Stabilization and Robustness Analysis under Feedback Delays

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Sponsor: NSF Energy, Power, Control, and Networks
Joint with Frederic Mazenc, Students, and Others

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Perturbed Systems with Feedback Delays

These are *doubly* parameterized families of ODEs of the form

$$Y'(t) = \mathcal{F}\big(t, Y(t), u(t, Y(t-\tau)), \delta(t)\big), \quad Y(t) \in \mathcal{Y} \subseteq \mathbb{R}^n. \qquad (1)$$

$\delta : [0, \infty) \to \mathcal{D}$ is (nonstochastic) uncertainty. $\mathcal{D} \subseteq \mathbb{R}^m$. $\tau =$ delay.

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$$Y'(t) = \mathcal{G}(t, Y(t), Y(t-\tau), \delta(t)), \quad Y(t) \in \mathcal{Y}, \tag{2}$$

where $\mathcal{G}(t, Y(t), Y(t-\tau), d) = \mathcal{F}(t, Y(t), u(t, Y(t-\tau)), d)$.

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Problem: For a given reference trajectory Y_r and delay τ, design u such that the dynamics for $\mathcal{E}(t) = Y(t) - Y_r(t)$ is ISS with respect to δ. This gives tracking of Y_r when $\delta = 0$.

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$$Y'(t) = \mathcal{G}(t, Y(t), Y(t-\tau)), \quad Y(t) \in \mathcal{Y} \tag{Σ}$$

$$|Y(t)| \leq \gamma_1 \left(e^{t_0 - t} \gamma_2(|Y|_{[t_0-\tau, t_0]}) \right) \tag{UGAS}$$

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When $\tau = 0$, a system is ISS iff it has an ISS LF (Sontag-Wang).

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3. Use the LKF to compute upper bounds on the delays that the feedback can tolerate while maintaining the stability property, and use the strictness to prove ISS.

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Mazenc, F., M. Malisoff, and Z. Lin, “Further results on input-to-state stability for nonlinear systems with delayed feedbacks,” *Automatica*, 44(9):2415-2421, 2008.

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Mazenc, F., M. Malisoff, and S.-I. Niculescu, "Reduction model approach for linear time-varying systems with delays," *IEEE Transactions on Automatic Control*, 59(8):2068-2082, 2014.

Emulation Approach

Transform a suitable Lyapunov function V for a UGAS system

$$\dot{x} = f(t,x) + g(t,x)u_s(t,x) \qquad (\Sigma_{\mathrm{nd}})$$

into an ISS Lyapunov-Krasovskii functional (LKF) for

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$U : [0,\infty) \times \mathcal{C}_n(\mathbb{R}) \to [0,\infty)$ is an ISS-LKF for (Σ_d) provided there are $\alpha_i \in \mathcal{K}_\infty$ and a $\kappa \in \mathbb{N}$ such that for all solutions $x(t)$ of (Σ_d), $U(t,x_t)$ is absolutely continuous in t and we have

(i) $\alpha_1(|\phi(0)|) \le U(t,\phi) \le \alpha_2(|\phi|_{[-\kappa\tau,0]})$ and

(ii) $D_t U(t,x_t) \le -\alpha_3(U(t,x_t)) + \alpha_4(|\delta|_{[t_o,t]})$

for all $\phi \in \mathcal{C}_n([-\kappa\tau,0])$ and almost all $t \ge t_o + \kappa\tau$.

Emulation Approach

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Assumption A: f and g are locally Lipschitz, $u_s \in C^1$, and there is an $\bar{L}$ such that for all $x \in \mathbb{R}^n$ and $t \geq 0$, (A1) $|f(t,x)| \leq \bar{L}|x|$, (A2) $|g(t,x)| \leq \bar{L}(|x|+1)$, and (A3) $|(\partial u_s/\partial x)(t,x)| \leq \bar{L}$ all hold.

