Model-Based Nonlinear Control of the Human Heart Rate During Treadmill Exercising

Frédéric Mazenc (INRIA-DISCO), Michael Malisoff* (LSU), and Marcio de Queiroz (LSU)

Biological and Biomedical Systems II
49th IEEE Conference on Decision and Control
December 15-17, 2010
Model (Cheng, Savkin, et al., IEEE-TBE)
Model (Cheng, Savkin, et al., IEEE-TBE)

\[ \begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2 \\
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}}.
\end{align*} \]
Model (Cheng, Savkin, et al., IEEE-TBE)

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2 \quad \text{(1a)} \\
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}}, \quad \text{(1b)}
\end{align*}
\]

\(x_1 = \text{deviation of the HR from the at rest heart rate.}\)
Model (Cheng, Savkin, et al., IEEE-TBE)

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2 \\
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}},
\end{align*}
\]

\(x_1\) = deviation of the HR from the at rest heart rate.
\(x_2\) = slower, local peripheral effects on the HR
Model (Cheng, Savkin, et al., IEEE-TBE)

\[ \dot{x}_1 = -a_1 x_1 + a_2 x_2 + a_2 u^2 \]  
\[ \dot{x}_2 = -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}} \]  

\( x_1 \) = deviation of the HR from the at rest heart rate.  
\( x_2 \) = slower, local peripheral effects on the HR (e.g., hormonal effects,
Model (Cheng, Savkin, et al., IEEE-TBE)

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2 \quad \text{(1a)} \\
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}}, \quad \text{(1b)}
\end{align*}
\]

\( x_1 \) = deviation of the HR from the at rest heart rate.
\( x_2 \) = slower, local peripheral effects on the HR (e.g., hormonal effects, increase in body temperature,
Model (Cheng, Savkin, et al., IEEE-TBE)

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2 \\ 
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}},
\end{align*}
\]  

(1a)  
(1b)

\(x_1\) = deviation of the HR from the at rest heart rate.  
\(x_2\) = slower, local peripheral effects on the HR (e.g., hormonal effects, increase in body temperature, and loss of body fluids).
Model (Cheng, Savkin, et al., IEEE-TBE)

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2 \\
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}},
\end{align*}
\]

(1a) \hspace{10cm} (1b)

\(x_1\) = deviation of the HR from the at rest heart rate.
\(x_2\) = slower, local peripheral effects on the HR (e.g., hormonal effects, increase in body temperature, and loss of body fluids).
\(u\) = treadmill speed.
Model (Cheng, Savkin, et al., IEEE-TBE)

\[ \dot{x}_1 = -a_1 x_1 + a_2 x_2 + a_2 u^2 \]  \hspace{1cm} (1a)

\[ \dot{x}_2 = -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1 - a_5)}}, \]  \hspace{1cm} (1b)

\( x_1 \) = deviation of the HR from the at rest heart rate.
\( x_2 \) = slower, local peripheral effects on the HR (e.g., hormonal effects, increase in body temperature, and loss of body fluids).
\( u \) = treadmill speed. \( a_i \) = constant parameter.
Model (Cheng, Savkin, et al., IEEE-TBE)

\begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2 \quad (1a) \\
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}}, \quad (1b)
\end{align*}

\(x_1\) = deviation of the HR from the at rest heart rate.
\(x_2\) = slower, local peripheral effects on the HR (e.g., hormonal effects, increase in body temperature, and loss of body fluids).
\(u\) = treadmill speed. \(a_i\) = constant parameter.

Motivation:
Model (Cheng, Savkin, et al., IEEE-TBE)

\[
\dot{x}_1 = -a_1 x_1 + a_2 x_2 + a_2 u^2 \tag{1a}
\]
\[
\dot{x}_2 = -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1 - a_5)}}, \tag{1b}
\]

\(x_1\) = deviation of the HR from the at rest heart rate.
\(x_2\) = slower, local peripheral effects on the HR (e.g., hormonal effects, increase in body temperature, and loss of body fluids).
\(u\) = treadmill speed. \(a_i\) = constant parameter.

