3: (CICS) There is a globally Lipschitz bounded ω such that $\omega(0) = 0$, and constants $T > 0$ and $q > 0$, such that for each continuous $\delta : [0, +\infty) \to \mathbb{R}$ that exponentially converges to 0, the following is true: All solutions $\xi : [0, +\infty) \to \mathbb{R}^n$ of

$$\dot{\xi}(t) = \mathcal{F}\left(t, \xi(t), \int_{t-T}^{t} \frac{e^{q(\ell-t)} Q(t,\ell,\ell+T)\omega(\xi(\ell))}{\int_{-T}^{0} e^{qr} r^{k-1}(r+T)^{k-1} \mathrm{d}r} \mathrm{d}\ell + \delta(t)\right) \tag{C}$$

satisfy $\lim_{t\to+\infty} \xi(t) = 0$, where $Q(t, a, b) = (t-a)^{k-1}(t-b)^{k-1}$.

Dynamic Extension

$$\begin{cases} \dot{x}(t) &= \mathcal{F}(t, x(t), z_1(t)) \\ \dot{z}_i(t) &= z_{i+1}(t), \quad i \in \{1, \ldots, k-1\} \\ \dot{z}_k(t) &= u(t) + \sum_{j=1}^{k} v_j z_j(t) \\ \dot{Y}(t) &= J_{2k-1} Y(t) + \frac{e_{2k-1}}{T} \frac{\omega(x(t))}{b_T}, \quad \text{where} \end{cases} \tag{AUG}$$

$$J_{2k-1} = \begin{bmatrix} -q & 1 & 0 & \ldots & 0 \\ 0 & -q & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -q & 1 \\ 0 & \ldots & \ldots & 0 & -q \end{bmatrix} \in \mathbb{R}^{(2k-1)\times(2k-1)}$$

and $b_T = \int_{-T}^{0} e^{q\ell} \ell^{k-1} (\ell + T)^{k-1} \mathrm{d}\ell$ with q and T as above.

Our Theorem

If Assumptions 1-3 hold, then we can construct positive constants a, b, and c and constant row vectors R_i such that all maximal solutions (x, z, Y) of (AUG) with the choices

$$\begin{array}{rcl} u(t) & = & \text{sat}_a(R_0\Psi(Y_t)) \\ & & +b\frac{\omega(x(t))}{b_T} + c\frac{\omega(x(t-T))}{b_T} + \vartheta(z_\star(t)), \\ \Psi(Y_t) & = & Y(t) - e^{TJ_{2k-1}}Y(t-T) \\ z_*(t) & = & (z_1(t) + R_1\Psi(Y_t), \ldots, z_k(t) + R_k\Psi(Y_t)) \end{array} \tag{2}$$

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Benefit: Bounded $u(t)$ under mild checkable (CICS) condition.

Sufficient Conditions for (CICS) for Small T

There exist f and g that are uniformly globally Lipschitz in x and continuous such that $\mathcal{F}(t, x, p) = f(t, x) + g(t, x)p$ holds for all $t \geq 0$, $x \in \mathbb{R}^n$, and $p \in \mathbb{R}$.

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$$\begin{aligned} & V_t(t,x) + V_x(t,x)\big(f(t,x) + g(t,x)\omega(x)\big) \leq -W(x), \\ & |V_x(t,x)g(t,x)| \leq r_0\sqrt{W(x)},\ |\omega(x)| \leq r_1\sqrt{W(x)}, \\ & |f(t,x)| \leq r_2\sqrt{W(x)},\ \text{ and }\ |g(t,x)| \leq r_3, \end{aligned} \tag{DB}$$

where ω is bounded, globally Lipschitz, and satisfies $\omega(0) = 0$.

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Mazenc, F., M. Malisoff, and L. Burlion. Bounded backstepping through a dynamic extension with delays. In *Proceedings of the 56th IEEE Conference on Decision and Control (Melbourne, Australia, 12-15 December 2017)*.

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Thank you for your attention!