Tracking Control and Robustness for Planar Vertical Takeoff and Landing Aircraft under Bounded Feedbacks

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1. PVTOL Model (Hauser-Sastry-Meyer, 1992)

This benchmark system models aircraft position and roll angle.

$$\begin{cases} \ddot{x} = -\bar{u}_1 \sin(\theta) + \varepsilon u_2 \cos(\theta) \\ \ddot{y} = \bar{u}_1 \cos(\theta) + \varepsilon u_2 \sin(\theta) - g \\ \ddot{\theta} = u_2 \end{cases} \tag{1}$$

- $(x, y) =$ lateral and vertical coordinates of center of mass.
- $\theta =$ roll angle relative to the horizon.
- $\bar{u}_1 =$ thrust control directed out of bottom.
- $g =$ gravitational constant. • $u_2 =$ rolling moment control.
- $\varepsilon =$ coupling between roll moment and lateral acceleration.

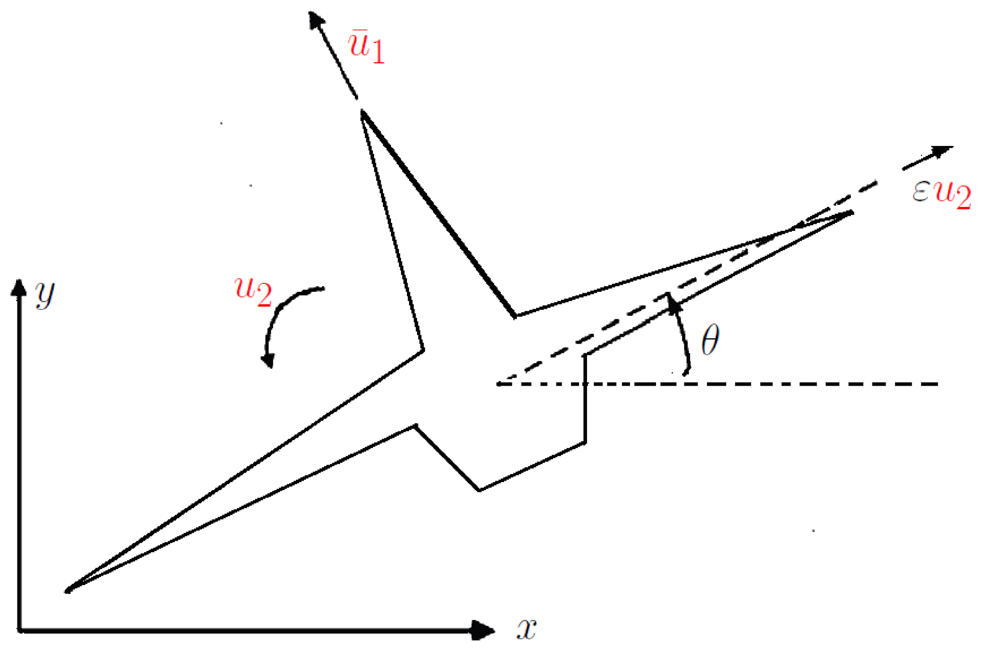

Figure 1: *Model*

2. Change of Coordinates (Olfati-Saber, 2002)

The Olfati-Saber changes of variables $z_1 = x - \varepsilon \sin(\theta)$, $z_2 = \dot{x} - \varepsilon\dot{\theta}\cos(\theta)$, $w_1 = y + \varepsilon(\cos(\theta) - 1)$, $w_2 = \dot{y} - \varepsilon\dot{\theta}\sin(\theta)$, $\xi_1 = \theta$, and $\xi_2 = \dot{\theta}$ and change of feedback $u_1 = \bar{u}_1 - \varepsilon\xi_2^2$ give

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -u_1 \sin(\xi_1) \\ \dot{w}_1 = w_2 \\ \dot{w}_2 = u_1 \cos(\xi_1) - g \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = u_2. \end{cases} \tag{2}$$

The new state variable is $Q = (z_1, z_2, w_1, w_2, \xi_1, \xi_2)^\top$.