**Motivation:** Metabolic cost from walking on level ground is approximately proportional to the square of the walking speed.
Model (Cheng, Savkin, et al., IEEE-TBE)

\begin{align}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2 \\
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}},
\end{align}

(1a) \hspace{1cm} (1b)

\(x_1\) = deviation of the HR from the at rest heart rate.

\(x_2\) = slower, local peripheral effects on the HR (e.g., hormonal effects, increase in body temperature, and loss of body fluids).

\(u\) = treadmill speed. \(a_i\) = constant parameter.

**Motivation:** Metabolic cost from walking on level ground is approximately proportional to the square of the walking speed.

Model has been validated with human subjects.
Model (Cheng, Savkin, et al., IEEE-TBE)

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2 \\
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1-a_5)}},
\end{align*}
\]  

\(x_1\) = deviation of the HR from the at rest heart rate.
\(x_2\) = slower, local peripheral effects on the HR (e.g., hormonal effects, increase in body temperature, and loss of body fluids).
\(u\) = treadmill speed. \(a_i\) = constant parameter.

Motivation: Metabolic cost from walking on level ground is approximately proportional to the square of the walking speed.

Model has been validated with human subjects. Unlike conventional linear models, it captures peripheral effects and is suitable for long duration exercise.
Control Objective
Control Objective

Given any bounded $C^0$ $x_{1r}, x_{2r}, u_r : [0, \infty) \rightarrow [0, \infty)$ such that

\begin{align*}
\dot{x}_{1r} &= -a_1 x_{1r} + a_2 x_{2r} + a_2 u_r^2, \tag{2a} \\
\dot{x}_{2r} &= -a_3 x_{2r} + a_4 \frac{x_{1r}}{1 + e^{-(x_{1r}-a_5)}}, \tag{2b}
\end{align*}
Control Objective

Given any bounded $C^0 \ x_1^r, x_2^r, u_r : [0, \infty) \rightarrow [0, \infty)$ such that

\[
\begin{align*}
\dot{x}_1^r &= -a_1 x_1^r + a_2 x_2^r + a_2 u_r^2 \\ 
\dot{x}_2^r &= -a_3 x_2^r + a_4 \frac{x_1^r}{1 + e^{-(x_1^r-a_5)}} 
\end{align*}
\]  

(2a)  

(2b)

design the controller $u$
Control Objective

Given any bounded $C^0 \ x_1, x_2, u : [0, \infty) \rightarrow [0, \infty)$ such that

\[
\begin{align*}
\dot{x}_1 &= -a_1 x_1 + a_2 x_2 + a_2 u^2, \\
\dot{x}_2 &= -a_3 x_2 + a_4 \frac{x_1}{1 + e^{-(x_1 - a_5)}},
\end{align*}
\]

(2a) (2b)

design the controller $u$ so that the tracking error variable $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) = (x_1 - x_1, x_2 - x_2)$ dynamics

\[
\begin{align*}
\dot{\tilde{x}}_1 &= -a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + a_2 [u^2 - u_r(t)^2], \\
\dot{\tilde{x}}_2 &= -a_3 \tilde{x}_2 + a_4 \left[ \frac{x_1}{1 + e^{-(x_1 - a_5)}} - \frac{x_1(t)}{1 + e^{-(x_1(t) - a_5)}} \right]
\end{align*}
\]

(3a) (3b)

is globally exponentially stable to zero.
Control Objective

Given any bounded $C^0 \ x_{1r}, x_{2r}, u_r : [0, \infty) \rightarrow [0, \infty)$ such that

$$\dot{x}_{1r} = -a_1 x_{1r} + a_2 x_{2r} + a_2 u_r^2$$ (2a)
$$\dot{x}_{2r} = -a_3 x_{2r} + a_4 \frac{x_{1r}}{1 + e^{-(x_{1r} - a_5)}}$$ (2b)

design the controller $u$ so that the tracking error variable

$\tilde{x} = (\tilde{x}_1, \tilde{x}_2) = (x_1 - x_{1r}, x_2 - x_{2r})$ dynamics

$$\dot{\tilde{x}}_1 = -a_1 \tilde{x}_1 + a_2 \tilde{x}_2 + a_2 [u^2 - u_r(t)^2]$$ (3a)
$$\dot{\tilde{x}}_2 = -a_3 \tilde{x}_2 + a_4 \left[ \frac{x_1}{1 + e^{-(x_{1r} - a_5)}} - \frac{x_{1r}(t)}{1 + e^{-(x_{1r}(t) - a_5)}} \right]$$ (3b)

is globally exponentially stable to zero, i.e., there are constants $c_i > 0$ so that $|\tilde{x}(t)| \leq c_1 e^{-c_2 t} |\tilde{x}(0)|$ for all trajectories of (1).
Standing Assumption
Standing Assumption

