Stabilization and Robustness Analysis for a Chemostat Model with Two Species

MICHAEL MALISOFF
Department of Mathematics
Louisiana State University

Joint with Frédéric Mazenc and Jérôme Harmand

Special Session on Mathematical Modeling in Biology, IV
March 2008 AMS Spring Southeastern Meeting
201 Tureaud, Louisiana State University, Baton Rouge

OUTLINE

- Review of Control Theory
- Model and Objectives
- Our Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Our Robustness Result
- Numerical Validation
- Further Research

OUTLINE

- Review of Control Theory
- Model and Objectives
- Our Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Our Robustness Result
- Numerical Validation
- Further Research

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

q = state variable

y = output

u = controller depending on y

$\mathbf{d}$ = unknown disturbance function

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

- q = state variable
- y = output
- u = controller depending on y
- $\mathbf{d}$ = unknown disturbance function

Trajectories for initial state q_0 are denoted $q(t) = \phi(t, q_0, \mathbf{d})$.

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

- q = state variable
- y = output
- u = controller depending on y
- $\mathbf{d}$ = unknown disturbance function

Trajectories for initial state q_0 are denoted $q(t) = \phi(t, q_0, \mathbf{d})$.

Goal of Control Theory: Find an explicit $u(y)$ so that all trajectories $q(t)$ meet some prescribed control objective.

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

- q = state variable
- y = output
- u = controller depending on y
- $\mathbf{d}$ = unknown disturbance function

Trajectories for initial state q_0 are denoted $q(t) = \phi(t, q_0, \mathbf{d})$.

Goal of Control Theory: Find an explicit $u(y)$ so that all trajectories $q(t)$ meet some prescribed control objective.

Main Method: Design $u(y)$ in conjunction with an explicit construction of a Lyapunov function for the control system.

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

- q = state variable
- y = output
- u = controller depending on y
- $\mathbf{d}$ = unknown disturbance function

Trajectories for initial state q_0 are denoted $q(t) = \phi(t, q_0, \mathbf{d})$.

Goal of Control Theory: Find an explicit $u(y)$ so that all trajectories $q(t)$ meet some prescribed control objective.

Main Method: Design $u(y)$ in conjunction with an explicit construction of a Lyapunov function for the control system.

Significance: Explicit Lyapunov functions allow us to precisely quantify the effect of the uncertainty $\mathbf{d}(t)$.

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

- q = state variable
- y = output
- u = controller depending on y
- $\mathbf{d}$ = unknown disturbance function

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

- q = state variable
- y = output
- u = controller depending on y
- $\mathbf{d}$ = unknown disturbance function

ISS [Sontag, 1989]: $\exists$ functions $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that $|q(t)| \leq \beta(|q(0)|, t) + \gamma(|\mathbf{d}|_\infty)$ along all trajectories.

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

- q = state variable
- y = output
- u = controller depending on y
- $\mathbf{d}$ = unknown disturbance function

ISS [Sontag, 1989]: $\exists$ functions $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that $|q(t)| \leq \beta(|q(0)|, t) + \gamma(|\mathbf{d}|_\infty)$ along all trajectories.

$\mathcal{KL}$: Means (0) β continuous, (1) $\beta(\cdot, t) \in \mathcal{K}_\infty$ $\forall t \geq 0$ and (2) $\forall r \geq 0$, $\beta(r, \cdot)$ is non-increasing and $\beta(r, t) \to 0$ as $t \to +\infty$.
$\mathcal{K}_\infty$: Means unbounded strictly increasing modulus.

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

- q = state variable
- y = output
- u = controller depending on y
- $\mathbf{d}$ = unknown disturbance function

ISS [Sontag, 1989]: $\exists$ functions $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that $|q(t)| \leq \beta(|q(0)|, t) + \gamma(|\mathbf{d}|_\infty)$ along all trajectories.

ISS Lyapunov Function: A C^1 proper positive definite function V for which there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\nabla V(q) f(q, u(q), d) \leq -\alpha_1(|q|) + \alpha_2(|d|)$ everywhere.

REVIEW of CONTROL THEORY

Control System: $\dot{q} = f(q, u, \mathbf{d})$, $y = H(q)$

- q = state variable
- y = output
- u = controller depending on y
- $\mathbf{d}$ = unknown disturbance function

ISS [Sontag, 1989]: $\exists$ functions $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that $|q(t)| \leq \beta(|q(0)|, t) + \gamma(|\mathbf{d}|_\infty)$ along all trajectories.

