Bounded Tracking Controllers
and Robustness Analysis for UAVs

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Uncontrolled UAV Model
Uncontrolled UAV Model

\[ v = v_0 \]

\[ h = (v_0 h) x (v_0 h) y = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} v_0 \]
Uncontrolled UAV Model

\[
\begin{align*}
 v_h &= \begin{bmatrix} v \\ 0 \end{bmatrix}
\end{align*}
\]
Uncontrolled UAV Model

\[ v_h = \begin{bmatrix} v \\ 0 \end{bmatrix} \]

\[ v_h^0 = \begin{bmatrix} (v_h^0)_x \\ (v_h^0)_y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix} \]
Uncontrolled UAV Model

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = 
\begin{bmatrix}
v \cos \theta \\
v \sin \theta
\end{bmatrix}
\]
Controlled UAV Model

\[
\begin{aligned}
\dot{x} &= v \cos(\theta), \quad \dot{y} = v \sin(\theta) \\
\dot{\theta} &= \alpha_\theta(\theta_c - \theta + \Delta), \quad \dot{v} = \alpha_v(v_c - v + \delta)
\end{aligned}
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Controlled UAV Model

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\end{aligned}
\]

(1)

\(x, y\) position of UAV at constant altitude
\(\theta, v\) heading angle and inertial velocity
\(\alpha_\theta, \alpha_v\) positive constants for autopilot
\(\theta_c, v_c\) controllers we will design
\(\Delta, \delta\) actuator disturbances
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Ailon, Chandler, Gu, Proud-Pachter-Azzo, Ren-Beard,...
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Omitted altitude dynamics: \(\ddot{h} = -\alpha_h \dot{h} + \alpha_h (h_c - h)\).
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Our Goal:
Controlled UAV Model

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\begin{align*}
\dot{x} &= v \cos(\theta), \\
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Our Goal: Tracking with input-to-state stability with respect to disturbances under controller amplitude and rate constraints.
Input-to-State Stability (Sontag, TAC’89)
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This generalizes uniform global asymptotic stability to systems

$$\dot{Y} = g(t, Y, \mu(t)), \quad Y \in \mathcal{X}.$$  \hspace{1cm} (2)
Input-to-State Stability (Sontag, TAC’89)

This generalizes uniform global asymptotic stability to systems

\[ \dot{Y} = G(t, Y, \mu(t)), \quad Y \in \mathcal{X}. \]  

(2)

It requires functions \( \gamma_i \in K_\infty \) such that all solutions of (2) satisfy

\[
|Y(t)| \leq \gamma_1 (e^{t_0 - t} \gamma_2(|Y(t_0)|)) + \gamma_3 (|\mu|_{[t_0, t]}) \quad \forall t \geq t_0 \geq 0. 
\]  

(3)
Input-to-State Stability (Sontag, TAC’89)

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(3)

Integral ISS (Sontag, ’98) is the same except with

\[ \gamma_0(|Y(t)|) \leq \gamma_1(e^{t_0-t} \gamma_2(|Y(t_0)|)) + \int_{t_0}^{t} \gamma_3(|\mu(r)|)dr. \]  

(4)
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In practical ISS or iISS, \( \gamma_3 \) can depend on \(|Y(t_0)|\).
Input-to-State Stability (Sontag, TAC’89)

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In practical ISS or iISS, \( \gamma_3 \) can depend on \( |Y(t_0)| \).

We show ISS and iISS properties with respect to \( \mu = (\delta, \Delta) \).
Reference Trajectories We Can Track

Definition: A $C^2$ function $\mathbf{r}^* : \mathbb{R} \to \mathbb{R}^3 \times (0, \infty)$ is called a trackable reference trajectory provided

1. $x^*, y^*, \dot{\theta}^*, \ddot{\theta}^*, v^*, \dot{v}^*$ are bounded,
2. $\dot{x}^*(t) = v^*(t) \cos(\theta^*(t))$ and $\dot{y}^*(t) = v^*(t) \sin(\theta^*(t))$ hold for all $t \in \mathbb{R}$, and
3. $\inf \{ v^*(t) : t \in \mathbb{R} \} > 0$.

Condition 3. is the no-stall condition.