We always assume that there is a constant $\varepsilon \in (0, 1]$ such that

$$\frac{a_1 a_3}{a_2 a_4} > P_\varepsilon \overset{\text{def}}{=} \max \left\{ \frac{1+\varepsilon}{\varepsilon}, \sup_{t \geq 0} \frac{1+b(1+(1+\varepsilon)x_1 r(t))e^{-x_1 r(t)}}{[1+be^{-(1+\varepsilon)x_1 r(t)}][1+be^{-x_1 r(t)}]} \right\} \quad \text{(SA)}$$

where $b = e^{a_5}$. 
Standing Assumption

We always assume that there is a constant $\varepsilon \in (0, 1]$ such that

$$\frac{a_1 a_3}{a_2 a_4} > P_\varepsilon \overset{\text{def}}{=} \max \left\{ \frac{1 + \varepsilon}{\varepsilon}, \sup_{t \geq 0} \frac{1 + b(1 + \{1 + \varepsilon\}x_{1r}(t))e^{-x_{1r}(t)}}{[1 + be^{-\{1 + \varepsilon\}x_{1r}(t)}][1 + be^{-x_{1r}(t)}]} \right\}$$  \hspace{1cm} (SA)

where $b = e^{a_5}$.

This condition is robust with respect to perturbations of the $a_i$'s.
Standing Assumption

We always assume that there is a constant $\varepsilon \in (0, 1]$ such that

$$\frac{a_1 a_3}{a_2 a_4} > P_\varepsilon \overset{\text{def}}{=} \max \left\{ \frac{1+\varepsilon}{\varepsilon}, \sup_{t \geq 0} \frac{1+b(1+(1+\varepsilon)x_{1r}(t))e^{-x_{1r}(t)}}{[1+be^{-(1+\varepsilon)x_{1r}(t)}][1+be^{-x_{1r}(t)}]} \right\} \quad (SA)$$

where $b = e^{a_5}$.

This condition is robust with respect to perturbations of the $a_i$’s. This robustness is important because the $a_i$’s are uncertain.
Standing Assumption

We always assume that there is a constant $\varepsilon \in (0, 1]$ such that

$$\frac{a_1 a_3}{a_2 a_4} > P_{\varepsilon} \overset{\text{def}}{=} \max \left\{ \frac{1+\varepsilon}{\varepsilon}, \sup_{t \geq 0} \frac{1 + b(1 + \{1+\varepsilon\} x_1(t)) e^{-x_1(t)}}{[1 + b e^{-\{1+\varepsilon\} x_1(t)}][1 + b e^{-x_1(t)}]} \right\}$$  \text{(SA)}

where $b = e^{a_5}$.

This condition is robust with respect to perturbations of the $a_i$'s. This robustness is important because the $a_i$'s are uncertain.

Cheng et al. use the Levenberg-Marquardt method to estimate the $a_i$'s.
Theorem 1
Theorem 1

The nonlinear controller

\[ u_c(x, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left( 1 + \frac{R(\tilde{x}_1, t)}{P_\varepsilon} \right) \tilde{x}_2 \right\}}, \]  

(4)
Theorem 1

The nonlinear controller

\[ u_c(x, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left( 1 + \frac{R(\tilde{x}_1, t)}{P_\varepsilon} \right) \tilde{x}_2 \right\}}, \]  

where

\[ R(\tilde{x}_1, t) = \frac{1 + be^{-x_1 r(t)} \left[ 1 + x_1 r(t) \int_{-1}^{0} e^{\tilde{x}_1 m} dm \right]}{(1 + be^{-(\tilde{x}_1 + x_1 r(t))})(1 + be^{-x_1 r(t)})}, \]
Theorem 1