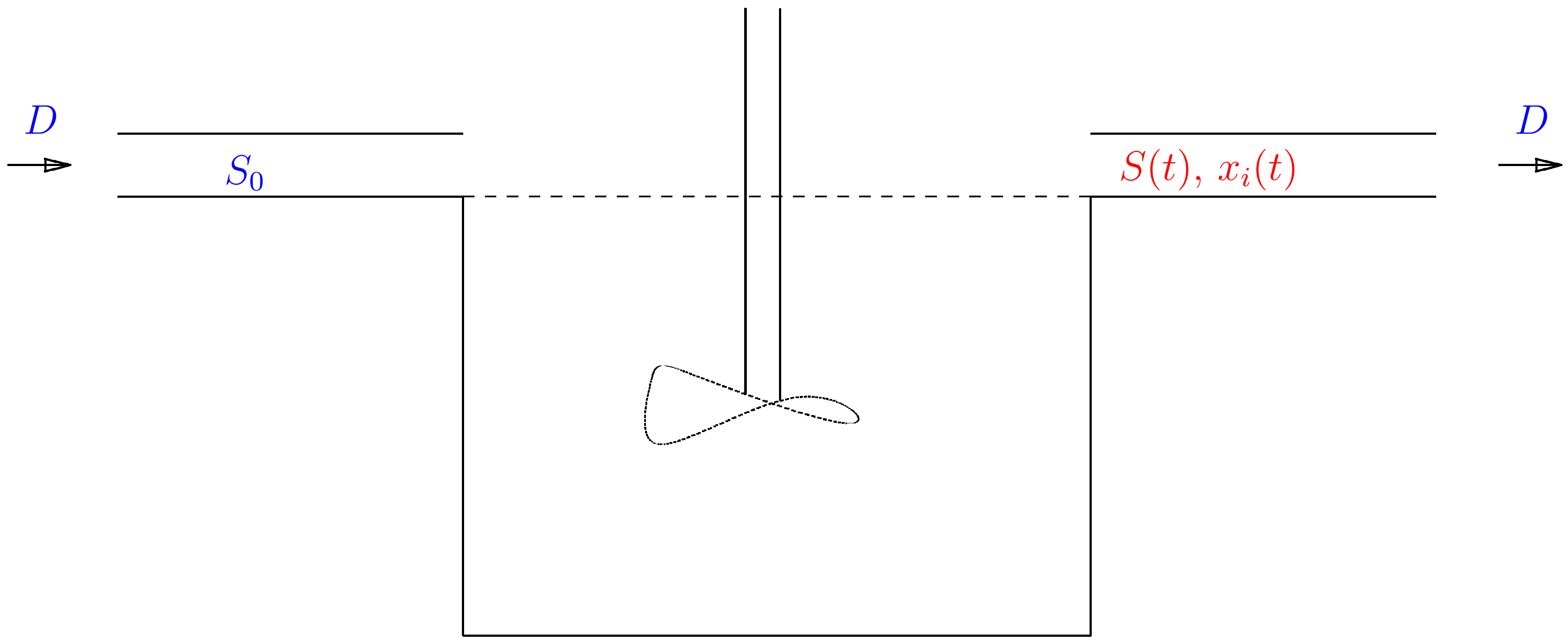
ISS Lyapunov Function: A C^1 proper positive definite function V for which there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\nabla V(q) f(q, u(q), d) \leq -\alpha_1(|q|) + \alpha_2(|d|)$ everywhere.

Lyapunov Characterizations [Sontag-Wang, 1995]: The system is ISS iff it admits an ISS Lyapunov function.

OUTLINE

- Review of Control Theory
- Model and Objectives
- Our Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Our Robustness Result
- Numerical Validation
- Further Research

CHEMOSTAT SET-UP



Feed Vessel $\longrightarrow$ Culture Vessel $\longrightarrow$ Collecting Receptacle

MODEL and GOAL

Basic Model: The two-species chemostat with nutrient concentration $S(t)$ and organism concentrations $X_i(t)$ evolving on $\mathcal{X} := (0, \infty)^3$ is

$$\begin{cases} \dot{S} = D[S_0 - S] - \frac{\mu_1(S)}{\mathcal{Y}_1} X_1 - \frac{\mu_2(S)}{\mathcal{Y}_2} X_2 \, , \\ \dot{X}_i = [\mu_i(S) - D] X_i \, , \quad i = 1, 2 \end{cases}$$

MODEL and GOAL

Basic Model: The two-species chemostat with nutrient concentration $S(t)$ and organism concentrations $X_i(t)$ evolving on $\mathcal{X} := (0,\infty)^3$ is

$$\begin{cases} \dot{S} &= D[S_0 - S] - \frac{\mu_1(S)}{\mathcal{Y}_1}X_1 - \frac{\mu_2(S)}{\mathcal{Y}_2}X_2 \, , \\ \dot{X}_i &= [\mu_i(S) - D]X_i \, , \quad i = 1, 2 \end{cases}$$

$D(\cdot)$ = dilution rate. $S_0(\cdot)$ = input nutrient concentration. $\mathcal{Y}_i$ = yield. $\mu_i(S) = \frac{K_i S}{L_i + S}$ = (Monod) uptake function, with $K_i, L_i > 0$ constants.

MODEL and GOAL

Basic Model: The two-species chemostat with nutrient concentration $S(t)$ and organism concentrations $X_i(t)$ evolving on $\mathcal{X} := (0, \infty)^3$ is

$$\begin{cases} \dot{S} &= D[S_0 - S] - \frac{\mu_1(S)}{\mathcal{Y}_1} X_1 - \frac{\mu_2(S)}{\mathcal{Y}_2} X_2 \,, \\ \dot{X}_i &= [\mu_i(S) - D] X_i \,, \quad i = 1, 2 \end{cases}$$

$D(\cdot)$ = dilution rate. $S_0(\cdot)$ = input nutrient concentration. $\mathcal{Y}_i$ = yield. $\mu_i(S) = \frac{K_i S}{L_i + S}$ = (Monod) uptake function, with $K_i, L_i > 0$ constants.

Goal: Given any $X_{i*} > 0$, design S_0 and $D(\cdot)$, depending only on $Y = X_1 + AX_2$ (where A is a given positive constant), that render $(S_*, X_{1*}, X_{2*}) \in \mathcal{X}$ robustly GAS.