3. Tracking Dynamics and Objective

Choose any C^2 reference trajectory-input pair for (2) such that

a) $\exists c_1 \in (0, \pi/2)$ such that $\xi_{1r}(t) \in [-\pi/2 + c_1, \pi/2 - c_1]$ $\forall t \geq 0$,
b) $\dot{\xi}_{1r}$ and $\ddot{\xi}_{1r}$ are bounded, and
c) $u_r = (u_{1r}, u_{2r})^\top$, $\dot{u}_r$, and $\ddot{u}_r$ are bounded and $\inf_{t\geq 0} u_{1r}(t) > 0$.

Taking $Q_r = (z_{1r}, z_{2r}, w_{1r}, w_{2r}, \xi_{1r}, \xi_{2r})^\top$ gives the tracking system

$$\dot{Q} - \dot{Q}_r = \begin{cases} \dot{\tilde{z}}_1 = \tilde{z}_2 \\ \dot{\tilde{z}}_2 = -u_1 \sin(\xi_1) + u_{1r}(t)\sin(\xi_{1r}(t)) \\ \dot{\tilde{w}}_1 = \tilde{w}_2 \\ \dot{\tilde{w}}_2 = u_1 \cos(\xi_1) - u_{1r}(t)\cos(\xi_{1r}(t)) \\ \dot{\tilde{\xi}}_1 = \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 = u_2 - u_{2r}(t) \end{cases} \tag{3}$$

We want bounded controllers u_i to make (3) UGAS and ULES to 0.
Main Challenges: u_1 must stay positive and (3) is underactuated.

4. Compactly Supported Smooth Indicator

$$\varphi_\ell(x) = 1 - \frac{1}{B_\ell}\int_{4\ell}^{\max\{4\ell, \min\{|x|, 6\ell\}\}} (q - 4\ell)^4 (q - 6\ell)^4 \mathrm{d}q \tag{4}$$

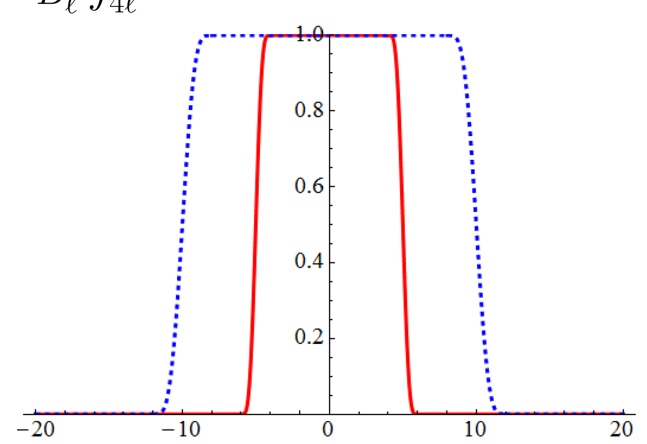

Figure 2: *Dashed and blue:* φ_2. *Solid and red:* φ_1.

5. Bounded Thrust (BT) Controller

$$\begin{aligned} u_1(t, \tilde{z}, \tilde{w}) &= \tfrac{1}{\cos(v)}\left[u_{1r}(t)\cos(\xi_{1r}(t)) + U_\lambda(\tilde{w})\right], \quad \text{where} \\ v(t, \tilde{z}) &= \arctan\left(\tan(\xi_{1r}(t)) - \frac{U_\lambda(\tilde{z})}{u_{1r}(t)\cos(\xi_{1r}(t))}\right) \\ U_\lambda(Z) &= \frac{-\sigma_\lambda\big(2Z_2 + \sigma_\lambda(\lambda Z_1)\varphi_\lambda(Z_2)\big) - \lambda\sigma'_\lambda(\lambda Z_1)\varphi_\lambda(Z_2)Z_2}{2 + \sigma_\lambda(\lambda Z_1)\varphi'_\lambda(Z_2)} \\ \sigma_\lambda(x) &= \tfrac{2\lambda}{\pi}\arctan\left(\tfrac{\pi x}{2\lambda}\right), \ \lambda > 0 \text{ is a small enough constant} \end{aligned} \tag{BT}$$

6. Transformed Tracking Dynamics (TTD)

Take $\varpi_1 = \xi_1 - v$, and suitable T_λ and S_λ such that $\dot{\varpi}_1 = \tilde{\xi}_2 - S_\lambda - T_\lambda$, and $\dot{S}_\lambda$ and T_λ are bounded. Set $u_3 = u_2 - u_{2r} - \dot{S}_\lambda$.