The nonlinear controller

\[ u_c(x, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left( 1 + \frac{R(\tilde{x}_1, t)}{P_\varepsilon} \right) \tilde{x}_2 \right\} } , \tag{4} \]

where

\[ R(\tilde{x}_1, t) = \frac{1 + be^{-x_1(t)}}{(1 + be^{-(\tilde{x}_1 + x_1(t))})(1 + be^{-x_1(t)})} \tag{5} \]

solves the aforementioned control problem.
Theorem 1

The nonlinear controller

\[ u_c(x, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left(1 + \frac{R(\tilde{x}_1, t)}{P_\varepsilon}\right) \tilde{x}_2 \right\}}, \quad (4) \]

where

\[ R(\tilde{x}_1, t) = \frac{1 + be^{-x_1(t)} \left[ 1 + x_1(t) \int_{-1}^{0} e^{\tilde{x}_1 m} dm \right]}{(1 + be^{-(\tilde{x}_1 + x_1(t))})(1 + be^{-x_1(t)})}, \quad (5) \]

solves the aforementioned control problem.

Proof:
Theorem 1

The nonlinear controller

\[ u_c(x, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left( 1 + \frac{R(\tilde{x}_1, t)}{P_\epsilon} \right) \tilde{x}_2 \right\} }, \quad (4) \]

where

\[ R(\tilde{x}_1, t) = \frac{1 + be^{-x_1r(t)} \left[ 1 + x_1r(t) \int_{-1}^{0} e^{\tilde{x}_1m} dm \right]}{(1 + be^{-(\tilde{x}_1+x_1r(t))})(1 + be^{-x_1r(t)})}, \quad (5) \]

solves the aforementioned control problem.

Proof: Take \( V(\tilde{x}_1, \tilde{x}_2) = \frac{1}{2} \tilde{x}_1^2 + \frac{k}{2} \tilde{x}_2^2 \)
Theorem 1

The nonlinear controller

\[
 u_c(x, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left( 1 + \frac{R(\tilde{x}_1, t)}{P_\epsilon} \right) \tilde{x}_2 \right\}}, \quad (4)
\]

where

\[
 R(\tilde{x}_1, t) = \frac{1 + b e^{-x_{1r}(t)} \left[ 1 + x_{1r}(t) \int_{-1}^{0} e^{\tilde{x}_1 m} dm \right]}{(1 + b e^{-(\tilde{x}_1+x_{1r}(t))})(1 + b e^{-x_{1r}(t)})}, \quad (5)
\]

solves the aforementioned control problem.

Proof: Take \( V(\tilde{x}_1, \tilde{x}_2) = \frac{1}{2} \tilde{x}_1^2 + \frac{k}{2} \tilde{x}_2^2 \), where \( k = \frac{a_2}{a_4 P_\epsilon} \).
Observer Design
Observer Design

Assume that $x_2(0)$ is unknown.
Observer Design

Assume that $x_2(0)$ is unknown. Use

$$u_c(x_1, \hat{x}_2, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left( 1 + \frac{R(\hat{x}_1, t)}{P_\varepsilon} \right) \hat{x}_2 \right\}}.$$  \hfill (6)
Observer Design

Assume that $x_2(0)$ is unknown. Use

$$u_c(x_1, \hat{x}_2, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left( 1 + \frac{R(\tilde{x}_1, t)}{P_\epsilon} \right) \tilde{x}_2 \right\}}. \quad (6)$$

The estimate $\hat{x}_2$ of $\tilde{x}_2$ is from the observer

$$\begin{align*}
\dot{\hat{x}}_1 &= -a_1 \hat{x}_1 + a_2 \hat{x}_2 + k_1 \tilde{x}_1 + a_2 [u_c^2(x_1, \hat{x}_2, t) - u_r(t)^2] \\
\dot{\hat{x}}_2 &= -a_3 \hat{x}_2 + a_4 R(\tilde{x}_1, t) \tilde{x}_1 + k_2 \tilde{x}_1. \quad (7)
\end{align*}$$
Observer Design

Assume that $x_2(0)$ is unknown. Use

$$u_c(x_1, \hat{x}_2, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left( 1 + \frac{R(\hat{x}_1, t)}{P_\varepsilon} \right) \hat{x}_2 \right\}}.$$  \hfill (6)