MODEL and GOAL

Basic Model: The two-species chemostat with nutrient concentration $S(t)$ and organism concentrations $X_i(t)$ evolving on $\mathcal{X} := (0,\infty)^3$ is

$$\begin{cases} \dot{S} & = & D[S_0 - S] - \frac{\mu_1(S)}{\mathcal{Y}_1} X_1 - \frac{\mu_2(S)}{\mathcal{Y}_2} X_2 \, , \\ \dot{X}_i & = & [\mu_i(S) - D] X_i \, , \quad i = 1, 2 \end{cases}$$

$D(\cdot)$ = dilution rate. $S_0(\cdot)$ = input nutrient concentration. $\mathcal{Y}_i$ = yield. $\mu_i(S) = \frac{K_i S}{L_i + S}$ = (Monod) uptake function, with $K_i, L_i > 0$ constants.

Competitive Exclusion: When $S_0(\cdot)$ and D are constant and the μ_i's are increasing, at most one species survives.

OVERVIEW of LITERATURE

Coexistence: In real ecological systems, $n > 1$ species can coexist on 1 substrate, so much of the literature aims at choosing S_0 and/or D to force coexistence.

OVERVIEW of LITERATURE

Coexistence: In real ecological systems, $n > 1$ species can coexist on 1 substrate, so much of the literature aims at choosing S_0 and/or D to force coexistence.

Time-Varying Controls: Have competitive exclusion if $n = 2$ and one of the controls is fixed and the other is periodic. See Hal Smith (*SIAP*'81), Hale-Somolinos (*JMB*'83),..

OVERVIEW of LITERATURE

Coexistence: In real ecological systems, $n > 1$ species can coexist on 1 substrate, so much of the literature aims at choosing S_0 and/or D to force coexistence.

Time-Varying Controls: Have competitive exclusion if $n = 2$ and one of the controls is fixed and the other is periodic. See Hal Smith (*SIAP*'81), Hale-Somolinos (*JMB*'83),..

Feedback Controls: De Leenheer-Smith (*JMB*'03) generated a coexistence equilibrium for $n = 2, 3$. See Mazenc-M-Harmand (*ACC'07*, *TCAS'08*) for $n = 2$ with explicit Lyapunov functions and tracking of oscillations.

OVERVIEW of LITERATURE

Coexistence: In real ecological systems, $n > 1$ species can coexist on 1 substrate, so much of the literature aims at choosing S_0 and/or D to force coexistence.

Time-Varying Controls: Have competitive exclusion if $n = 2$ and one of the controls is fixed and the other is periodic. See Hal Smith (*SIAP*'81), Hale-Somolinos (*JMB*'83),..

Feedback Controls: De Leenheer-Smith (*JMB*'03) generated a coexistence equilibrium for $n = 2, 3$. See Mazenc-M-Harmand (*ACC'07*, *TCAS'08*) for $n = 2$ with explicit Lyapunov functions and tracking of oscillations.

Outputs: De Leenheer-Smith and Gouzé-Robledo (*IJRNC*'06..) stabilized using only $X_1 + X_2$ or S. No ISS.

OUTLINE

- Review of Control Theory
- Model and Objectives
- Our Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Our Robustness Result
- Numerical Validation
- Further Research

STANDING ASSUMPTION

$\exists\, S_* > 0$ such that (i) $\mu_1(S_*) = \mu_2(S_*)$, (ii) $\mu_2(S) < \mu_1(S)$ if $0 < S < S_*$, and (iii) $\mu_2(S) > \mu_1(S)$ if $S > S_*$.

STANDING ASSUMPTION

$\exists\ S_* > 0$ such that (i) $\mu_1(S_*) = \mu_2(S_*)$, (ii) $\mu_2(S) < \mu_1(S)$ if $0 < S < S_*$, and (iii) $\mu_2(S) > \mu_1(S)$ if $S > S_*$.

Example: Take $\mu_1(S) = \frac{0.5S}{0.05+S}$ and $\mu_2(S) = \frac{S}{1+S}$.

STANDING ASSUMPTION

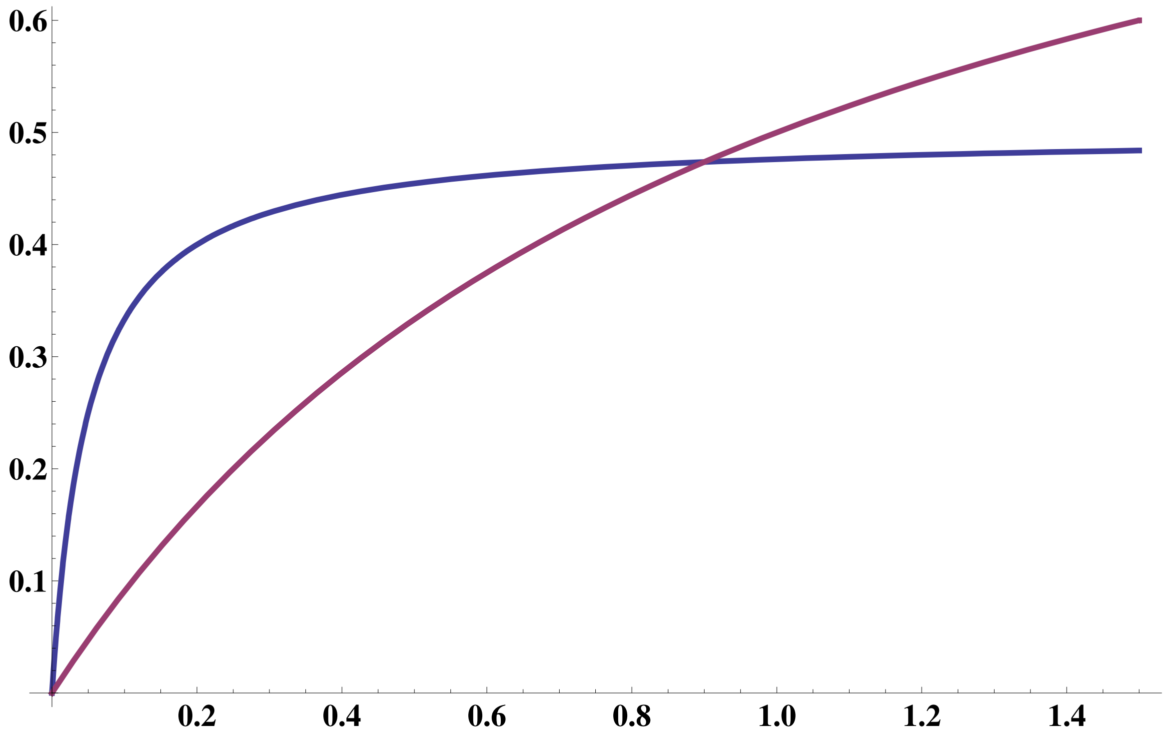
$\exists\, S_* > 0$ such that (i) $\mu_1(S_*) = \mu_2(S_*)$, (ii) $\mu_2(S) < \mu_1(S)$ if $0 < S < S_*$, and (iii) $\mu_2(S) > \mu_1(S)$ if $S > S_*$.