$$\begin{cases} \dot{\tilde{z}}_1 = \tilde{z}_2 \\ \dot{\tilde{z}}_2 = -u_1 \sin(\varpi_1 + v) + u_{1r}(t)\sin(\xi_{1r}(t)) \\ \dot{\tilde{w}}_1 = \tilde{w}_2 \\ \dot{\tilde{w}}_2 = u_1 \cos(\varpi_1 + v) - u_{1r}(t)\cos(\xi_{1r}(t)) \\ \dot{\varpi}_1 = \varpi_2 - \mathcal{T}_\lambda(t, \varpi_1, \tilde{z}, \tilde{w}) \\ \dot{\varpi}_2 = u_3 \end{cases} \tag{TTD}$$

It suffices to design u_3 to make (TTD) UGAS and ULES to 0.

7. Designing u_3

We can choose constants $a > 0$ and $\lambda > 0$ such that

$$u_3(t, \tilde{z}, \tilde{w}, \varpi) = \frac{-\sigma_a\big(2\varpi_2 + \sigma_a(a\varpi_1)\varphi_a(\varpi_2)\big) - a\sigma'_a(a\varpi_1)\varphi_a(\varpi_2)\left[\varpi_2 - \mathcal{T}_\lambda(t, \varpi_1, \tilde{z}, \tilde{w})\right]}{2 + \sigma_a(a\varpi_1)\varphi'_a(\varpi_2)} \tag{5}$$

is bounded, C^1, and renders (TTD) both UGAS and ULES to 0.

8. Input-to-State Stability (ISS)

For any $\bar{\delta} > 0$, we can scale σ_λ and σ_a to prove ISS for (TTD) under disturbances $\delta = (\delta_1, \delta_2) : [0, \infty) \to \mathcal{B}_2(\bar{\delta})$ added to (u_1, u_3).

Our ISS result provides $\bar{\gamma}_i \in \mathcal{K}_\infty$ and a constant $\bar{c} > 0$ such that all trajectories of (TTD) satisfy the following for all $t \geq t_0 \geq 0$:

$$|(\tilde{z}(t), \tilde{w}(t), \varpi(t))| \ \leq \ \bar{\gamma}_1(|(\tilde{z}(t_o), \tilde{w}(t_o), \varpi(t_o))|)e^{-\bar{c}(t - t_o)} + \bar{\gamma}_2(|\delta|_\infty)$$

The ISS paradigm was introduced by Sontag in T-AC in 1989.
It agrees with UGAS when the perturbation δ is not present.

9. Key Technical Bounded Backstepping Lemma

Standard backstepping has been used for PVTOL controller design but does not ensure boundedness of controllers and so does not apply in our case. Instead, we use bounded backstepping.

Under certain conditions on a subsystem $\dot{S} = E(t, S)$ evolving on a Euclidean state space that is UGAS and ULES to 0, and on real valued functions Θ and L, we show that

$$\begin{cases} \dot{X}_1 = X_2 + \Theta(t, X) \\ \dot{X}_2 = \beta_{\ell,\bar{\eta}}(t, X) + L(t, X, S) + \eta \\ \dot{S} = E(t, S) \end{cases} \tag{6}$$

in closed loop with the bounded C^1 feedback

$$\beta_{\ell,\bar{\eta}}(t, X) = \frac{-[1 + 172\bar{\eta}/\ell]\sigma_\ell\big(2X_2 + \sigma_\ell(\ell X_1)\varphi_\ell(X_2)\big) - \ell\sigma'_\ell(\ell X_1)\varphi_\ell(X_2)[X_2 + \Theta(t, X)]}{2 + \sigma_\ell(\ell X_1)\varphi'_\ell(X_2)} \tag{7}$$

is UGAS and ULES to 0 when $\eta \equiv 0$, and ISS with respect to disturbances $\eta : [0, \infty) \to \mathrm{Ball}_{\bar{\eta}}$. This works for any bound $\bar{\eta} > 0$.

We apply this three times for different subsystems to cover (TTD). Take $(X, S) = (\varpi, 0)$, then $(X, S) = (\tilde{w}, \varpi)$, and then $(X, S) = (\tilde{z}, (\tilde{w}, \varpi))$ for appropriate choices of Θ and L.

