# Stability and Robustness Analysis for Human Pointing Motions with Acceleration under Feedback Delays

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Joint with Paul Varnell and Fumin Zhang

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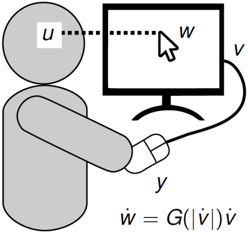
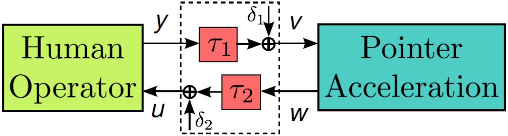
First use of systems perspective to study pointer acceleration

# Background

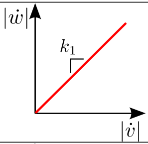
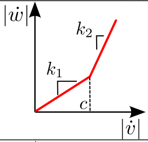
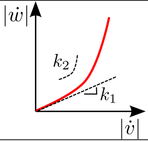



$$\dot{w} = G(|\dot{v}|)\dot{v}$$




$y$ : true pointer position
$v$ : pointer position specified by user
$w$ : pointer acceleration output
$u$ : feedback of perceived pointer position

# Background

| Name | Scaling Function | I/O Velocity plot |
|---|---|---|
| No Accel. | $G(\lvert\dot{v}\rvert) = k_1$ |  |
| Threshold | $G(\lvert\dot{v}\rvert) = \begin{cases} k_1, & \text{if } 0 \leq \lvert\dot{v}\rvert < c \\ k_2, & \text{if } \lvert\dot{v}\rvert \geq c \end{cases}$ |  |
| Linear | $G(\lvert\dot{v}\rvert) = k_1 + k_2\lvert\dot{v}\rvert$ |  |

Common Acceleration Profiles. $\dot{w} = G(|\dot{v}|)\dot{v}$

# One Dimensional Pointing Model

Vertical integration to endpoint (VITE) model:

$$\begin{cases} \dot{\nu} &= \gamma(-\nu + \rho - u) \\ \dot{y} &= g(t) \max\{\langle \nu, d \rangle, 0\} \end{cases} \tag{VITE}$$

$g(t)$ is go signal, $\rho$ is target pointer position, $d = \rho(0) - u(0)$.
(Bullock-Grossberg, *Psychological Review*, 1988)

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Interconnect (VITE) with $\dot{w} = G(|\dot{v}|)\dot{v}$ by setting $(u, v) = (w, y)$.

$$\dot{x}(t) = \begin{bmatrix} \tilde{G}(x_2^+(t-\tau_1))\, x_2^+(t-\tau_1) + \delta_1(t) \\ -\gamma(x_1(t-\tau_2) + x_2(t) + \delta_2(t)) \end{bmatrix} \tag{1D}$$

$x_1 = w =$ displayed pointer position. $\gamma =$ internal parameter $> 0$.
$x_2 = \nu =$ difference vector. $\tilde{G}(\cdot) = gG(g\cdot)$. $\dot{w} = G(|\dot{v}|)\dot{v}$.

# Key Ingredient: Strict Lyapunov Function

Lemma: Let $\tilde{G} : [0, \infty) \to \mathbb{R}$ be any locally Lipschitz function that is positive on $(0, \infty)$ and such that $\liminf_{r\to\infty} \tilde{G}(r) > 0$. Then we can find a constant $\underline{\gamma} > 0$ such that for each $\gamma \geq \underline{\gamma}$, the function

$$V_{\mathrm{new}}(x) = \int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+ \mathrm{d}\zeta + \frac{\gamma}{2}(x_1 + x_2)^2 - x_2^+ \tilde{G}(x_2^+)(x_1 + x_2) \quad (1)$$

is a strict Lyapunov function for

$$\dot{x}(t) = \begin{bmatrix} \tilde{G}(x_2^+(t))\, x_2^+(t) \\ -\gamma(x_1(t) + x_2(t)) \end{bmatrix} \quad \text{(1D)}$$

with respect to its equilibrium $\mathcal{E} = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 = -x_1\}$ on its state space $\mathbb{R}^2$.

