Stabilization of Two-Species Chemostats with Delayed Measurements and Haldane Growth Functions

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Biological Systems Session
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Chemostat Apparatus

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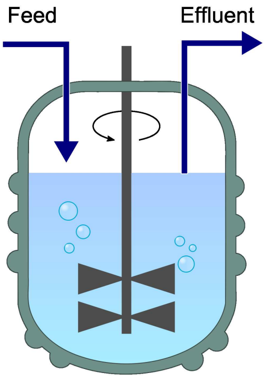
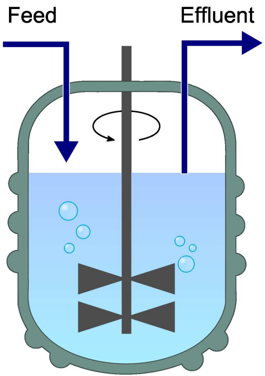


Figure 1: Chemostat

Chemostat Apparatus



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Bioreactor.

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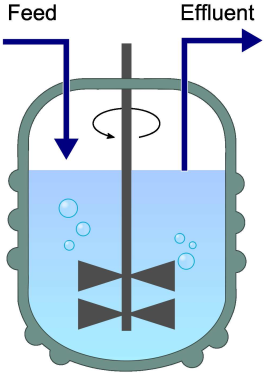

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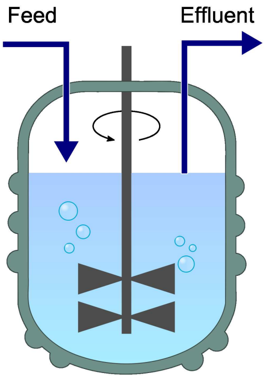

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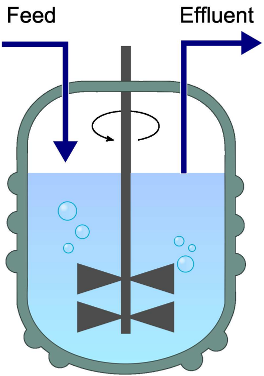

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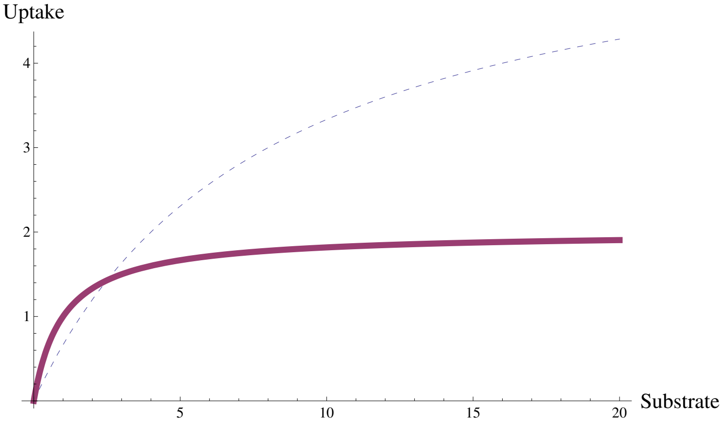

Figure 2: $(K_1, L_1, g_1) = (6, 8, 0.00)$ and $(K_2, L_2, g_2) = (2, 1, 0.00)$

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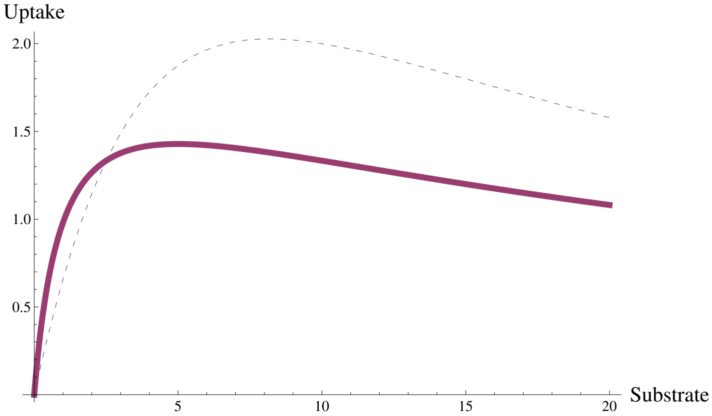

Figure 3: $(K_1, L_1, g_1) = (6, 8, 0.12)$ and $(K_2, L_2, g_2) = (2, 1, 0.04)$

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$x_{1*}, x_{2*} > 0$ are any constants such that $s_* + x_{1*} + x_{2*} = s_{\text{in}}$.

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- Standard Poincaré-Bendixson and Lyapunov function methods do not apply under delays.

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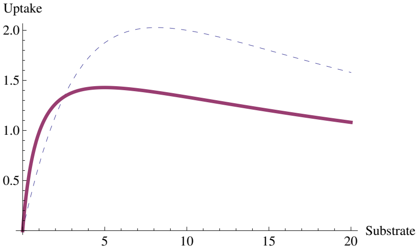

Figure 4: μ_1 (Dashed) and μ_2 (Solid).