10. Trackable Trajectories

If $(z_{1r}, w_{1r}) : [0, \infty) \to \mathbb{R}^2$ is any C^4 time-periodic function such that $\ddot{w}_{1r}(t) + g$ is positive valued, then we can track using

$$u_{1r} = \sqrt{(\ddot{z}_{1r})^2 + (\ddot{w}_{1r} + g)^2} \quad \text{and} \quad u_{2r} = \ddot{\xi}_{1r} \ , \tag{8}$$

and with $\xi_{2r} = \dot{\xi}_{1r}$, $z_{2r} = \dot{z}_{1r}$, $w_{2r} = \dot{w}_{1r}$, and

$$\xi_{1r} = \arcsin\left(\frac{-\ddot{z}_{1r}}{\sqrt{(\ddot{z}_{1r})^2 + (\ddot{w}_{1r} + g)^2}}\right) . \tag{9}$$

We applied this to

$$\big(z_{1r}(t), w_{1r}(t)\big)^\top = 5\big(1.5 + \cos(t), 1.5 + \sin(t)\big)^\top \tag{10}$$

$\lambda = 2$, and $a = 10.14$. We simulated with and without δ_2 added to u_3.

For the disturbance bound $\bar{\delta} = 0.25$, we switched to

$$u_3 = \frac{-[1 + 172\bar{\delta}/a]\sigma_a\big(2\varpi_2 + \sigma_a(a\varpi_1)\varphi_a(\varpi_2)\big) - a\sigma'_a(a\varpi_1)\varphi_a(\varpi_2)\left[\varpi_2 - \mathcal{T}_\lambda(t, \varpi_1, \tilde{z}, \tilde{w})\right]}{2 + \sigma_a(a\varpi_1)\varphi'_a(\varpi_2)} .$$

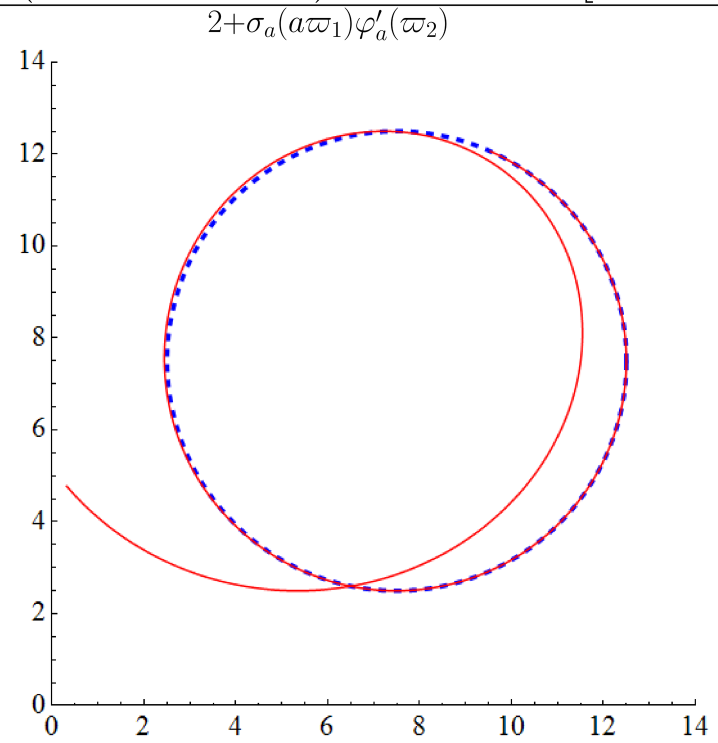

Figure 3: $(z_1, w_1)^\top$ *Tracking* $(z_{1r}(t), w_{1r}(t))^\top$ *with* $\delta = 0$.