# Key Ingredient: Strict Lyapunov Function

Key Properties:

$$\frac{1}{10}\int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+ \mathrm{d}\zeta + \frac{\gamma}{4}(x_1+x_2)^2 \leq V_{\mathrm{new}}(x) \leq \int_0^{x_2} \tilde{G}(\zeta^+)\zeta^+ \mathrm{d}\zeta + \frac{(\tilde{G}(x_2^+)x_2^+)^2}{2} + \frac{\gamma+1}{2}(x_1+x_2)^2 \qquad \text{(PPD)}$$

and

$$\dot{V}_{\mathrm{new}}(x) \leq -\frac{\gamma^2}{4}(x_1+x_2)^2 - \frac{1}{2}\left(\tilde{G}(x_2^+)x_2^+\right)^2 \qquad \text{(SLD)}$$

for all $x \in \mathbb{R}^2$.

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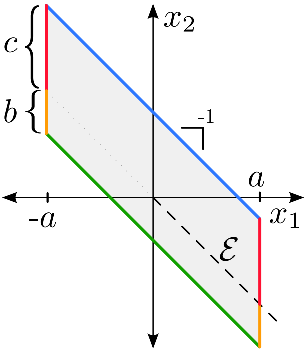
Can prove input-to-state stability with respect to $\delta_i$'s.

Can convert $V_{\mathrm{new}}$ into a Lyapunov-Krasovskii functional to understand effects of the time delays $\tau_i$'s.

# Robust Forward Invariance

We can find maximum allowable bounds on $\delta_2$'s such that sets of this form are forward invariant for (1D) for zero $\delta_1$'s and zero $\tau_i$'s.

$$S_{a,b,c} = \{x \in \mathbb{R}^2 : |x_1| \leq a, -b \leq x_1 + x_2 \leq c\} \qquad \text{(FI)}$$



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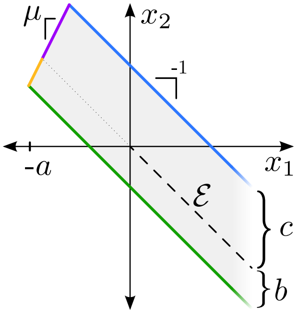
We have analogs for cases where both $\delta_i$'s and $\tau_i$'s are nonzero that produce unbounded forwardly invariant sets of this form.

$$S^{\mu}_{a,b,c} = \left\{ \begin{array}{l} x \in \mathbb{R}^2 : -b \leq x_1 + x_2 \leq c, x_1 \geq -a, \\ \text{and } \; x_2 \leq \mu x_1 + (\mu + 1)a - b \end{array} \right\} \qquad \text{(FID)}$$



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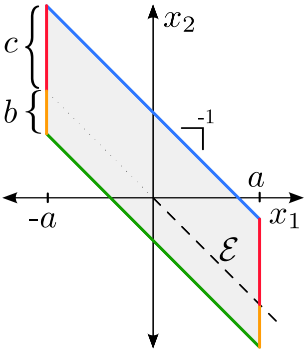

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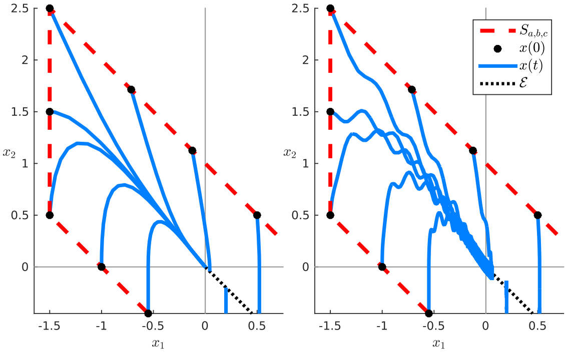

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Thanks for your interest!