Consequence of Trackability: There are constants $c_0 > 0$ and $T > 0$ such that

$$\int_{t+T}^{t} [\dot{\theta}^*(s)]^2 ds \geq c_0$$

for all $t \in \mathbb{R}$.
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Definition: A $C^2$ function $\mathcal{R}_* = (x_*, y_*, \theta_*, v_*) : \mathbb{R} \rightarrow \mathbb{R}^3 \times (0, \infty)$ is called a trackable reference trajectory provided

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Consequence of Trackability: There are constants $c_0 > 0$ and $T > 0$ such that $\int_t^{t+T} [\dot{\theta}_*(s)]^2 ds \geq c_0$ for all $t \in \mathbb{R}$. 
Tracking a Given Trackable Reference Trajectory

\[ \psi = -\sin(\theta)x + \cos(\theta)y, \quad \xi = \cos(\theta)x + \sin(\theta)y \]
Tracking a Given Trackable Reference Trajectory

\[ \psi = -\sin(\theta)x + \cos(\theta)y, \quad \xi = \cos(\theta)x + \sin(\theta)y \]
\[ \tilde{\psi} = \psi - \psi_*(t), \quad \tilde{\xi} = \xi - \xi_*(t), \quad \tilde{\theta} = \theta - \theta_*(t), \quad \tilde{v} = v - v_*(t). \]
Tracking a Given Trackable Reference Trajectory

\[ \psi = -\sin(\theta)x + \cos(\theta)y, \quad \xi = \cos(\theta)x + \sin(\theta)y \]

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Tracking variable: \( \mathcal{E} = (\tilde{\psi}, \tilde{\xi}, \tilde{\theta}, \tilde{v}). \)
Tracking a Given Trackable Reference Trajectory

\[ \psi = -\sin(\theta)x + \cos(\theta)y, \quad \xi = \cos(\theta)x + \sin(\theta)y \]
\[ \tilde{\psi} = \psi - \psi^*(t), \quad \tilde{\xi} = \xi - \xi^*(t), \quad \tilde{\theta} = \theta - \theta^*(t), \quad \tilde{\nu} = \nu - \nu^*(t). \]

Tracking variable: \( \mathcal{E} = (\tilde{\psi}, \tilde{\xi}, \tilde{\theta}, \tilde{\nu}) \).

\[ v_c(t, \mathcal{E}) = v_N(\mathcal{E}) + v^*(t) + \dot{v}^*(t)/\alpha_v \]
\[ \theta_c(t, \mathcal{E}) = \theta_N(t, \mathcal{E}) + \theta^*(t) + \dot{\theta}^*(t)/\alpha_{\theta} \]
Tracking a Given Trackable Reference Trajectory

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Tracking variable: \( \mathbf{E} = (\tilde{\psi}, \tilde{\xi}, \tilde{\theta}, \tilde{v}) \).

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\begin{align*}
\nu_c(t, \mathbf{E}) &= \nu_N(\mathbf{E}) + \nu_*(t) + \dot{\nu}_*(t)/\alpha_v \\
\theta_c(t, \mathbf{E}) &= \theta_N(t, \mathbf{E}) + \theta_*(t) + \dot{\theta}_*(t)/\alpha_\theta
\end{align*}
\]  

Tracking Dynamics:

\[
\begin{align*}
\dot{\tilde{\psi}} &= -\dot{\theta}_*(t)\tilde{\xi} + \alpha_\theta[\tilde{\xi} + \xi_*(t)][\tilde{\theta} - \theta_N - \Delta] \\
\dot{\tilde{\xi}} &= \dot{\theta}_*(t)\tilde{\psi} + \tilde{v} - \alpha_\theta[\tilde{\psi} + \psi_*(t)][\tilde{\theta} - \theta_N - \Delta] \\
\dot{\tilde{\theta}} &= \alpha_\theta(-\tilde{\theta} + \theta_N + \Delta) \\
\dot{\tilde{v}} &= \alpha_v(-\tilde{v} + v_N + \delta)
\end{align*}
\]
Theorem (Gruszka-M-Mazenc, TAC’12)

Let $k > 0$ be any constant. Choose any constant $\bar{\Delta} > 0$ such that $\alpha\theta \|\dot{\theta}_*\|\bar{\Delta} < c_0/(2T)$.
Theorem (Gruszka-M-Mazenc, TAC’12)

