Recent Results on Control Problems for Chemostats

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Joint work with Frédéric Mazenc from INRIA

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Chemostat Apparatus

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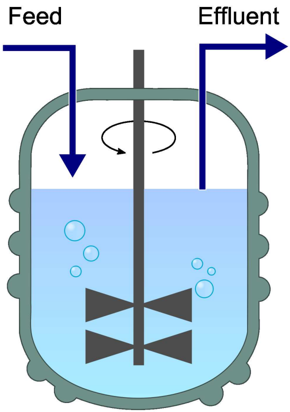

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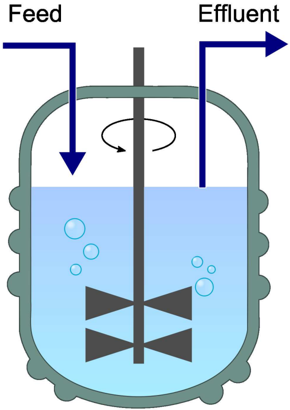

Bioreactor.

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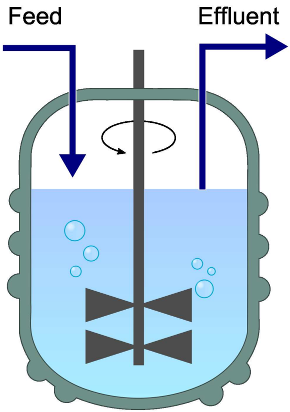

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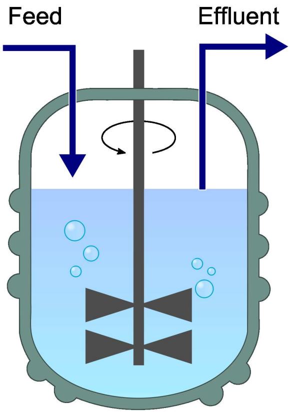

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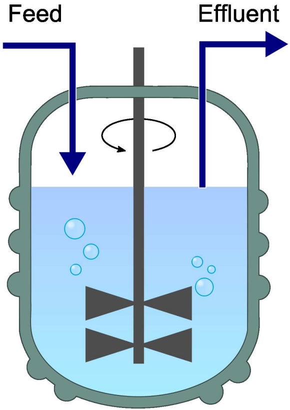

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Two-Species Chemostat Model

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- D cannot be constant if the μ_i's are monotone.

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- There is a constant $s_* \in (0, s_{\rm in})$ such that $\mu_1(s_*) = \mu_2(s_*)$.
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$$\aleph = \delta_1 \mu_1(s_*) \left[1 + \frac{1}{s_*} \sum_{i=1}^{2} \frac{\delta_i}{L_i + s_{\rm in} + g_i s_{\rm in}^2} x_{i*} \right] + K_2 s_* x_{2*} \frac{L_2 g_1 - L_1 g_2}{\delta_2} > 0 \qquad \text{(SG)}$$

$$\mho = \frac{s_*}{(1-a)\mu_1(s_*)} \left[-K_1 + \frac{L_1 - g_1 s_*^2}{L_2 - g_2 s_*^2} K_2 \right] \neq 0 \qquad \text{(SC)}$$

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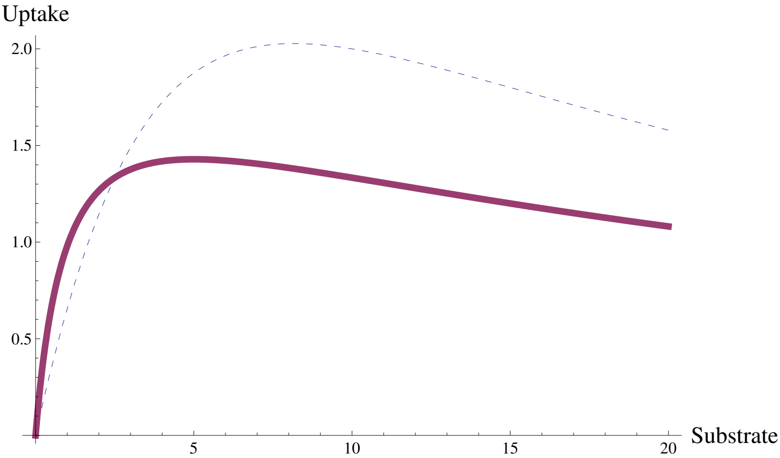
$x_{1*}, x_{2*} > 0$ are any constants such that $s_* + x_{1*} + x_{2*} = s_{\rm in}$.