Example: Take $\mu_1(S) = \frac{0.5S}{0.05+S}$ and $\mu_2(S) = \frac{S}{1+S}$. $S_* = 0.9$.

STANDING ASSUMPTION

$\exists\, S_* > 0$ such that (i) $\mu_1(S_*) = \mu_2(S_*)$, (ii) $\mu_2(S) < \mu_1(S)$ if $0 < S < S_*$, and (iii) $\mu_2(S) > \mu_1(S)$ if $S > S_*$.

Example: Take $\mu_1(S) = \frac{0.5S}{0.05+S}$ and $\mu_2(S) = \frac{S}{1+S}$. $S_* = 0.9$.



OUR MAIN STABILITY THEOREM

Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/\mathcal{Y}_i$, $y = x_1 + a x_2$, $a = A\mathcal{Y}_2/\mathcal{Y}_1$.

OUR MAIN STABILITY THEOREM

Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/\mathcal{Y}_i$, $y = x_1 + ax_2$, $a = A\mathcal{Y}_2/\mathcal{Y}_1$.
Fix any $x_{i*} > 0$. Errors: $\xi_i = \ln(x_i/x_{i*})$ and $\Sigma = \ln(S/S_*)$.

OUR MAIN STABILITY THEOREM

Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/\mathcal{Y}_i$, $y = x_1 + ax_2$, $a = A\mathcal{Y}_2/\mathcal{Y}_1$.
Fix any $x_{i*} > 0$. Errors: $\xi_i = \ln(x_i/x_{i*})$ and $\Sigma = \ln(S/S_*)$.

Theorem 1: Assume $\varepsilon \in (0, \bar{\varepsilon}]$ and $a \neq 1$. Then (S_*, x_{1*}, x_{2*}) is a GAS equilibrium for the (S, x_1, x_2) dynamics when

$$S_0 = S_* + x_{1*} + x_{2*}$$
$$D(y) = \mu_1(S_*) - \varepsilon(a-1)\sigma\left(y - x_{1*} - ax_{2*}\right).$$

OUR MAIN STABILITY THEOREM

Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/\mathcal{Y}_i$, $y = x_1 + ax_2$, $a = A\mathcal{Y}_2/\mathcal{Y}_1$.
Fix any $x_{i*} > 0$. Errors: $\xi_i = \ln(x_i/x_{i*})$ and $\Sigma = \ln(S/S_*)$.

Theorem 1: Assume $\varepsilon \in (0, \bar{\varepsilon}]$ and $a \neq 1$. Then (S_*, x_{1*}, x_{2*}) is a GAS equilibrium for the (S, x_1, x_2) dynamics when

$$S_0 = S_* + x_{1*} + x_{2*}$$
$$D(y) = \mu_1(S_*) - \varepsilon(a-1)\sigma\left(y - x_{1*} - ax_{2*}\right).$$

More precisely, we can construct a function $\beta \in \mathcal{KL}$ such that $|(\Sigma, \xi_1, \xi_2)(t)| \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t)$ for all $t \geq 0$ along all trajectories $(S, x_1, x_2)(t)$ of the closed loop dynamics.

OUR MAIN STABILITY THEOREM

Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/\mathcal{Y}_i$, $y = x_1 + ax_2$, $a = A\mathcal{Y}_2/\mathcal{Y}_1$. Fix any $x_{i*} > 0$. Errors: $\xi_i = \ln(x_i/x_{i*})$ and $\Sigma = \ln(S/S_*)$.

Theorem 1: Assume $\varepsilon \in (0, \bar{\varepsilon}]$ and $a \neq 1$. Then (S_*, x_{1*}, x_{2*}) is a GAS equilibrium for the (S, x_1, x_2) dynamics when

$$S_0 = S_* + x_{1*} + x_{2*}$$
$$D(y) = \mu_1(S_*) - \varepsilon(a-1)\sigma\left(y - x_{1*} - ax_{2*}\right).$$

More precisely, we can construct a function $\beta \in \mathcal{KL}$ such that $|(\Sigma, \xi_1, \xi_2)(t)| \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t)$ for all $t \geq 0$ along all trajectories $(S, x_1, x_2)(t)$ of the closed loop dynamics.