Let $k > 0$ be any constant. Choose any constant $\bar{\Delta} > 0$ such that $\alpha_\theta \| \dot{\theta}_* \| \bar{\Delta} < c_0 / (2T)$. Choose $Q_1 = \frac{1}{2} |(\tilde{\psi}, \tilde{\xi})|^2$,

$$v_N(\mathcal{E}) = -k \frac{\tilde{\xi}}{2\alpha_v \sqrt{Q_1} + 1} \quad \text{and} \quad \theta_N(t, \mathcal{E}) = k \frac{\tilde{\psi} \xi_*(t) - \tilde{\xi} \psi_*(t)}{2 \sqrt{Q_1} + 1}.$$
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\nu_N(\mathcal{E}) = -k \frac{\tilde{\xi}}{2 \alpha_v \sqrt{Q_1 + 1}} \quad \text{and} \quad \theta_N(t, \mathcal{E}) = k \frac{\tilde{\psi} \xi_*(t) - \tilde{\xi} \psi_*(t)}{2 \sqrt{Q_1 + 1}}.
\]

Then (TD) are iISS with respect to \( \delta \in \mathcal{M}_\mathbb{R} \) and practically iISS with respect to \( (\delta, \Delta) \in \mathcal{M}_\mathbb{R} \times [-\bar{\Delta}, \bar{\Delta}] \).
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Then (TD) are iISS with respect to \( \delta \in \mathcal{M}_\mathbb{R} \) and practically iISS with respect to \( (\delta, \Delta) \in \mathcal{M}_{\mathbb{R} \times [-\bar{\Delta}, \bar{\Delta}]} \). There is a constant \( \delta_M > 0 \) such that (TD) are ISS with respect to \( \delta \in \mathcal{M}_{[-\delta_M, \delta_M]} \) and practically ISS with respect to \( (\delta, \Delta) \in \mathcal{M}_{[-\delta_M, \delta_M]^2} \).
Theorem (Gruszka-M-Mazenc, TAC’12)

Let $k > 0$ be any constant. Choose any constant $\tilde{\Delta} > 0$ such that $\alpha_{\theta} \| \dot{\theta}_* \| \tilde{\Delta} < c_0/(2T)$. Choose $Q_1 = \frac{1}{2} |(\tilde{\psi}, \tilde{\xi})|^2$,

$$v_N(\mathcal{E}) = -k \frac{\tilde{\xi}}{2\alpha_v \sqrt{Q_1} + 1} \quad \text{and} \quad \theta_N(t, \mathcal{E}) = k \frac{\tilde{\psi} \xi_*(t) - \tilde{\xi} \psi_*(t)}{2\sqrt{Q_1} + 1}.$$

Then (TD) are iISS with respect to $\delta \in \mathcal{M}_R$ and practically iISS with respect to $(\delta, \Delta) \in \mathcal{M}_{\mathbb{R} \times [-\tilde{\Delta}, \tilde{\Delta}]}$. There is a constant $\delta_M > 0$ such that (TD) are ISS with respect to $\delta \in \mathcal{M}_{[-\delta_M, \delta_M]}$ and practically ISS with respect to $(\delta, \Delta) \in \mathcal{M}_{[-\delta_M, \delta_M]^2}$.

Key Features:
Theorem (Gruszka-M-Mazenc, TAC’12)

Let $k > 0$ be any constant. Choose any constant $\Delta > 0$ such that $\alpha_\theta \|\dot{\theta}_*\|\Delta < c_0/(2T)$. Choose $Q_1 = \frac{1}{2}|(\tilde{\psi}, \tilde{\xi})|^2$,

$$v_N(E) = -k \frac{\tilde{\xi}}{2\alpha_N \sqrt{Q_1} + 1}$$

and

$$\theta_N(t, E) = k \frac{\tilde{\psi} \xi_*(t) - \tilde{\xi} \psi_*(t)}{2 \sqrt{Q_1} + 1}.$$

Then (TD) are iISS with respect to $\delta \in \mathcal{M}_{\mathbb{R}}$ and practically iISS with respect to $(\delta, \Delta) \in \mathcal{M}_{\mathbb{R} \times [\Delta, \Delta]}$. There is a constant $\delta_M > 0$ such that (TD) are ISS with respect to $\delta \in \mathcal{M}_{[-\delta_M, \delta_M]}$ and practically ISS with respect to $(\delta, \Delta) \in \mathcal{M}_{[-\delta_M, \delta_M]^2}$.

