Stability and Stabilization for Chemostat Models: A Survey

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Joint with Frédéric Mazenc from INRIA DISCO
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Chemostat Apparatus

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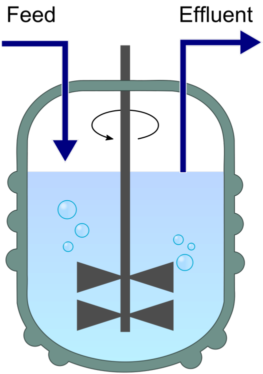

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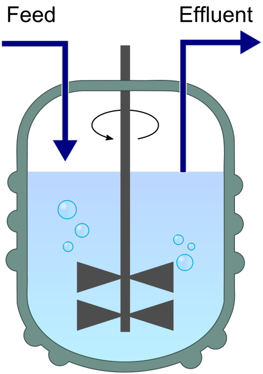

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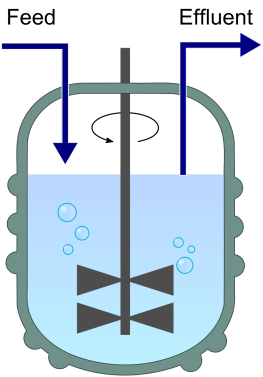

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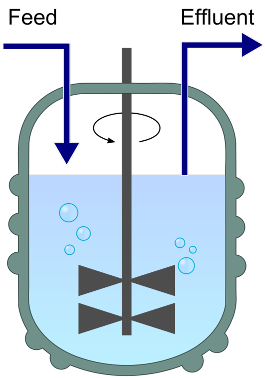

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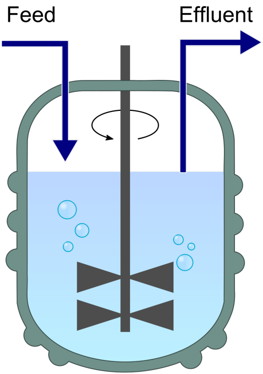

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Chemostat with N competing species and M limiting nutrients:

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This talk focuses on stability and stabilization of componentwise positive points that apply for any value of $N = M$. JBD'12.

Reminder of Model

Specializing the model to the $N = M$ case gives

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We first address the nontrivial problem of finding componentwise positive equilibria of the dynamics.

Then we study the stability and stabilization of such points.

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$$\sup_{S\in[\epsilon,B]^N} \sum_{p=1,\, p\neq i}^{N} \frac{\mathcal{G}_{p,i}(S)D_p^s(s_p^{in}-\epsilon)}{\mathcal{G}_{p,p}(S)} \; < \; D_i^s(s_i^{in} - B) \quad . \tag{4}$$

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Interacting species cases:

$$\mathcal{G}_{i,j}(S) = \int_0^{\mathcal{R}_{i,j}(s_j)} \mathcal{J}_{i,j}\big(r, S - P_j(S)\big)\mathrm{d}r.$$

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Use Brouwer degrees and the homotopy invariance property.

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Idea of Proof: Use Barbalat's Lemma and the Lyapunov function

$$L(\tilde{S}, \tilde{X}) = \sum_{j=1}^{N} \frac{1}{1 + g_j s_{j*}} \varphi_{s_{j*}}(\tilde{s}_j) + \sum_{k=1}^{N} \varphi_{x_{k*}}(\tilde{x}_k) , \qquad (7)$$

where $S_* = (s_{1*}, s_{2*}, \ldots, s_{N*})$, $X_* = (x_{1*}, x_{2*}, \ldots, x_{N*})$, and

$$\varphi_{\xi_*}(\xi) = \xi - \xi_* \ln\left(1 + \frac{\xi}{\xi_*}\right) . \qquad (8)$$

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Theorem A: Let (1) satisfy Assumptions 1-2 with Monod $\mathcal{G}_{i,j}$'s and $C = [c_{i,j}]$ invertible. Assume there is a constant $D > 0$ such that $D_j^s = D_i^x = D$ and $\eta_{i,j} = 1$ for all $i, j \in \{1, 2, \ldots, N\}$. Then (1) has a GAS equilibrium (S_*, X_*) relative to $(0, \infty)^{2N}$. $\square$

Idea of Proof: Use Barbalat's Lemma and the Lyapunov function

$$L(\tilde{S}, \tilde{X}) = \sum_{j=1}^{N} \frac{1}{1 + g_j s_{j*}} \varphi_{s_{j*}}(\tilde{s}_j) + \sum_{k=1}^{N} \varphi_{x_{k*}}(\tilde{x}_k) \; , \qquad (7)$$

where $S_* = (s_{1*}, s_{2*}, \ldots, s_{N*})$, $X_* = (x_{1*}, x_{2*}, \ldots, x_{N*})$, and

$$\varphi_{\xi_*}(\xi) = \xi - \xi_* \ln\left(1 + \frac{\xi}{\xi_*}\right) . \qquad (8)$$

We can solve for (S_*, X_*) explicitly using the Monod formulas.