See full paper for the explicit construction of $\bar{\varepsilon} > 0$ and β.

OUR MAIN STABILITY THEOREM

Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/\mathcal{Y}_i$, $y = x_1 + ax_2$, $a = A\mathcal{Y}_2/\mathcal{Y}_1$. Fix any $x_{i*} > 0$. Errors: $\xi_i = \ln(x_i/x_{i*})$ and $\Sigma = \ln(S/S_*)$.

Theorem 1: Assume $\varepsilon \in (0, \bar{\varepsilon}]$ and $a \neq 1$. Then (S_*, x_{1*}, x_{2*}) is a GAS equilibrium for the (S, x_1, x_2) dynamics when

$$S_0 = S_* + x_{1*} + x_{2*}$$
$$D(y) = \mu_1(S_*) - \varepsilon(a-1)\sigma\left(y - x_{1*} - ax_{2*}\right).$$

More precisely, we can construct a function $\beta \in \mathcal{KL}$ such that $|(\Sigma, \xi_1, \xi_2)(t)| \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t)$ for all $t \geq 0$ along all trajectories $(S, x_1, x_2)(t)$ of the closed loop dynamics.

Remark: *Cannot* pick $\varepsilon = 0$.

OUR MAIN STABILITY THEOREM

Set $\sigma(r) = \frac{r}{\sqrt{1+r^2}}$, $x_i = X_i/\mathcal{Y}_i$, $y = x_1 + ax_2$, $a = A\mathcal{Y}_2/\mathcal{Y}_1$. Fix any $x_{i*} > 0$. Errors: $\xi_i = \ln(x_i/x_{i*})$ and $\Sigma = \ln(S/S_*)$.

Theorem 1: Assume $\varepsilon \in (0, \bar{\varepsilon}]$ and $a \neq 1$. Then (S_*, x_{1*}, x_{2*}) is a GAS equilibrium for the (S, x_1, x_2) dynamics when

$$S_0 = S_* + x_{1*} + x_{2*}$$
$$D(y) = \mu_1(S_*) - \varepsilon(a-1)\sigma\left(y - x_{1*} - ax_{2*}\right).$$

More precisely, we can construct a function $\beta \in \mathcal{KL}$ such that $|(\Sigma, \xi_1, \xi_2)(t)| \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t)$ for all $t \geq 0$ along all trajectories $(S, x_1, x_2)(t)$ of the closed loop dynamics.

Simpler than Mazenc-M-Harmand (*ACC'07*, *TCAS'08*), outputs, robust stability, explicit strict Lyapunov function.

OUTLINE

- Review of Control Theory
- Model and Objectives
- Our Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Our Robustness Result
- Numerical Validation
- Further Research

OUTLINE

- Review of Control Theory
- Model and Objectives
- Our Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Our Robustness Result
- Numerical Validation
- Further Research

OUR ROBUSTNESS RESULTS

Using a suitable bound $\bar{\Delta}$ on $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2)$, we can design $\beta \in \mathcal{KL}$, $\alpha \in \mathcal{K}_\infty$ so that along the trajectories of

$$\dot{S} = [D(y)+\mathbf{d}_2](S_0+\mathbf{d}_1-S) - \mu_1(S)x_1 - \mu_2(S)x_2$$
$$\dot{x}_i = [\mu_i(S) - D(y) - \mathbf{d}_2]x_i, \quad i = 1, 2$$

the errors satisfy an iISS [Sontag, 1998] estimate of the form

$$\alpha(|(\Sigma, \xi_1, \xi_2)(t)|) \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t) + \int_0^t |\mathbf{d}(r)|\mathrm{d}r.$$

OUR ROBUSTNESS RESULTS

Using a suitable bound $\bar{\Delta}$ on $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2)$, we can design $\beta \in \mathcal{KL}$, $\alpha \in \mathcal{K}_\infty$ so that along the trajectories of

$$\dot{S} = [D(y) + \mathbf{d}_2](S_0 + \mathbf{d}_1 - S) - \mu_1(S)x_1 - \mu_2(S)x_2$$
$$\dot{x}_i = [\mu_i(S) - D(y) - \mathbf{d}_2]x_i, \quad i = 1, 2$$

the errors satisfy an iISS [Sontag, 1998] estimate of the form

$$\alpha(|(\Sigma, \xi_1, \xi_2)(t)|) \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t) + \int_0^t |\mathbf{d}(r)| \mathrm{d}r.$$

In the special case where $\mathbf{d}_2 \equiv 0$, we get iISS if

$$\bar{\Delta} = \frac{0.16\mu_1(S_*)S_*}{\mu_1(S_*) + \varepsilon|a-1|}.$$

OUR ROBUSTNESS RESULTS

Using a suitable bound $\bar{\Delta}$ on $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2)$, we can design $\beta \in \mathcal{KL}$, $\alpha \in \mathcal{K}_\infty$ so that along the trajectories of

$$\dot{S} = [D(y)+\mathbf{d}_2](S_0+\mathbf{d}_1-S) - \mu_1(S)x_1 - \mu_2(S)x_2$$
$$\dot{x}_i = [\mu_i(S) - D(y) - \mathbf{d}_2]x_i, \quad i = 1, 2$$

the errors satisfy an iISS [Sontag, 1998] estimate of the form

$$\alpha(|(\Sigma, \xi_1, \xi_2)(t)|) \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t) + \int_0^t |\mathbf{d}(r)| \mathrm{d}r.$$