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Assumption B: There are $\sigma \in \mathcal{K}_\infty$ such that $\sigma(r) \leq r$ for all $r \geq 0$; constants $K_1 \geq 1$ and $K_i \geq 0$ for $i = 2,3,4$; and a C^1 uniformly proper and positive definite $V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ such that for all $x \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, $l \geq 0$, and $t \geq 0$, we have

- H1 $V_t(t,x) + V_x(t,x)[f(t,x) + g(t,x)u_s(t,x)] \leq -\sigma(|x|)^2$;
- H2 $|V_x(t,x)g(t,x)| \leq K_1\sigma(|x|)$, $\left|\frac{\partial u_s}{\partial x}(t,x)f(l,x)\right|^2 \leq K_2\sigma(|x|)^2$;
- H3 $\left|\frac{\partial u_s}{\partial x}(t,x)g(l,x)\right|^2 \leq K_3(\sigma(|x|)+1)$; and
- H4 $\left[\left|\frac{\partial u_s}{\partial x}(t,x)g(l,x)\right|\,|u_s(l,q)|\right]^2 \leq K_4[\sigma^2(|x|) + \sigma^2(|q|)]$.

Sample Result (F. Mazenc, M., Z. Lin)

Theorem 1: If Assumptions A and B are satisfied, then

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))[u_s(t, x(t-\tau)) + \delta(t)] \qquad (\Sigma_{\rm d})$$

with any constant feedback delay $\tau \in (0, \bar{\tau}]$ where

$$\bar{\tau} = \frac{1}{4K_1\sqrt{3K_2+3K_4+1}}$$

admits the ISS-LKF

$$U(t, x_t) = V(t, x(t)) + \frac{1}{8\bar{\tau}} \int_{t-2\bar{\tau}}^{t} \left(\int_r^t \sigma^2(|x(p)|)\mathrm{d}p \right) \mathrm{d}r$$

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Remark: When $V_t \equiv 0$ and the drift $f \equiv 0$, we can make the delay bound $\bar{\tau}$ arbitrarily large by taking $K_2 = 0$ and scaling u_s.

Application of Emulation Approach

When $m : \mathbb{R} \to \mathbb{R}^n$ is continuous, we build an ISS-LKF for

$$\dot{x}(t) = -m(t)m^T(t)[x(t-\tau) + \delta(t)]. \qquad (\Sigma_{\mathrm{id}})$$

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Assume $|m(t)| = 1$ for all $t \in \mathbb{R}$ and that we know constants $\alpha' \in (0, 1)$, $\beta' > 0$, and $\tilde{c} > 0$ such that

$$\alpha' I_n \leq \int_t^{t+\tilde{c}} m(r)m^T(r)\mathrm{d}r \leq \beta' I_n \text{ for all } t \in \mathbb{R}.$$

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Corollary: Let $\tau \in (0, \bar{\tau}]$. Then (Σ_{id}) admits the ISS-LKF

$$U(t, x_t) = x^T(t)P(t)x(t) + \frac{\alpha'}{8\bar{\tau}} \int_{t-2\tau}^{t} \left(\int_r^t |x(l)|^2 \mathrm{d}l \right) \mathrm{d}r,$$

where

$$P(t) = \kappa I_n + \int_{t-\tilde{c}}^{t} \int_s^t m(l)m^T(l)\,\mathrm{d}l\,\mathrm{d}s$$

and $\kappa = 1 + \frac{\tilde{c}}{2} + \frac{1}{4\alpha'}\tilde{c}^4$.

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$$\dot{x}(t) = M(t)x(t) + N(t)u(t - \tau) + \delta(t). \tag{3}$$

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$$\dot{x}(t) = M(t)x(t) + N(t)u(t-\tau) + \delta(t). \tag{3}$$

Theorem 2: If there is a bounded continuous K such that

$$\dot{z}(t) = \left[M(t) + \lambda(t, t+\tau)N(t+\tau)K(t)\right]z(t) \tag{4}$$

is UGAS, where λ is the fundamental matrix for M, then there are $\overline{\beta} \in \mathcal{KL}$ and $\overline{\gamma} \in \mathcal{K}_\infty$ such that all trajectories of (3) with

$$u(t) = K(t)\left[x(t) + \int_{t-\tau}^{t} \lambda(t, r+\tau)N(r+\tau)u(r)\mathrm{d}r\right] \tag{5}$$

satisfy

$$|x(t)| + |u|_{[t-\tau,t]} \le \overline{\beta}\big(|x(t_0)| + |u|_{[t_0-\tau,t_0]}, t - t_0\big) + \overline{\gamma}(|\delta|_{[t_0,t]}) \tag{6}$$

for all initial times $t_0 \ge 0$ and all $t \ge t_0$.

Reduction Approach

Next consider

$$\dot{x}(t) = F(t)x(t) + G(t)u(t - \tau) + \delta(t) \ . \qquad \text{(RS)}$$

F and G continuous, F has some period $\overline{T} > 0$, and G bounded.