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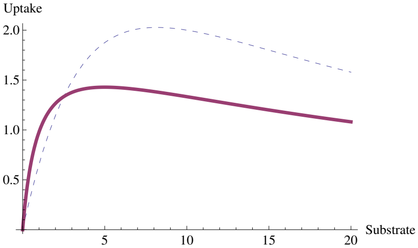

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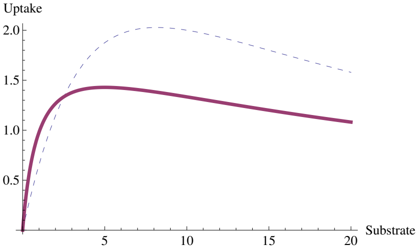

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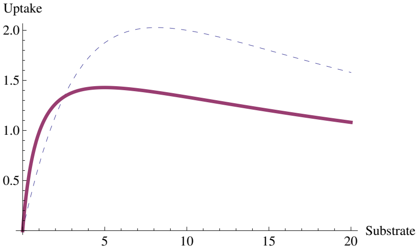

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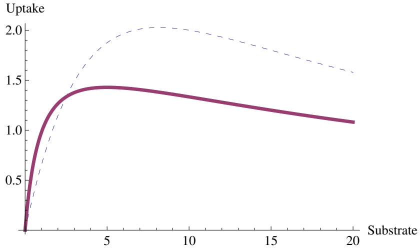

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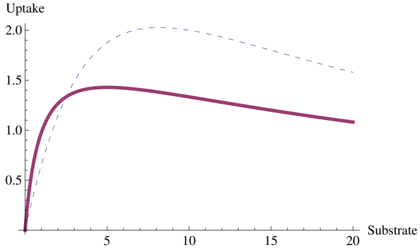

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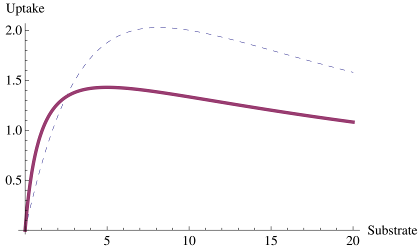

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We took $(s(t), x_1(t), x_2(t)) \equiv (2.5, 1, 0.1)$ on $[-0.5, 0]$.

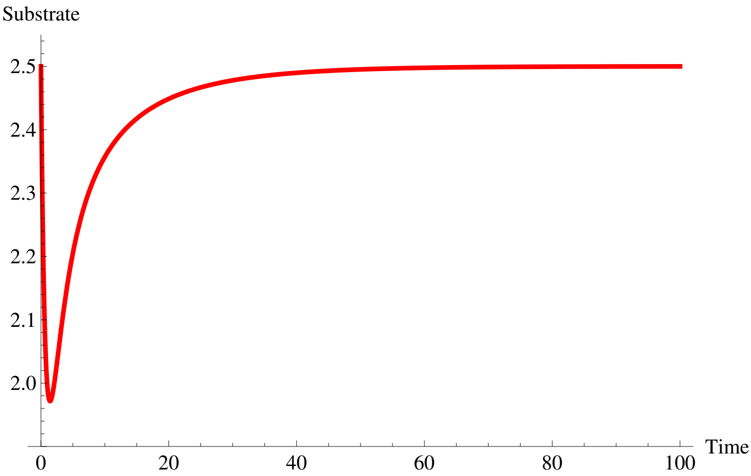

Figure 5: Convergence of s towards $s_* = 2.5$.

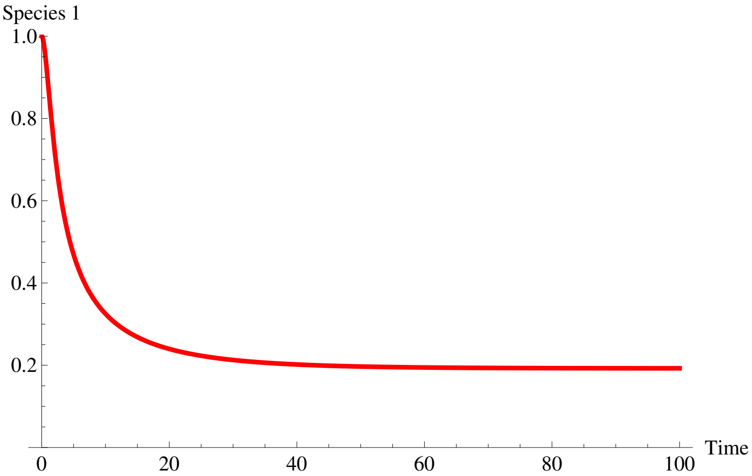

Figure 6: Convergence of x_1 towards $x_{1*} = 0.19$.

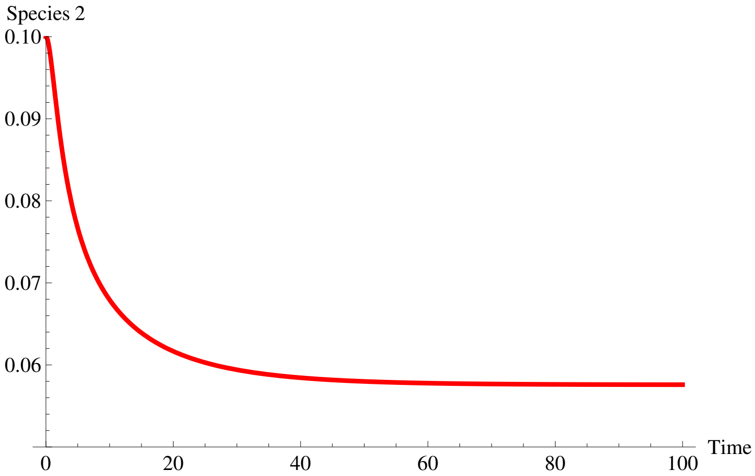

Figure 7: Convergence of x_2 towards $x_{2*} = 0.06$.

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- For details and proofs, see [Mazenc, F., and M. Malisoff, *Automatica*, Vol. 46, No. 9, Sept. 2010, regular paper.]