Reminder of the Model

Now we view D and s_j^{in} as controllers.

$$\left\{ \begin{array}{rcl} \dot{s}_j & = & D(s_j^{in} - s_j) - \sum_{i=1}^{N} \mathcal{G}_{i,j}(S) x_i \ , \ 1 \leq j \leq N \\ \dot{x}_i & = & \left[-D + \sum_{j=1}^{N} \eta_{i,j} \mathcal{G}_{i,j}(S) \right] x_i \ , \ 1 \leq i \leq N \ . \end{array} \right. \tag{1}$$

Regulation of Species and Nutrient (M&M'12)

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Theorem B: Assume that the $\mathcal{G}_{i,j}$'s are Monod, that $C = [c_{i,j}]$ is invertible, and that $C^{-1}\nu = (k_1, k_2, \ldots, k_N)^\top \in (0, \infty)^N$.

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$$D \in \left(0, \min_j \frac{1}{k_j g_j}\right) \quad \text{and} \quad \varpi_j = \frac{D k_j}{1 - D k_j g_j} \quad \forall j \in \{1, 2, \ldots, N\}\,. \quad (9)$$

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Then (1) with the dilution rate $D_j^s \equiv D_i^x \equiv D$ and the constants

$$s_j^{in} = \varpi_j + k_j \sum_{i=1}^{N} c_{i,j}\xi_i\,, \quad j = 1, 2, \ldots, N \qquad (10)$$

admits $(\varpi_1, ..., \varpi_N, \xi_1, ..., \xi_N)$ as a globally asymptotically stable componentwise positive equilibrium point relative to $(0,\infty)^{2N}$. $\square$

Simulations for Theorem B

We took the Monod uptake functions

$$\mathcal{G}_{i,j}(S) = \frac{c_{i,j} s_j}{1+g_j s_j} \tag{11}$$

with $N = 3$ and the parameters

$$\begin{aligned} & c_{k,k} = 2 \ \ \forall k \in \{1,2,3\} \ , \quad c_{i,k} = \tfrac{1}{12} \ \text{ for } \ i \neq k \ , \\ & \text{and } \ g_k = \tfrac{1}{4} \ \ \forall k \in \{1,2,3\} \ . \end{aligned} \tag{12}$$

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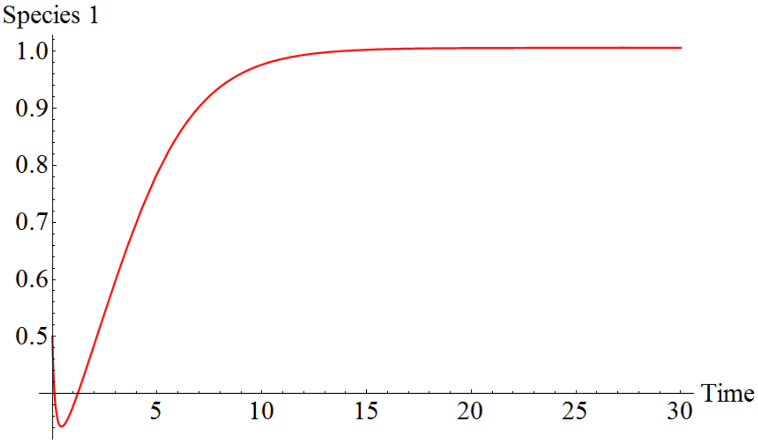
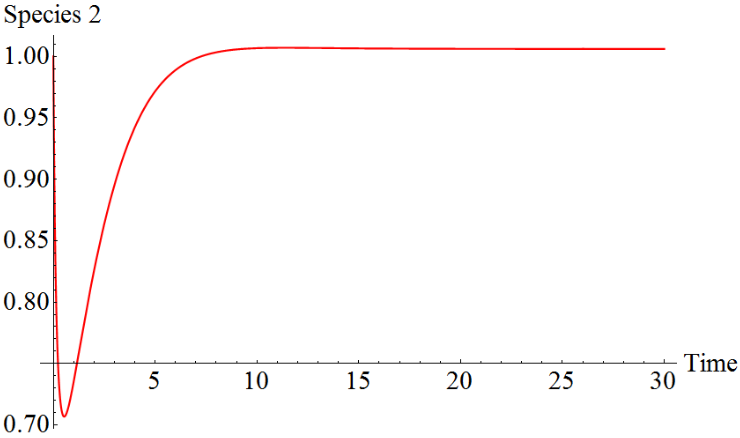
These controller values satisfy the requirements from Theorem B for stabilizing the species levels to $X_* = (1, 1, 1)$.

