Lyapunov Functions, Point Stabilization, and Strictification

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- Strict and nonstrict Lyapunov functions

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M. Malisoff and F. Mazenc. Constructions of Strict Lyapunov Functions. Communications and Control Engineering Series, Springer-Verlag London Ltd., London, UK, 2009.

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A Lyapunov function for a system $\dot{x} = \mathcal{F}(t, x)$ with state space $\mathcal{X}$ is a positive definite proper function $V : [0, \infty) \times \mathcal{X} \to [0, \infty)$ such that $\dot{V} := V_t + V_x\mathcal{F} \leq 0$ on $[0, \infty) \times \mathcal{X}$.

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For example, $V(x) = \ln(1 + x^2)$ is a Lyapunov function for $\dot{x} = -x/(1 + x^2)$ because $\dot{V} \leq -x^2/(1 + x^2)^2$, which gives global asymptotic stability, i.e., attractivity and local stability.

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However, for each constant $\bar{\delta} > 0$, we can find an x_0 such that the trajectory for $\dot{x} = -x/(1 + x^2) + \bar{\delta}$ starting at x_0 is unbounded, which means we lack input-to-state stability.

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Using LaSalle Invariance, we can often use nonstrict Lyapunov functions to prove stability.

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For example, take $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2^3$. Use $V(x) = 0.5|x|^2$.
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For example, take $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2^3$. Use $V(x) = 0.5|x|^2$.
Then $\dot{V} = -x_2^4$. The largest invariant set in $\{x : x_2 = 0\}$ is $\{0\}$.

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However, explicit strict Lyapunov function *constructions* are often needed in applications to certify robustness.

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This has led to significant research on explicitly constructing strict Lyapunov functions.

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We assume standard assumptions on the dynamics which hold under smooth forward completeness and time-periodicity.

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We say that (1) is ISS provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ and a modulus $\bar{\alpha}$ with respect to $\mathcal{X}$ s.t. for all initial conditions $x(t_0) = x_0 \in \mathcal{X}$ and all disturbances d, the corresponding trajectories $t \mapsto \zeta(t; t_0, x_0, d)$ satisfy

$$|\zeta(t; t_0, x_0, d)| \leq \beta\Big(\bar{\alpha}(x_0), t - t_0\Big) + \gamma(|d|_\infty) \;\; \forall t \geq t_0 \, . \qquad (2)$$

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ISS Lyapunov function decay: $\dot{V} \leq -\alpha_1(V) + \alpha_2(|d|)$, $\alpha_i \in \mathcal{K}_\infty$.

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Need $K(t, x)$ and $D_x V(t, x) \cdot g(t, x)$.

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Q: Can we transform V into a strict Lyapunov function?

A: Yes, and we can allow time varying systems and relax NDC.

Let $V \in C^\infty$ be a nonstrict Lyapunov function for $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, with f and V having period T in t.

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A: Yes, and we can allow time varying systems and relax NDC.

Let $V \in C^\infty$ be a nonstrict Lyapunov function for $\dot{x} = f(t, x)$, $x \in \mathbb{R}^n$, with f and V having period T in t. Goal: Strictify it.

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Then we can explicitly determine functions $\mathcal{F}_j$ *and* $\mathcal{G}$ *such that*

$$V^\sharp(t,x) = \sum_{j=1}^{\ell-1} \mathcal{F}_j\big(V(t,x)\big)A_j(t,x) + \mathcal{G}\big(t, V(t,x)\big) \tag{6}$$

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Hence, (5) holds with $\tau = \frac{\pi}{4}$ and $\rho(r) = r^2/\{200(r+1)\}$.

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Assumption A *There exist a storage function* $V_1 : \mathcal{X} \to [0, \infty)$*; functions* h_j *such that* $h_j(0) = 0$ *for all* j*; everywhere positive functions* $r_1, \ldots, r_m$ *and* ρ*; and an integer* $N > 0$ *for which*

$$\nabla V_1(x) f(x) \ \leq \ -r_1(x) h_1^2(x) - ... - r_m(x) h_m^2(x) \ \ \forall x \in \mathcal{X} \qquad (8)$$

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Also, $f \in C^\infty(\mathbb{R}^n)$, and V_1 has a positive definite quadratic lower bound in some neighborhood of $0 \in \mathbb{R}^n$.

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Assume $\alpha > d$. Want a global strict Lyapunov function for (13).

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Along the trajectories of the L-V error dynamics,

$$\dot{S} \le -\frac{1}{4}\left[\tilde{x}^2 + \left\{(\tilde{x}+\alpha\tilde{y})(\tilde{x}+x_*)\right\}^2\right]. \tag{17}$$

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- We aim to extend strictification to general classes of adaptive time delayed systems with state constraints.