Further reducing $\bar{\Delta}$ gives usual ISS [Sontag, 1989] estimate

$$|(\Sigma, \xi_1, \xi_2)(t)| \leq \beta(|(\Sigma, \xi_1, \xi_2)(0)|, t) + \gamma(|\mathbf{d}|_\infty).$$

OUTLINE

- Review of Control Theory
- Model and Objectives
- Our Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Our Robustness Result
- Numerical Validation
- Further Research

EXAMPLE

$$\begin{cases} \dot{S} = (D(y)+\mathbf{d}_2)(S_0+\mathbf{d}_1-S) - \frac{0.05Sx_1}{20+S} - \frac{.052Sx_2}{25+S} \\ \dot{x}_1 = \left[\frac{.05S}{20+S} - D(y) - \mathbf{d}_2\right] x_1 \\ \dot{x}_2 = \left[\frac{.052S}{25+S} - D(y) - \mathbf{d}_2\right] x_2 \end{cases}$$

Choose $y = x_1 + 0.8x_2$, and $x_{1*} = 0.05$ and $x_{2*} = 0.02$.

EXAMPLE

$$\begin{cases} \dot{S} = (D(y)+\mathbf{d}_2)(S_0+\mathbf{d}_1-S) - \frac{0.05Sx_1}{20+S} - \frac{.052Sx_2}{25+S} \\ \dot{x}_1 = \left[\frac{.05S}{20+S} - D(y) - \mathbf{d}_2\right] x_1 \\ \dot{x}_2 = \left[\frac{.052S}{25+S} - D(y) - \mathbf{d}_2\right] x_2 \end{cases}$$

Choose $y = x_1 + 0.8x_2$, and $x_{1*} = 0.05$ and $x_{2*} = 0.02$.

- Our assumptions hold with $S_* = 105$, $\varepsilon \in (0, .00753]$, $S_0 = 105.07$, and $D(y) = .042 + 0.001506\sigma(y - 0.066)$. Hence, all closed loop trajectories converge to $(105, 0.05, 0.02)$ when $\mathbf{d} = 0$.

EXAMPLE

$$\begin{cases} \dot{S} = (D(y)+\mathbf{d}_2)(S_0+\mathbf{d}_1-S) - \frac{0.05 S x_1}{20+S} - \frac{.052 S x_2}{25+S} \\ \dot{x}_1 = \left[\frac{.05S}{20+S} - D(y) - \mathbf{d}_2\right] x_1 \\ \dot{x}_2 = \left[\frac{.052S}{25+S} - D(y) - \mathbf{d}_2\right] x_2 \end{cases}$$

Choose $y = x_1 + 0.8x_2$, and $x_{1*} = 0.05$ and $x_{2*} = 0.02$.

- Our assumptions hold with $S_* = 105$, $\varepsilon \in (0, .00753]$, $S_0 = 105.07$, and $D(y) = .042 + 0.001506\sigma(y - 0.066)$. Hence, all closed loop trajectories converge to $(105, 0.05, 0.02)$ when $\mathbf{d} = 0$.
- When $\mathbf{d}_1 \equiv 0$, we get iISS to disturbances $\mathbf{d}_2(t)$ bounded by $\bar{\Delta} \approx 0.20\mu_1(S_*)$ i.e. about 20% of D.

EXAMPLE

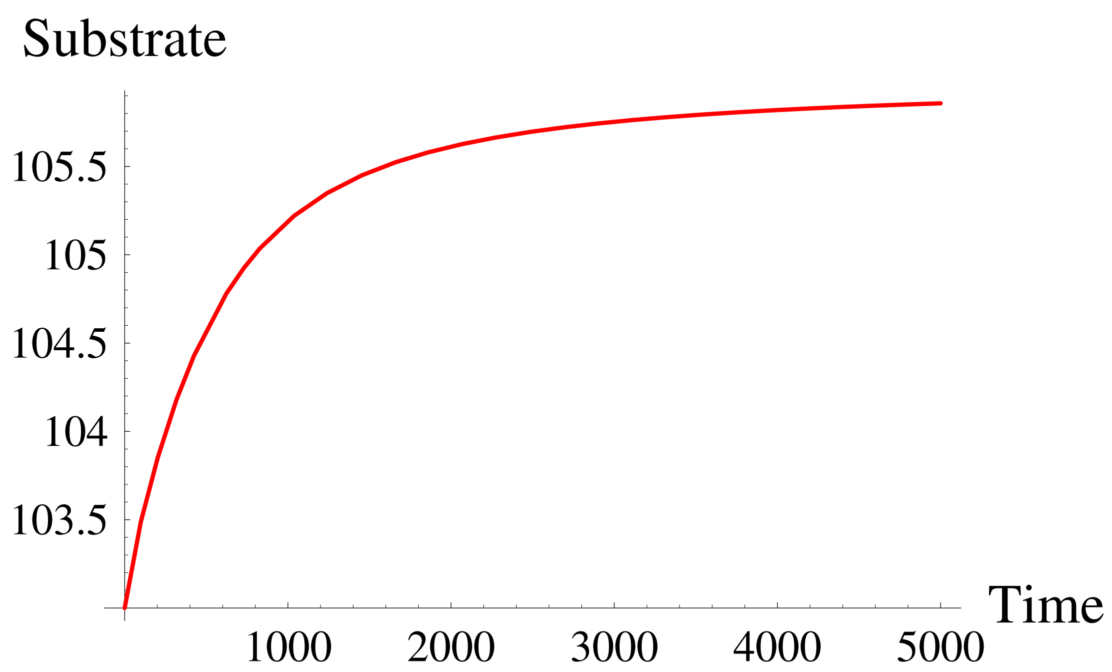
$$\begin{cases} \dot{S} = (D(y)+\mathbf{d}_2)(S_0+\mathbf{d}_1-S) - \frac{0.05 S x_1}{20+S} - \frac{.052 S x_2}{25+S} \\ \dot{x}_1 = \left[\frac{.05 S}{20+S} - D(y) - \mathbf{d}_2\right] x_1 \\ \dot{x}_2 = \left[\frac{.052 S}{25+S} - D(y) - \mathbf{d}_2\right] x_2 \end{cases}$$