Key Features: iISS Lyapunov functions.
Theorem (Gruszka-M-Mazenc, TAC’12)

Let $k > 0$ be any constant. Choose any constant $\bar{\Delta} > 0$ such that $\alpha_{\theta} \| \dot{\theta}_* \| \bar{\Delta} < c_0/(2T)$. Choose $Q_1 = \frac{1}{2}|(\tilde{\psi}, \tilde{\xi})|^2$,

$$v_N(\mathcal{E}) = -k \frac{\tilde{\xi}}{2\alpha_{\nu} \sqrt{Q_1} + 1} \quad \text{and} \quad \theta_N(t, \mathcal{E}) = k \frac{\tilde{\psi} \xi_*(t) - \tilde{\xi} \psi_*(t)}{2 \sqrt{Q_1} + 1}.$$

Then (TD) are iISS with respect to $\delta \in \mathcal{M}_\mathbb{R}$ and practically iISS with respect to $(\delta, \Delta) \in \mathcal{M}_{\mathbb{R}^2}[-\bar{\Delta}, \bar{\Delta}]$. There is a constant $\delta_M > 0$ such that (TD) are ISS with respect to $\delta \in \mathcal{M}[-\delta_M, \delta_M]$ and practically ISS with respect to $(\delta, \Delta) \in \mathcal{M}_{\mathbb{R}^2}[-\delta_M, \delta_M]^2$.

Key Features: iISS Lyapunov functions. Controls do not depend on $\nu$ and satisfy certain amplitude and rate constraints.
Theorem (Gruszka-M-Mazenc, TAC’12)

Let $k > 0$ be any constant. Choose any constant $\tilde{\Delta} > 0$ such that $\alpha_\theta \|\dot{\theta}_*\|\tilde{\Delta} < c_0/(2T)$. Choose $Q_1 = \frac{1}{2}|(\tilde{\psi}, \tilde{\xi})|^2$,

$$v_N(\mathcal{E}) = -k \frac{\tilde{\xi}}{2\alpha_v \sqrt{Q_1} + 1} \quad \text{and} \quad \theta_N(t, \mathcal{E}) = k \frac{\tilde{\psi}_\xi(t) - \tilde{\xi}_\psi(t)}{2 \sqrt{Q_1} + 1}.$$

Then (TD) are iISS with respect to $\delta \in \mathcal{M}_\mathbb{R}$ and practically iISS with respect to $(\delta, \Delta) \in \mathcal{M}_\mathbb{R} \times [-\tilde{\Delta}, \tilde{\Delta}]$. There is a constant $\delta_M > 0$ such that (TD) are ISS with respect to $\delta \in \mathcal{M}_{[-\delta_M, \delta_M]}$ and practically ISS with respect to $(\delta, \Delta) \in \mathcal{M}_{[-\delta_M, \delta_M]^2}$.

Key Features: iISS Lyapunov functions. Controls do not depend on $\nu$ and satisfy certain amplitude and rate constraints.

$||v_N|| \leq k/\{\sqrt{2}\alpha_v\}$ and $||\theta_N|| \leq \sqrt{2}k \max\{||\xi_*||, ||\psi_*||\}$. 
Control Amplitude and Rate Constraints

\[
\begin{align*}
\nu_c(t, \mathcal{E}) &= \nu_N(\mathcal{E}) + \nu_*(t) + \dot{\nu}_*(t)/\alpha_v \\
\theta_c(t, \mathcal{E}) &= \theta_N(t, \mathcal{E}) + \theta_*(t) + \dot{\theta}_*(t)/\alpha_\theta
\end{align*}
\]
Control Amplitude and Rate Constraints

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\begin{align*}
\nu_c(t, \mathcal{E}) &= \nu_N(\mathcal{E}) + v_*(t) + \dot{v}_*(t)/\alpha_v \\
\theta_c(t, \mathcal{E}) &= \theta_N(t, \mathcal{E}) + \theta_*(t) + \dot{\theta}_*(t)/\alpha_\theta
\end{align*}
\]

Let \( \varepsilon > 0 \) be a constant and \([\nu_a, \bar{\nu}_a]\) be the desired \( \nu_c \) envelope.
Control Amplitude and Rate Constraints

\[ v_c(t, \varepsilon) = v_N(\varepsilon) + v_*(t) + \dot{v}_*(t)/\alpha_v \]
\[ \theta_c(t, \varepsilon) = \theta_N(t, \varepsilon) + \theta_*(t) + \dot{\theta}_*(t)/\alpha_{\theta} \]

Let \( \varepsilon > 0 \) be a constant and \([v_a, \bar{v}_a] \) be the desired \( v_c \) envelope.