The estimate $\hat{x}_2$ of $\ddot{x}_2$ is from the observer

$$\begin{align*}
\dot{\hat{x}}_1 &= -a_1 \hat{x}_1 + a_2 \hat{x}_2 + k_1 \ddot{x}_1 + a_2 [u_c^2(x_1, \hat{x}_2, t) - u_r(t)^2] \\
\dot{\hat{x}}_2 &= -a_3 \hat{x}_2 + a_4 R(\ddot{x}_1, t) \ddot{x}_1 + k_2 \ddot{x}_1.
\end{align*}$$  \hfill (7)

Here $k_1 > 0$ and $k_2 > 0$ are tuning constants, and $\ddot{x}_1 = \ddot{x}_1 - \dot{x}_1$. 
Observer Design

Assume that $x_2(0)$ is unknown. Use

$$u_c(x_1, \hat{x}_2, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left(1 + \frac{R(\hat{x}_1, t)}{P_{\varepsilon}}\right) \hat{x}_2 \right\}}.$$  \hspace{1cm} (6)

The estimate $\hat{x}_2$ of $\tilde{x}_2$ is from the observer

$$\begin{align*}
\dot{\hat{x}}_1 &= -a_1 \hat{x}_1 + a_2 \hat{x}_2 + k_1 \bar{x}_1 + a_2 [u_c^2(x_1, \hat{x}_2, t) - u_r(t)^2] \\
\dot{\hat{x}}_2 &= -a_3 \hat{x}_2 + a_4 R(\hat{x}_1, t) \bar{x}_1 + k_2 \bar{x}_1.
\end{align*}$$  \hspace{1cm} (7)

Here $k_1 > 0$ and $k_2 > 0$ are tuning constants, and $\bar{x}_1 = \hat{x}_1 - \hat{x}_1$.

**Proposition.** The $(\hat{x}, \bar{x})$ dynamics in closed loop with (6) is globally exponentially stable to the origin.
Observer Design

Assume that $x_2(0)$ is unknown. Use

$$u_c(x_1, \hat{x}_2, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left(1 + \frac{R(\hat{x}_1, t)}{P_{\epsilon}}\right) \hat{x}_2 \right\}}. \quad (6)$$

The estimate $\hat{x}_2$ of $\tilde{x}_2$ is from the observer

$$\begin{align*}
\dot{\hat{x}}_1 &= -a_1 \hat{x}_1 + a_2 \hat{x}_2 + k_1 \bar{x}_1 + a_2 [u_c^2(x_1, \hat{x}_2, t) - u_r(t)^2] \\
\dot{\hat{x}}_2 &= -a_3 \hat{x}_2 + a_4 R(\tilde{x}_1, t) \tilde{x}_1 + k_2 \bar{x}_1.
\end{align*} \quad (7)$$

Here $k_1 > 0$ and $k_2 > 0$ are tuning constants, and $\bar{x}_1 = \tilde{x}_1 - \hat{x}_1$.

**Proposition.** The $(\tilde{x}, \bar{x})$ dynamics in closed loop with (6) is globally exponentially stable to the origin.

**Proof:**
Observer Design

Assume that $x_2(0)$ is unknown. Use

$$u_c(x_1, \hat{x}_2, t) = \sqrt{\max \left\{ 0, u_r(t)^2 - \left( 1 + \frac{R(\hat{x}_1, t)}{P_{\varepsilon}} \right) \hat{x}_2 \right\}}.$$  \hspace{1cm} (6)

The estimate $\hat{x}_2$ of $\tilde{x}_2$ is from the observer

$$\begin{align*}
\dot{\hat{x}}_1 &= -a_1 \hat{x}_1 + a_2 \hat{x}_2 + k_1 \bar{x}_1 + a_2 [u_c^2(x_1, \hat{x}_2, t) - u_r(t)^2] \\
\dot{\hat{x}}_2 &= -a_3 \hat{x}_2 + a_4 R(\hat{x}_1, t) \hat{x}_1 + k_2 \bar{x}_1.
\end{align*} \hspace{1cm} (7)$$

Here $k_1 > 0$ and $k_2 > 0$ are tuning constants, and $\bar{x}_1 = \tilde{x}_1 - \hat{x}_1$.

