Stability and Robustness Analysis for a Multispecies Chemostat Model with Delays in the Growth Rates and Uncertainties

Frederic Mazenc
Michael Malisoff
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Background on Chemostats

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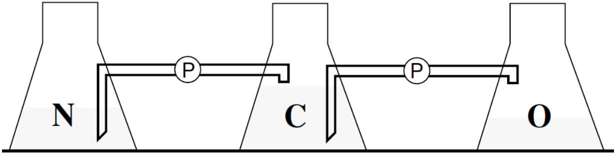

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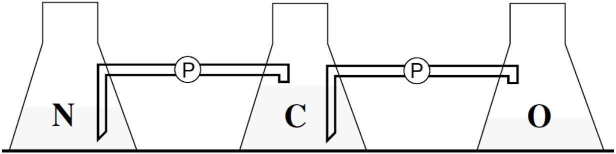

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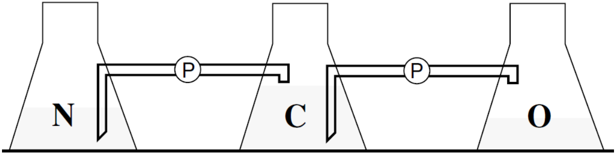

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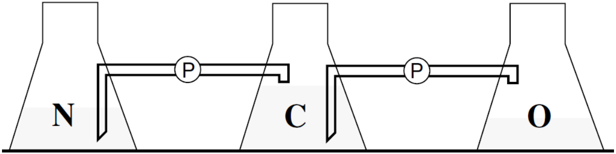

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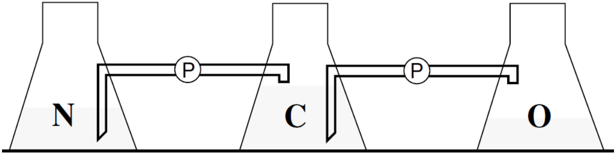

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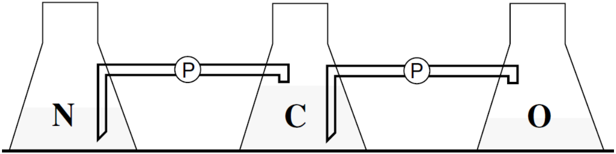

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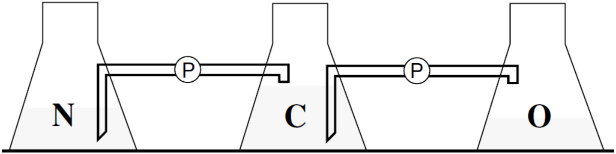

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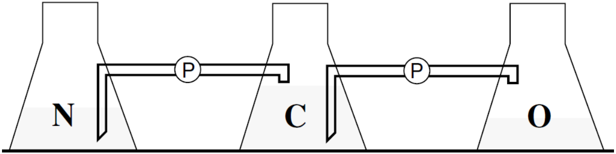

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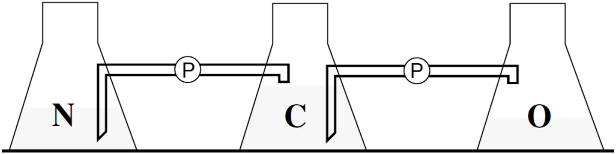

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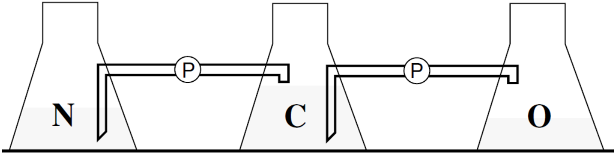

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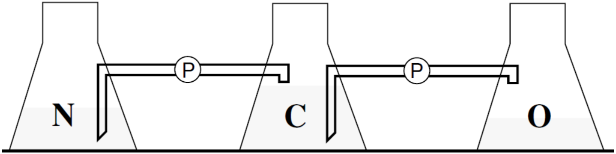

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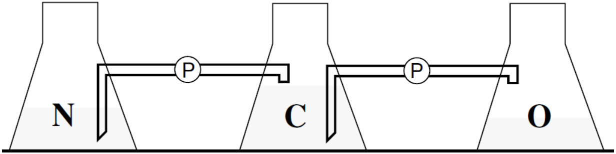

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$$\begin{cases} \dot{s}(t) = D[s_{\text{in}} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t))x_i(t) + \delta_0(t) \\ \dot{x}_i(t) = x_i(t)\mu_i(s(t-\tau_i)) + D[x_i^0 - x_i(t)] + \delta_i(t),\ 1 \le i \le n \end{cases} \qquad \text{(M)}$$

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2) $\delta_i(t) \in [\underline{d}_i, \bar{d}_i]$ for all i where $Ds_{\text{in}} + \underline{d}_0 > 0$, $\bar{d}_0 < 0.5Ds_*$, $Dx_i^0 + \underline{d}_i > 0$ for all indices $i \in \mathcal{P}$, and $\underline{d}_i = 0$ for all indices $i \in \{1, 2, \dots, n\} \setminus \mathcal{P}$, where $\mathcal{P} = \{i \in \{1, 2, \dots, n\} : x_i^0 > 0\}$.