Simulation for First Species x_1



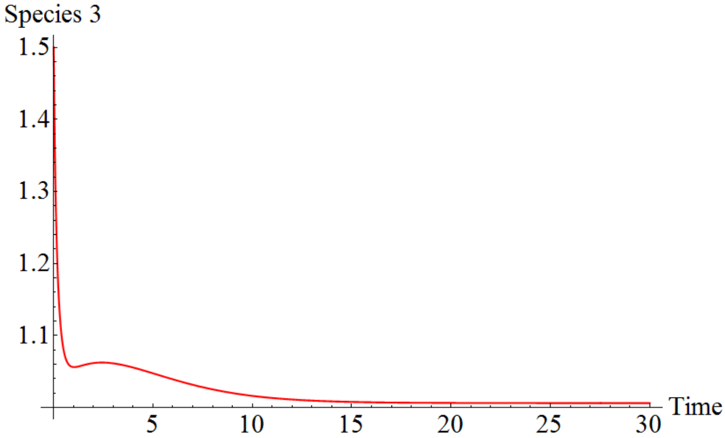
Initial value $x_1(0) = 0.5$.

Simulation for First Species x_2



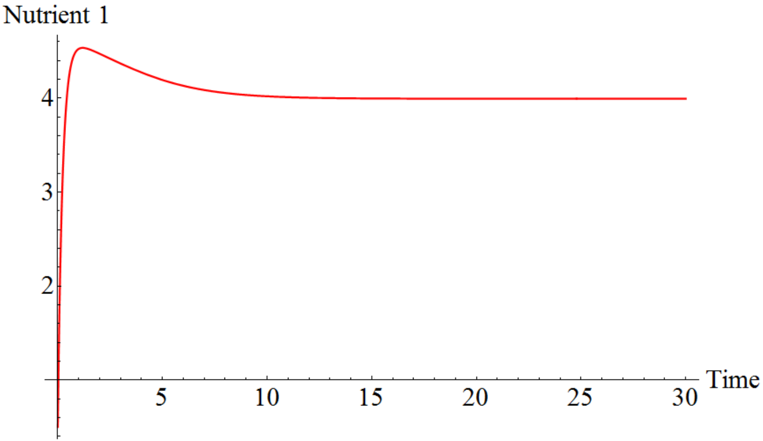
Initial value $x_2(0) = 1$.

Simulation for First Species x_3



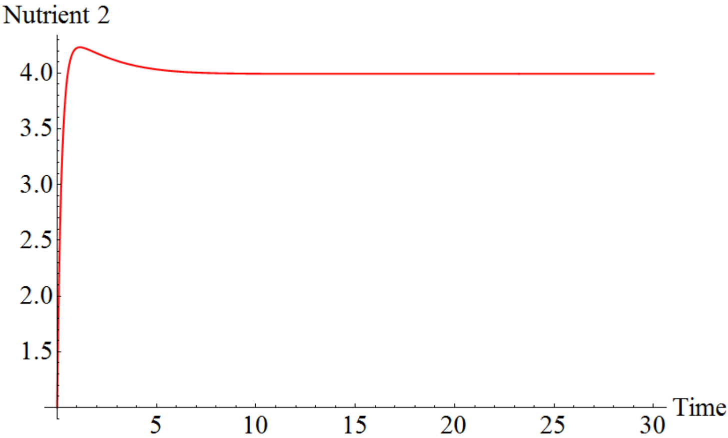
Initial value $x_3(0) = 1.5$.

Simulation for First Substrate s_1



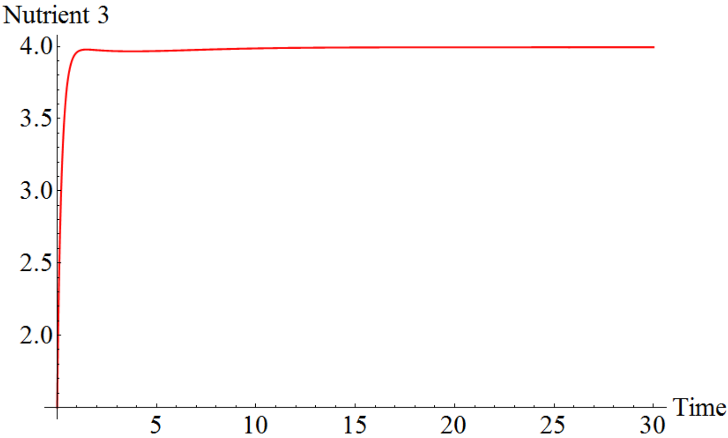
Initial value $s_1(0) = 0.5$.

Simulation for First Substrate s_2



Initial value $s_2(0) = 1$.

Simulation for First Substrate s_3



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- Other conditions ensure stabilizability of desired componentwise positive equilibrium points.
- We aim for extensions that prove robustness to unknown perturbations in the sense of input-to-state stability.