**Proposition.** The $(\tilde{x}, \bar{x})$ dynamics in closed loop with (6) is globally exponentially stable to the origin.

Proof: Take $V^\#(\tilde{x}, \bar{x}) = V(\tilde{x}) + \bar{L}|\bar{x}|^2$ for a big enough $\bar{L} > 0$. 
Simulations
Simulations

We took $a_1 = 2.2$, $a_2 = 19.96$, $a_3 = 0.0831$, $a_4 = 0.002526$, $a_5 = 8.32$ (Cheng et al., IEEE-TBE).
Simulations

We took $a_1 = 2.2$, $a_2 = 19.96$, $a_3 = 0.0831$, $a_4 = 0.002526$, $a_5 = 8.32$ (Cheng et al., IEEE-TBE).

We generated the reference trajectory $x_r$ by designing $u_r$ and then solving the reference dynamics with $x_r(0) = 0$. 
Simulations

We took $a_1 = 2.2$, $a_2 = 19.96$, $a_3 = 0.0831$, $a_4 = 0.002526$, $a_5 = 8.32$ (Cheng et al., IEEE-TBE).

We generated the reference trajectory $x_r$ by designing $u_r$ and then solving the reference dynamics with $x_r(0) = 0$. 

Reference treadmill speed (km/hr)
Simulations

We took $a_1 = 2.2$, $a_2 = 19.96$, $a_3 = 0.0831$, $a_4 = 0.002526$, $a_5 = 8.32$ (Cheng et al., IEEE-TBE).

We generated the reference trajectory $x_r$ by designing $u_r$ and then solving the reference dynamics with $x_r(0) = 0$.

The resulting $x_{1r}$ satisfies (SA) with $\varepsilon = 0.5$ so our results apply.
Tracking using State Feedback Control $u_c(x, t)$
Tracking using State Feedback Control $u_c(x, t)$
Tracking using State Feedback Control $u_c(x, t)$

$\chi_1 r$ (blue and dashed) and state $\chi_1$ (red and solid).
Tracking using State Feedback Control \( u_c(x, t) \)

\[ x_{1r} \] (blue and dashed) and state \( x_1 \) (red and solid).

Initial state: \( x(0) = (2, 0) \).
Tracking using Output Control $u_c(x_1, \hat{x}_2, t)$
Tracking using Output Control $u_c(x_1, \hat{x}_2, t)$
Tracking using Output Control $u_c(x_1, \hat{x}_2, t)$

$x_{1r}$ (blue and dashed) and state $x_1$ (red and solid).
Tracking using Output Control $u_c(x_1, \hat{x}_2, t)$

$x_{1r}$ (blue and dashed) and state $x_1$ (red and solid). Initial states: $x(0) = (0.01, 0.05), \hat{x}(0) = (2, 0.3)$. 
Certifying Good Performance under Uncertainty

Since our global Lyapunov functions are strict, we can prove input-to-state stability of the augmented tracking error dynamics with respect to additive uncertainty on the controller.
Certifying Good Performance under Uncertainty

Since our global Lyapunov functions are strict, we can prove input-to-state stability of the augmented tracking error dynamics with respect to additive uncertainty on the controller.

\[
\begin{align*}
\dot{x}_1 &= -a_1 \ddot{x}_1 + a_2 \ddot{x}_2 + a_2 \left[ (u_c(x_1, \ddot{x}_2 - \bar{x}_2, t) + d)^2 - u_r(t)^2 \right] \\
\dot{x}_2 &= -a_3 \ddot{x}_2 + a_4 R(\ddot{x}_1, t) \ddot{x}_1 \\
\dot{\bar{x}}_1 &= -a_1 \bar{x}_1 + a_2 \bar{x}_2 - k_1 \bar{x}_1 \\
\dot{\bar{x}}_2 &= -a_3 \bar{x}_2 - k_2 \bar{x}_1
\end{align*}
\]
Certifying Good Performance under Uncertainty

Since our global Lyapunov functions are strict, we can prove input-to-state stability of the augmented tracking error dynamics with respect to additive uncertainty on the controller.