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Assumption 2) maintains forward invariance of $(0, \infty)^{n+1}$ for (M).

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Controls: x_i^0 and $s_{\rm in}$. x_i^0: substrate inputs from other chemostats.

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We also have an Assumption 3) with a bound $\bar{\tau}$ on the delays τ_i.

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Significance: Uniform persistence of all species for which $x_i^0 > 0$. ISS for arbitrarily large upper bounds $\bar{d}_i$ on $\delta_i(t)$ for $i \ge 1$.

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Significance: Since $\underline{x} > 0$ and $\bar{s} \ge s_{\text{in}}$ are arbitrary, we get ISS properties on all of $(0,\infty)^{n+1}$ under our disturbance bounds.

Main Idea of Proof: Undelayed Case

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Construct a function $T \in \mathcal{K}_\infty$ and constants $c_i > 0$ and $k_i > 0$ such that the time derivative of

$$V(\mathcal{E}) = \tilde{s} - s_* \ln\left(\frac{\tilde{s}+s_*}{s_*}\right) + \sum_{i=1}^{n} \frac{1}{c_i}\Psi_i(\tilde{x}_i), \text{ where}$$

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$$\text{and } \Psi_i(\tilde{x}_i) = x_i \text{ for all } i \in \{1, 2, \ldots, n\} \setminus \mathcal{P}$$

along all solutions of (M) starting in any $\mathcal{S}_{\bar{s},\underline{x}}$ with $\tau_i = 0$ satisfies

$$\frac{d}{dt}V(\mathcal{E}(t)) \ \le\ -k_1\left(\frac{\tilde{s}^2(t)}{s(t)} + \sum_{i=1}^{n}\frac{\tilde{x}_i^2(t)}{x_i(t)}\right) + k_2|\delta|_{[0,t]} \qquad (1)$$

for all $t \geq T(|\mathcal{E}(0)|)$, where $\tilde{x}_i = x_i - x_{i*}$ for all i and $\tilde{s} = s - s_*$.

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for all $t \geq T(|\mathcal{E}(0)|)$, where $\tilde{x}_i = x_i - x_{i*}$ for all i and $\tilde{s} = s - s_*$. Extend this to ISS estimate on $[0, \infty)$ by a trajectory analysis.

Simulations

$n = 2,\ D = 0.4,\ s_* = 0.5,\ x_1^0 = 1,\ x_2^0 = 0.55,\ s_{\text{in}} = 1.34412,$
$\mu_1(s) = \frac{s}{5+s},\ \mu_2(s) = \frac{s}{2+s},\ x_{1*} = 1.29412,\ x_{2*} = 1.1, \tau = (0.14, 0)$
$\delta(t) = (\delta_0(t), \delta_1(t), \delta_2(t)) = (0, -0.1\sin(t), 0.1\cos(t)).$

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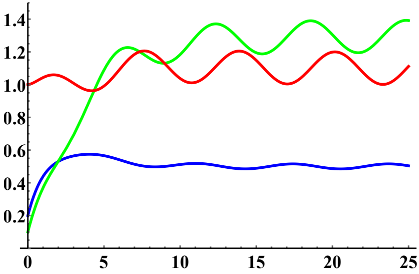

$x_1(t)$ and $x_2(t)$ are Green and Red Curves, Respectively. $s(t)$ is Blue Curve. Initial State $(s(0), x_1(0), x_2(0)) = (0.2, 0.1, 1)$.

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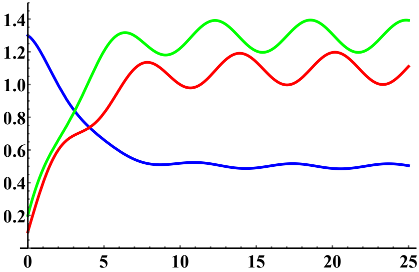

$x_1(t)$ and $x_2(t)$ are Green and Red Curves, Respectively. $s(t)$ is Blue Curve. Initial State $(s(0),x_1(0),x_2(0))=(1.3,0.2,0.1)$.

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Thank you for your attention!

Backup Slides to Use if Time Allows or Questions Warrant

Our Models and Theorem

$$\begin{cases} \dot{s}(t) = D[s_{\text{in}} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t))x_i(t) + \delta_0(t) \\ \dot{x}_i(t) = x_i(t)\mu_i(s(t-\tau_i)) + D[x_i^0 - x_i(t)] + \delta_i(t), \; 1 \leq i \leq n \end{cases} \qquad \text{(M)}$$

$\mu_i(s) = \frac{m_i s}{a_i + s}$. Equilibria: $\mathcal{E}_* = (s_*, x_{1*}, \ldots, x_{n*}) \in (0, \infty) \times [0, \infty)^n$.