Choose $y = x_1 + 0.8 x_2$, and $x_{1*} = 0.05$ and $x_{2*} = 0.02$.

- Our assumptions hold with $S_* = 105$, $\varepsilon \in (0, .00753]$, $S_0 = 105.07$, and $D(y) = .042 + 0.001506\sigma(y - 0.066)$. Hence, all closed loop trajectories converge to $(105, 0.05, 0.02)$ when $\mathbf{d} = 0$.
- If instead $\mathbf{d}_2 \equiv 0$, then we have iISS to disturbances $\mathbf{d}_1(t)$ bounded by $\bar{\Delta} \approx 16$, or about 15% of $S_0 = 105.07$.

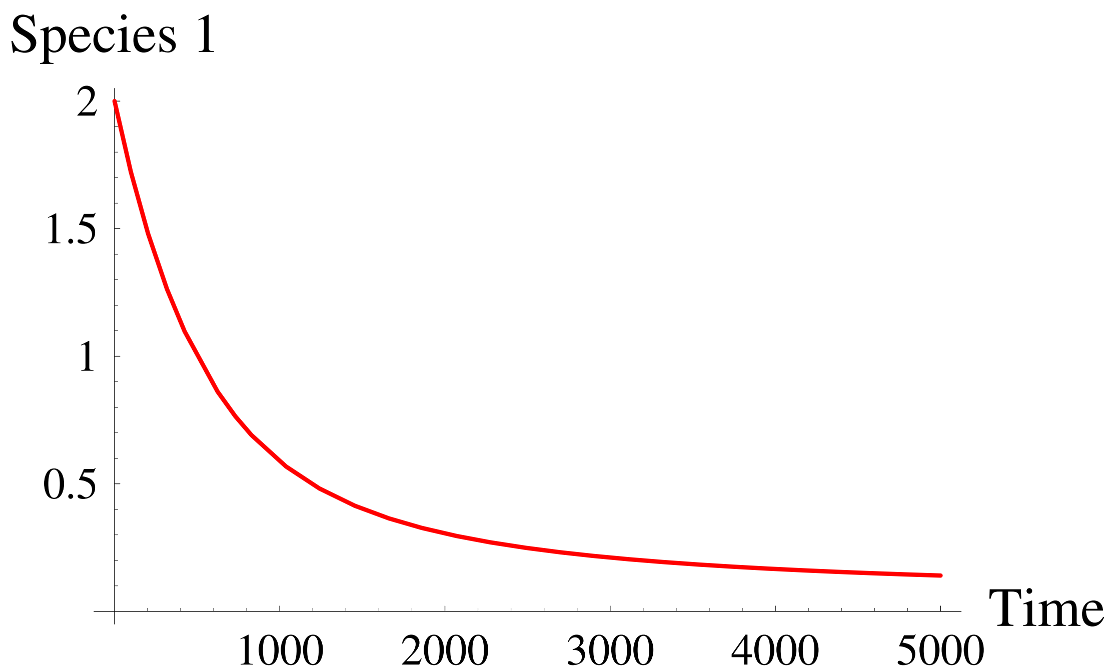
SIMULATION of EXAMPLE

We used $\mathbf{d}(t) \equiv (1, 0)$ and $(S, x_1, x_2)(0) = (103, 2, 1)$.