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$$\begin{array}{l} M_F = \frac{1}{\overline{T}} \int_0^{\overline{T}} F(\ell)\mathrm{d}\ell \quad \text{and} \\ \mathcal{F}(t) = \frac{1}{\overline{T}} \int_{t-\overline{T}}^{t} \left(\int_m^t F(\ell)\mathrm{d}\ell \right) \mathrm{d}m - L^0, \end{array} \tag{7}$$

where the (i,j) entry of $L^0 \in \mathbb{R}^{n\times n}$ is $\frac{1}{2}(\varphi^\sharp_{i,j} + \varphi^\flat_{i,j})$ for all i and j, and $\varphi^\sharp_{i,j}$ (resp., $\varphi^\flat_{i,j}$) is the maximum (resp., minimum) of

$$\frac{1}{\overline{T}} \int_{t-\overline{T}}^{t} \left(\int_m^t F_{i,j}(\ell)\mathrm{d}\ell \right) \mathrm{d}m$$

over all t.

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Assumption 1: There exist a bounded continuous function K and a C^1 function P such that the time derivative of

$$Q(t,z) = z^\top P(t)z \tag{8}$$

along $\dot{z}(t) = (F(t) + e^{-M_F\tau}G(t+\tau)K(t))z(t)$ satisfies

$$\dot{Q}(t) \leq -|z(t)|^2. \tag{9}$$

Also, there are positive constants p_* and p_s such that

$$|P(t)| \leq p_* \quad \text{and} \quad p_s I_n \leq P(t) \leq p_* I_n \tag{10}$$

hold for all $t \in \mathbb{R}$.

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Assumption 2: The inequalities

$$\begin{aligned} &|\mathcal{F}|_\infty |K|_\infty p_* e^{|F|_\infty \tau} |G|_\infty \ \leq\ \tfrac{1}{16}\ , \\ &|G|_\infty |\mathcal{F}|_\infty |K|_\infty e^{(|F|_\infty+1)\tau} \sqrt{\tau}\ \leq\ \tfrac{1}{\sqrt{2}}\ ,\ \text{and} \\ &|\mathcal{F}|_\infty |K|_\infty p_* |G|_\infty e^{\tau} \max\left\{1, J_* e^{|F|_\infty \tau} \sqrt{\tau}\right\}\ \leq\ 0.19 \end{aligned} \tag{11}$$

hold, where $J_* = 2|F|_\infty + e^{|F|_\infty \tau} |G|_\infty |K|_\infty (1 + |\mathcal{F}|_\infty)$.

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Theorem 3: If Assumptions 1-2 hold, then (RS) with the control

$$u(t) = K(t)\left[x(t) + \int_{t-\tau}^{t} e^{M_F(t-r-\tau)} G(r+\tau) u(r) \mathrm{d}r\right] \qquad (12)$$

is exponentially ISS.

One of Our Applications of Reduction Approach

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Simple pendulum:

$$\begin{cases} \dot{r}_1(t) & = & r_2(t) \\ \dot{r}_2(t) & = & -\frac{g}{\ell}\sin(r_1(t)) + \frac{1}{m\ell^2} v(t-\tau) \end{cases} \tag{13}$$

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We wish to track with $r_{1,s}(t) = \omega t$ and $\tau = 1$ when $\omega > 0$ is a large enough constant, which gives a rapidly time-varying system. $m =$ mass, $\ell =$ pendulum length, $g = 9.8$.

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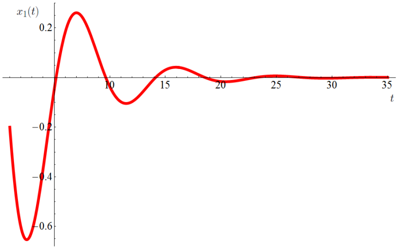
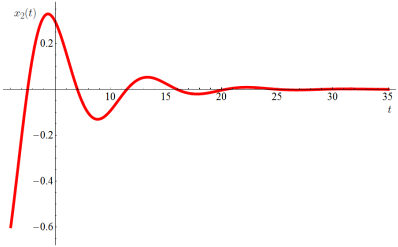
Corollary: The control $v(t-1) = m\ell^2(u(t-1) + \frac{g}{\ell}\sin(\omega t))$ with