Reduces to Gouze-Robledo model when uncertainties δ_i and delays τ_i are 0 and usual model when the inputs $x_i^0 \geq 0$ are zero.

Theorem: Under our assumptions, for all constants $\underline{x} > 0$ and $\bar{s} \geq s_{\text{in}}$, the dynamics for the error vector $\mathcal{E} = (s, x) - \mathcal{E}_*$ satisfy ISS on the set $\mathcal{S}_{\bar{s},\underline{x}} = \{\mathcal{E} : \mathcal{E} + \mathcal{E}_* \in (0, \bar{s}] \times (0, \infty)^{n-1} \times (\underline{x}, \infty)\}$.$\square$

Significance: Since $\underline{x} > 0$ and $\bar{s} \geq s_{\text{in}}$ are arbitrary, we get ISS properties on all of $(0, \infty)^{n+1}$ under our disturbance bounds.

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Main Idea of Proof: Delayed Case

Build $T \in \mathcal{K}_\infty$ and positive constants $\mathcal{M}_*$, v_0, and $\bar{N}$ such that

$$V^\sharp(\tilde{s}_t, \tilde{\alpha}(t)) = V(\tilde{s}(t), \tilde{\alpha}(t)) + \mathcal{M}_* \int_{t-\overline{\tau}}^{t} \int_{\ell}^{t} \frac{\tilde{s}^2(r)}{s(r)} \mathrm{d}r \, \mathrm{d}\ell \qquad (2)$$

satisfies

$$\overbrace{V^\sharp(\tilde{s}_t, \tilde{\alpha}(t))}^{\cdot} \leq -v_0 \left(\frac{\tilde{s}^2(t)}{s(t)} + \sum_{i=1}^{n} \frac{\tilde{\alpha}_i^2(t)}{\alpha_i(t)} + \int_{t-\overline{\tau}}^{t} \int_{\ell}^{t} \frac{\tilde{s}^2(r)}{s(r)} \mathrm{d}r \, \mathrm{d}\ell \right) + \bar{N}|\delta|_{[0,t]}$$

along all solutions of the $(\tilde{s}, \tilde{\alpha})$ system for all $t \geq T(|\mathcal{E}(0)|)$ where

$$\alpha_i(t) = x_i(t) e^{\int_{t-\tau_i}^{t} [\mu_i(s(\ell)) - \mu_i(s_*)] \mathrm{d}\ell} \qquad (3)$$

for all i, V is the Lyapunov construction from the undelayed case, $\tilde{s} = s - s_*$ is as before, and $\tilde{\alpha}_i = \alpha_i - x_{i*}$ and $\tau_i \leq \overline{\tau}$ for all i.

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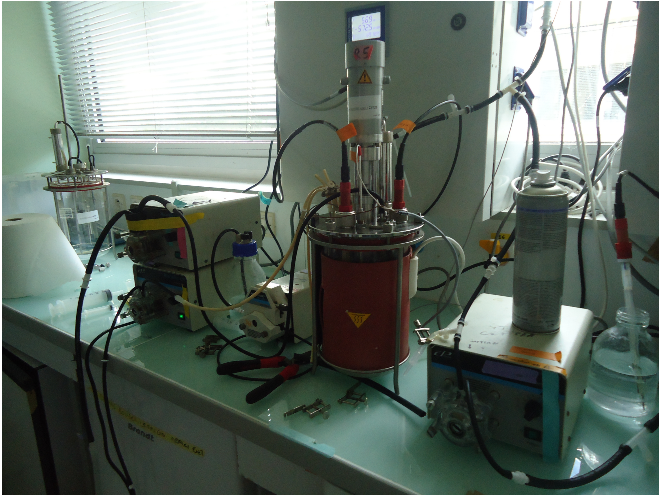

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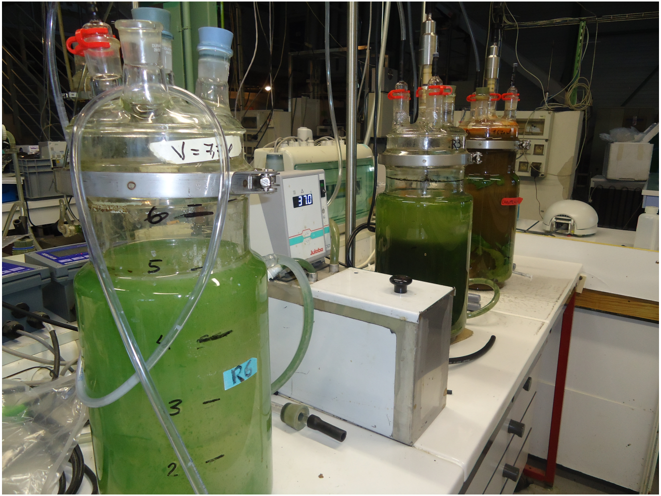

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