Assume that \( v_a + \varepsilon < v_*(t) + \dot{v}_*(t)/\alpha_v < \bar{v}_a - \varepsilon \) holds for all \( t \).
Control Amplitude and Rate Constraints

\[
\begin{align*}
\nu_c(t, \mathcal{E}) &= \nu_N(\mathcal{E}) + \nu_*(t) + \dot{\nu}_*(t)/\alpha_v \\
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\end{align*}
\]

Let \( \varepsilon > 0 \) be a constant and \([\underline{\nu}_a, \bar{\nu}_a]\) be the desired \( \nu_c \) envelope.

Assume that \( \underline{\nu}_a + \varepsilon < \nu_*(t) + \dot{\nu}_*(t)/\alpha_v < \bar{\nu}_a - \varepsilon \) holds for all \( t \).

We can choose the constant \( k > 0 \) small enough such that \( \underline{\nu}_a < \nu_c(t, \mathcal{E}(t)) < \bar{\nu}_a \) along all trajectories, and similarly for \( \dot{\theta}_c \).
Control Amplitude and Rate Constraints

\[ v_c(t, \mathcal{E}) = v_N(\mathcal{E}) + v_*(t) + \dot{v}_*(t)/\alpha_v \]
\[ \theta_c(t, \mathcal{E}) = \theta_N(t, \mathcal{E}) + \theta_*(t) + \dot{\theta}_*(t)/\alpha_\theta \]

Let \( \varepsilon > 0 \) be a constant and \([v_a, \bar{v}_a]\) be the desired \( v_c \) envelope.

Assume that \( v_a + \varepsilon < v_*(t) + \dot{v}_*(t)/\alpha_v < \bar{v}_a - \varepsilon \) holds for all \( t \).

We can choose the constant \( k > 0 \) small enough such that
\[ v_a < v_c(t, \mathcal{E}(t)) < \bar{v}_a \] along all trajectories, and similarly for \( \theta_c \).

Let \([\theta_r, \bar{\theta}_r]\) and \([v_r, \bar{v}_r]\) be the desired rate envelopes.
Control Amplitude and Rate Constraints

\[ v_c(t, \mathcal{E}) = v_N(\mathcal{E}) + v_*(t) + \dot{v}_*(t)/\alpha_v \]
\[ \theta_c(t, \mathcal{E}) = \theta_N(t, \mathcal{E}) + \theta_*(t) + \dot{\theta}_*(t)/\alpha_{\theta} \]

Let \( \varepsilon > 0 \) be a constant and \([\underline{v}_a, \overline{v}_a]\) be the desired \( v_c \) envelope.

Assume that \( \underline{v}_a + \varepsilon < v_*(t) + \dot{v}_*(t)/\alpha_v < \overline{v}_a - \varepsilon \) holds for all \( t \).

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Let \([\underline{\theta}_r, \overline{\theta}_r]\) and \([\underline{v}_r, \overline{v}_r]\) be the desired rate envelopes.

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Let $\varepsilon > 0$ be a constant and $[v_a, \bar{v}_a]$ be the desired $v_c$ envelope.

Assume that $v_a + \varepsilon < v_*(t) + \dot{v}_*(t)/\alpha_v < \bar{v}_a - \varepsilon$ holds for all $t$.

We can choose the constant $k > 0$ small enough such that $v_a < v_c(t, \mathcal{E}(t)) < \bar{v}_a$ along all trajectories, and similarly for $\theta_c$.

Let $[\theta_r, \bar{\theta}_r]$ and $[\bar{v}_r, \bar{v}_r]$ be the desired rate envelopes.

Assume that $\bar{v}_r + \varepsilon < \dot{\theta}_*(t) + \ddot{\theta}_*(t)/\alpha_\theta < \bar{v}_r - \varepsilon$ holds for all $t$.

For each constant $B > 0$, we can find a constant $\bar{K}(B)$ such that if $|(\tilde{\theta}(t_0), \tilde{v}(t_0))| \leq B$ and $k \in (0, \bar{K}(B))$ both hold, then $v_r < \dot{v}_c(t, \mathcal{E}(t)) < \bar{v}_r$ along all trajectories, and similarly for $\dot{\theta}_c$. 
Conclusions

▶ The benchmark model for controlled UAVs includes uncertainty in both controls.
▶ Our controls give input-to-state stability estimates whose overshoot terms quantify the effects of the uncertainty.
▶ They satisfy command amplitude, command rate, and state constraints, e.g., coordinated turning conditions \[ |\dot{\theta}| \leq c^* / v. \]
▶ It may be useful to obtain more information on the behavior of the trajectories of the closed loop (TD) with \( v_N \) and \( \theta_N \).
▶ We also aim to extend our work to coordinated control of uncertain UAVs under time delays in the controls.
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