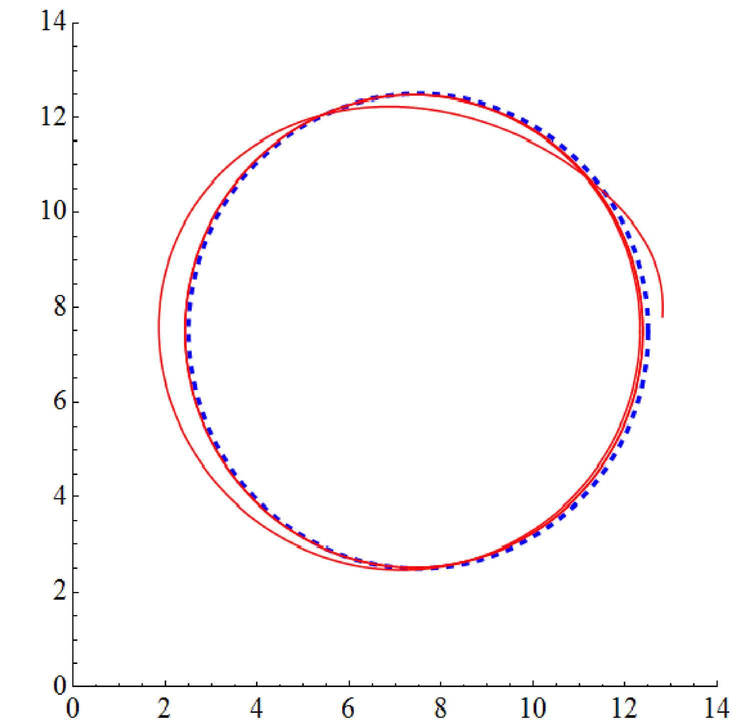

Figure 4: $(z_1, w_1)^\top$ *Tracking* $(z_{1r}, w_{1r})^\top$ *with* $\delta_2 = 0.15$.

We can also track along Cassini's Oval

$$\begin{aligned} \big(z_{1r}(t), w_{1r}(t)\big)^\top &= R(t)\big(\cos(t), \sin(t)\big)^\top, \quad \text{where} \\ R(t) &= \sqrt{a^2\cos(2t) + \sqrt{b^4 - (a^2\sin(2t))^2}} \ . \end{aligned} \tag{11}$$

If $a = 2.65$ and $b = 2.9$, then $\ddot{w}_{1r}(t) + g \geq 0.552321$ for all $t \geq 0$.

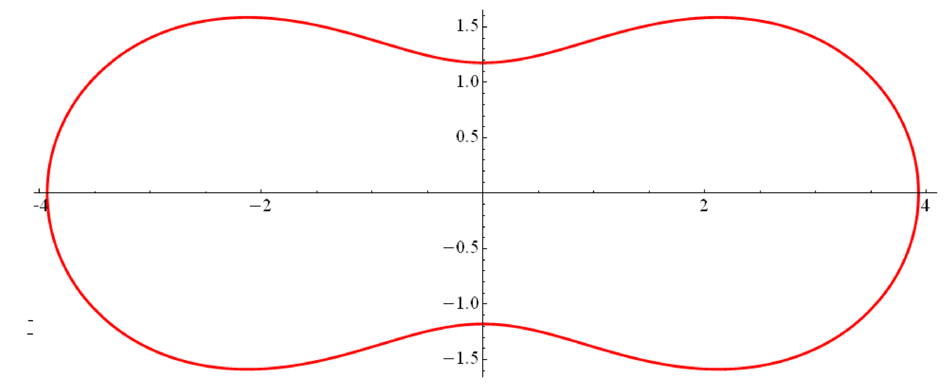

Figure 5: *Cassini's Oval with* $a = 2.65$ *and* $b = 2.9$.

11. Conclusions

- The PVTOL aircraft dynamics is a benchmark model that is of continuing ongoing research interest.
- We developed a new bounded tracking feedback design that gives UGAS and ULES for a large class of reference trajectories.
- Combined with the Do-Jiang-Pan observer design, our feedbacks apply when the velocity measurements are unavailable.
- Our feedbacks give ISS performance to actuator disturbances for any a priori bound on the admissible disturbances.
- Our proofs used a new bounded backstepping method which we anticipate being useful for other models in feedforward form.

12. References

- Gruszka, A., M. Malisoff, and F. Mazenc, "On tracking for the PVTOL model with bounded feedbacks," in Proceedings of the 2011 American Control Conference, accepted as regular paper. [Finalist for Student Best Paper Award]
- Gruszka, A., M. Malisoff, and F. Mazenc, "Tracking control and robustness analysis for PVTOL aircraft under bounded feedbacks," submitted in November 2010, in review.

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