$$\begin{aligned}
\dot{x}_1 &= -a_1 \ddot{x}_1 + a_2 \ddot{x}_2 + a_2 \left[ (u_c(x_1, \ddot{x}_2 - \bar{x}_2, t) + d)^2 - u_r(t)^2 \right] \\
\dot{x}_2 &= -a_3 \ddot{x}_2 + a_4 R(\bar{x}_1, t) \ddot{x}_1 \\
\ddot{x}_1 &= -a_1 \ddot{x}_1 + a_2 \ddot{x}_2 - k_1 \ddot{x}_1 \\
\ddot{x}_2 &= -a_3 \ddot{x}_2 - k_2 \ddot{x}_1
\end{aligned}$$

(8)

**Theorem:** For each constant $\bar{\delta} > 0$, we can find constants $\bar{c}_i > 0$ depending on $\bar{\delta}$ so that along all trajectories of (8) for all measurable functions $d : [0, \infty) \rightarrow [-\bar{\delta}, \bar{\delta}]$, we have

$$|\langle \ddot{x}(t), \bar{x}(t) \rangle| \leq \bar{c}_1 |\langle \ddot{x}(0), \bar{x}(0) \rangle| e^{-\bar{c}_2 t} + \bar{c}_3 |d|_{[0,t]}$$

for all $t \geq 0$. 
Certifying Good Performance under Uncertainty

Since our global Lyapunov functions are strict, we can prove input-to-state stability of the augmented tracking error dynamics with respect to additive uncertainty on the controller.

\[
\begin{align*}
\dot{x}_1 &= -a_1 \dot{x}_1 + a_2 \dot{x}_2 + a_2 \left[ (u_c(x_1, \dot{x}_2 - \bar{x}_2, t) + d)^2 - u_r(t)^2 \right] \\
\dot{x}_2 &= -a_3 \dot{x}_2 + a_4 R(\dot{x}_1, t) \dot{x}_1 \\
\ddot{x}_1 &= -a_1 \ddot{x}_1 + a_2 \ddot{x}_2 - k_1 \bar{x}_1 \\
\ddot{x}_2 &= -a_3 \ddot{x}_2 - k_2 \bar{x}_1
\end{align*}
\] (8)

**Theorem:** For each constant \( \tilde{\delta} > 0 \), we can find constants \( \tilde{c}_i > 0 \) depending on \( \tilde{\delta} \) so that along all trajectories of (8) for all measurable functions \( d : [0, \infty) \rightarrow [-\tilde{\delta}, \tilde{\delta}] \), we have

\[ |(\ddot{x}(t), \dot{x}(t))| \leq \tilde{c}_1 |(\ddot{x}(0), \dot{x}(0))| e^{-\tilde{c}_2 t} + \tilde{c}_3 |d|_{[0,t]} \] for all \( t \geq 0 \).

**Proof:** Pick \( \bar{L} > 0 \) so that \( V^\# \) is an ISS Lyapunov function.
Conclusions
Conclusions

- The control of human heart rate in real time during exercise is an important problem in biomedical engineering.
Conclusions

- The control of human heart rate in real time during exercise is an important problem in biomedical engineering.
- We designed a bounded exponentially stabilizing controller for a nonlinear human heart rate dynamics.
Conclusions

- The control of human heart rate in real time during exercise is an important problem in biomedical engineering.
- We designed a bounded exponentially stabilizing controller for a nonlinear human heart rate dynamics.
- The reference trajectory gives a desired heart rate profile, and the control input is the treadmill speed.
Conclusions

- The control of human heart rate in real time during exercise is an important problem in biomedical engineering.
- We designed a bounded exponentially stabilizing controller for a nonlinear human heart rate dynamics.
- The reference trajectory gives a desired heart rate profile, and the control input is the treadmill speed.
- Using an observer, the tracking is guaranteed for all possible initial values and gives ISS to actuator errors.
Conclusions

- The control of human heart rate in real time during exercise is an important problem in biomedical engineering.

- We designed a bounded exponentially stabilizing controller for a nonlinear human heart rate dynamics.

- The reference trajectory gives a desired heart rate profile, and the control input is the treadmill speed.

- Using an observer, the tracking is guaranteed for all possible initial values and gives ISS to actuator errors.

- For complete proofs, see [FM, MM, and MdQ, “Tracking control and robustness analysis for a nonlinear model of human heart rate during exercise," *Automatica*, accepted.]