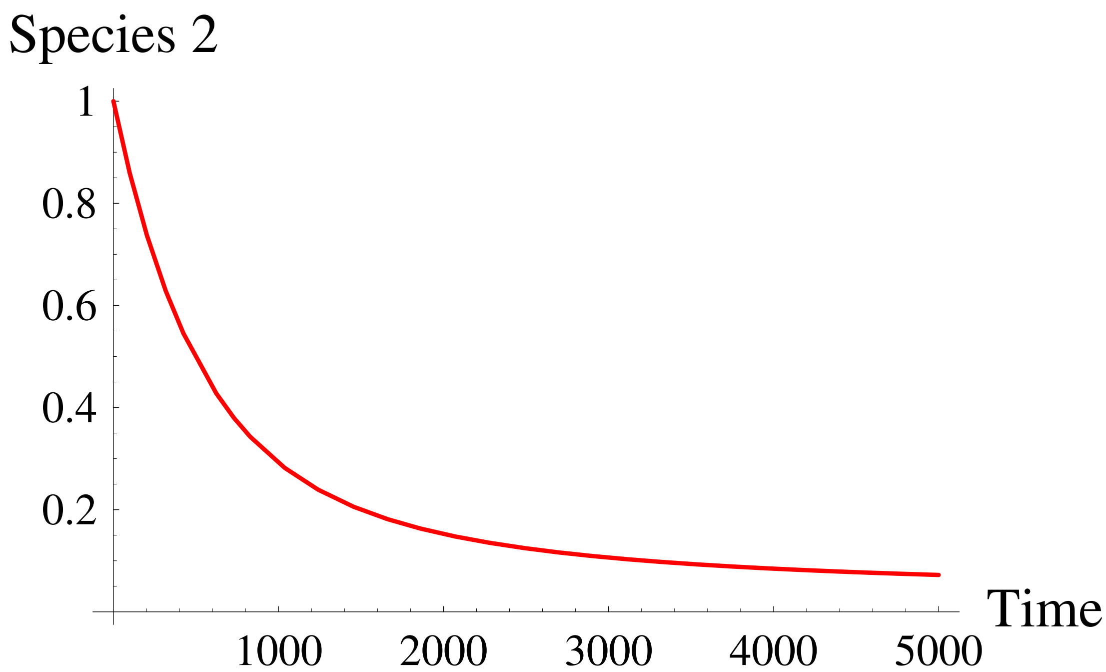
SIMULATION of EXAMPLE

We used $\mathbf{d}(t) \equiv (1, 0)$ and $(S, x_1, x_2)(0) = (103, 2, 1)$.



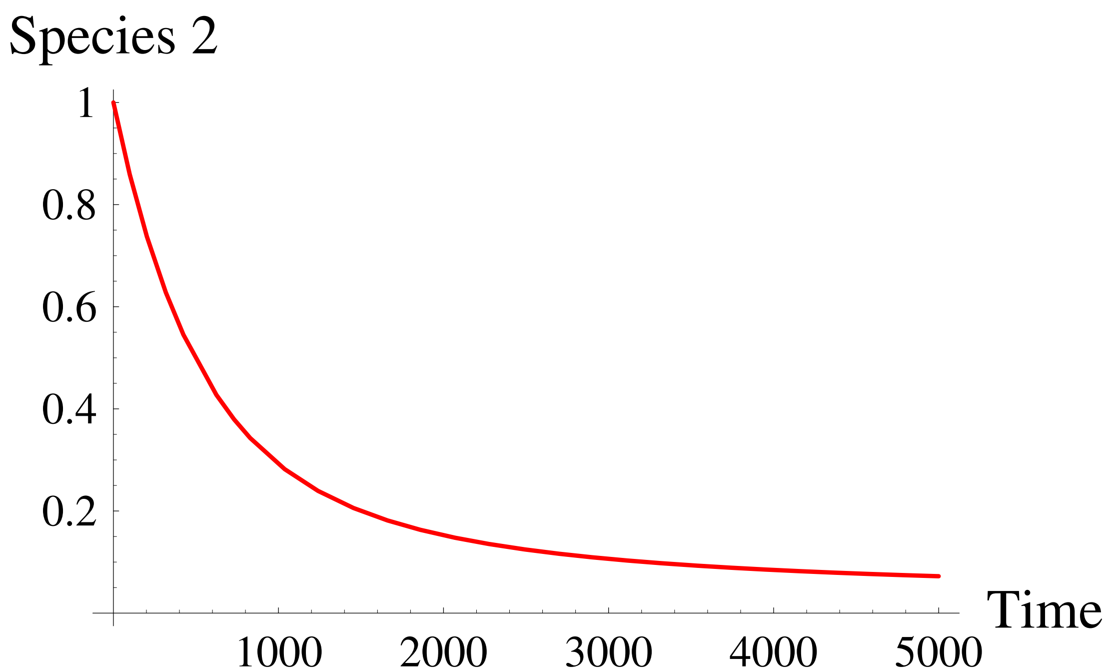
SIMULATION of EXAMPLE

We used $\mathbf{d}(t) \equiv (1, 0)$ and $(S, x_1, x_2)(0) = (103, 2, 1)$.



SIMULATION of EXAMPLE

We used $\mathbf{d}(t) \equiv (1, 0)$ and $(S, x_1, x_2)(0) = (103, 2, 1)$.



Persistence. $(S(t), x_1(t), x_2(t)) \to (105, 0.05, 0.02)$, but with an overshoot determined by iISS and the magnitude of $\mathbf{d}_1$.

OUTLINE

- Review of Control Theory
- Model and Objectives
- Our Main Stability Theorem
- Proof Ideas: Explicit Lyapunov Function
- Our Robustness Result
- Numerical Validation
- Further Research

SUGGESTIONS

- It would be of interest to extend our work to tracking of prescribed oscillations. This would explain oscillatory behaviors observed in nature and suggest feedback mechanisms for achieving them.

SUGGESTIONS

- It would be of interest to extend our work to tracking of prescribed oscillations. This would explain oscillatory behaviors observed in nature and suggest feedback mechanisms for achieving them.
- Another desirable extension would be to models with three or more competing species, or more than one limiting substrate.

SUGGESTIONS

- It would be of interest to extend our work to tracking of prescribed oscillations. This would explain oscillatory behaviors observed in nature and suggest feedback mechanisms for achieving them.
- Another desirable extension would be to models with three or more competing species, or more than one limiting substrate.
- The authors thank Patrick De Leenheer for illuminating discussions and the NSF for support for this work under DMS grant 0424011.

SUGGESTIONS

- It would be of interest to extend our work to tracking of prescribed oscillations. This would explain oscillatory behaviors observed in nature and suggest feedback mechanisms for achieving them.
- Another desirable extension would be to models with three or more competing species, or more than one limiting substrate.
- The authors thank Patrick De Leenheer for illuminating discussions and the NSF for support for this work under DMS grant 0424011.
- Full paper at http://www.math.lsu.edu/∼malisoff/.