Haldane Growth Functions



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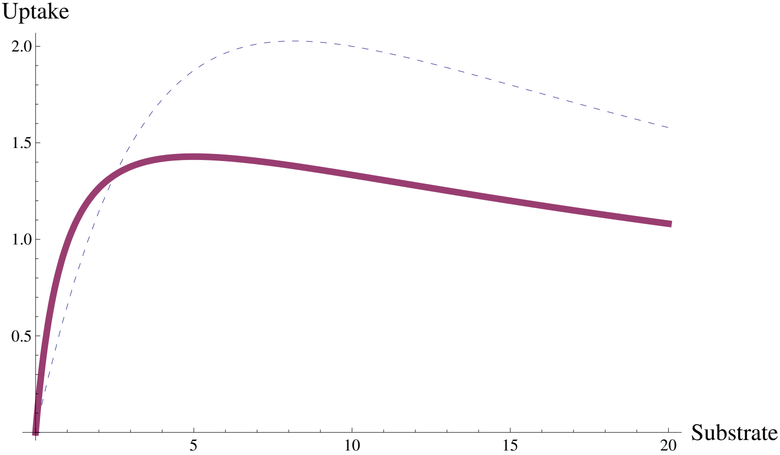

$(K_1, L_1, g_1) = (6, 8, 0.12)$ and $(K_2, L_2, g_2) = (2, 1, 0.04)$

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We can compute constants $\bar{\varepsilon}_i$ depending on τ_M so that for any constants $\varepsilon_i \in (0, \bar{\varepsilon}_i)$ for $i = 1, 2$ such that $\varepsilon_1 \varepsilon_2 \leq \bar{\varepsilon}_3$, the control

$$D = \mu_1(s_*) - \text{sign}(\mho)\varepsilon_1 \sigma\Big(\varepsilon_2\{x_1(t-\tau) + a x_2(t-\tau) - x_{1*} - a x_{2*}\}\Big)$$

globally asymptotically stabilizes (s_*, x_{1*}, x_{2*}) for all initializations $(\phi_s, \phi_{x_1}, \phi_{x_2}) \in C([-2\tau_M, 0], (0, \infty)^3)$.

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$$\dot{U}_1 \leq -(\tilde{s} + \tilde{x}_1 + \tilde{x}_2)^2 - \frac{\aleph}{5}\frac{\tilde{s}^2}{s} - \frac{\varepsilon_1\varepsilon_2|\mho|}{8}(\tilde{x}_1 + a\tilde{x}_2)^2, \quad t \geq \tau \,.$$

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The case $a = 1$ would give a nonisolated equilibrium.

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Corollary: We can choose K and a constant $\varepsilon > 0$ such that if $\mathcal{T}(\mu, \nu) = \max\{|\mu_i'(s) - \nu_i'(s)| : i = 1, 2; s \in [0, s_{\text{in}}]\} < \varepsilon$, then

$$\begin{cases} \dot{s} &= D(y)[s_{\text{in}} - s] - \nu_1(s)x_1 - \nu_2(s)x_2 \\ \dot{x}_i &= [\nu_i(s) - D(y)]\, x_i, \ \ i = 1, 2 \end{cases} \tag{RC}$$

is GAS to some point $(s_v, x_{1v}, x_{2v}) \in (0, \infty)^3$.

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- Competitive exclusion required us to use a nonconstant controller to get permanence of both species.
- We dropped the usual assumption on the relative sizes of the growth yields G_i.
- When $G_1 < G_2$, we can allow uncertain monotone uptake functions μ_i that are not necessary concave.

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- We achieved output feedback GAS of componentwise positive equilibria using only the sum of the species levels.
- Competitive exclusion required us to use a nonconstant controller to get permanence of both species.
- We dropped the usual assumption on the relative sizes of the growth yields G_i.
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- Desirable extensions would allow more than two species, or multiple limiting substrates.