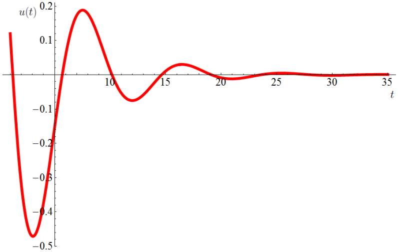
$$\begin{aligned} u(t) &= -0.6x_1(t) - 0.4x_2(t) \\ &\quad - \int_{t-1}^{t} \big(0.6(t-s-1) + 0.4\big)u(s)\mathrm{d}s \end{aligned} \tag{14}$$

ensures exponential ISS of the linearized tracking dynamics to 0.

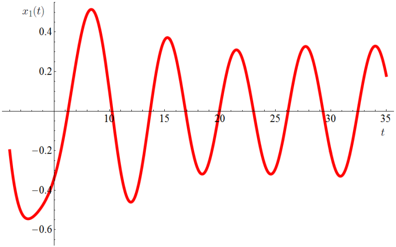
Pendulum Simulations with $\delta = 0$

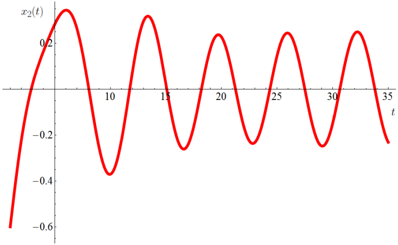


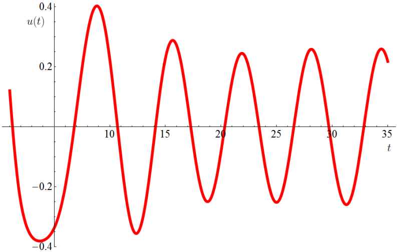




Pendulum Simulations with $\delta = 0.1(\sin, \cos)$







Reduction Approach for Nonlinear Systems

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Main Assumptions: (a) F admits a decomposition of the form

$$F(t,x) = \lambda(t, t+\tau)B(t+\tau)f_1(t,\tau,x) + f_2(t,x), \qquad (15)$$

and suitable continuous functions α_1 and α_2 such that

$$|f_1(t,\tau,x)| \le |x|^2\alpha_1(\tau, |x|^2) \ \text{ and } \ |f_2(t,x)| \le |x|^2\alpha_2(|x|^2) \qquad (16)$$

for all $t \in \mathbb{R}$, $\tau \ge 0$, and $x \in \mathbb{R}^n$, where λ is the fundamental solution of $\dot{x} = A(t)x$.

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for all $t \in \mathbb{R}$, $\tau \geq 0$, and $x \in \mathbb{R}^n$, where λ is the fundamental solution of $\dot{x} = A(t)x$. (b) There is a matrix K such that

$$\dot{x} = \big(A(t) + \lambda(t, t+\tau)B(t+\tau)K(t, \tau)\big)x \qquad (17)$$

satisfies appropriate stability properties.

Reduction Approach for Nonlinear Systems

For suitable q, v, and a, we can then prove:

Theorem 3: For each constant $\tau > 0$ and each initial function $(\phi_x, \phi_u) \in C^0([-\tau, 0], \mathbb{R}^n \times \mathbb{R}^p)$ satisfying

$$\begin{aligned} &\sqrt{q(\tau)} \left| \phi_x(0) + \int_{-\tau}^{0} \lambda(0, r+\tau) B(r+\tau) \phi_u(r) \mathrm{d}r \right| \\ &+ \frac{a}{\tau} \int_{-\tau}^{0} (r + 2\tau) |\phi_u(r)| \mathrm{d}r < v(\tau), \end{aligned} \tag{18}$$

the unique solution of (LS), in closed loop with

$$\begin{aligned} u(t) = &-f_1(t, \tau, x(t)) + K(t, \tau) \Big[x(t) \\ &+ \int_{t-\tau}^{t} \lambda(t, r+\tau) B(r+\tau) u(r) \mathrm{d}r \Big], \end{aligned} \tag{19}$$

converges to 0 as $t \to \infty$.

Joint work with V. Andrieu, M. Krstic, and Others

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We also have feedback delays and state constraints in our SICON paper on 3D curve tracking, where the state constraints are chosen to compute maximal allowable perturbations.

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Promising future research directions involve adaptive predictive control and parameter identification for nonlinear systems.