Stabilization in a Chemostat with Sampled and Delayed Measurements

Frederic Mazenc

Jerome Harmand

Michael Malisoff

Background and Motivation

Chemostat: Laboratory apparatus for continuous culture of microorganisms, many biotechnological applications..

Models: Represent cell or microorganism growth, wastewater treatment, or natural environments like lakes..

States: Microorganism and substrate concentrations, prone to incomplete measurements and model uncertainties..

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O. Bernard, D. Dochain, J. Gouze, J. Monod, H. Smith

Input-to-State Stable (ISS)

ISS (Sontag, '89) generalizes uniform global asymptotic stability.

$$Y'(t) = \mathcal{G}(t, Y(t), Y(t - \tau(t))), \quad Y(t) \in \mathcal{Y} \qquad (\Sigma)$$

$$|Y(t)| \leq \gamma_1 \left(e^{t_0 - t} \gamma_2(|Y|_{[t_0 - \bar{\tau}, t_0]}) \right) \qquad \text{(UGAS)}$$

γ_i's are 0 at 0, strictly increasing, unbounded. $\sup_{t \geq 0} \tau(t) \leq \bar{\tau}$.

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$$Y'(t) = \mathcal{G}\big(t, Y(t), Y(t - \tau(t)), \delta(t)\big), \quad Y(t) \in \mathcal{Y} \qquad (\Sigma_{\text{pert}})$$

$$|Y(t)| \leq \gamma_1 \left(e^{t_0 - t} \gamma_2(|Y|_{[t_0 - \bar{\tau}, t_0]}) \right) + \gamma_3(|\delta|_{[t_0, t]}) \qquad \text{(ISS)}$$

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Find γ_i's by building Lyapunov-Krasovskii functionals (LKFs).

Equivalent to $\mathcal{KL}$ formulation; see Sontag 1998 SCL paper.

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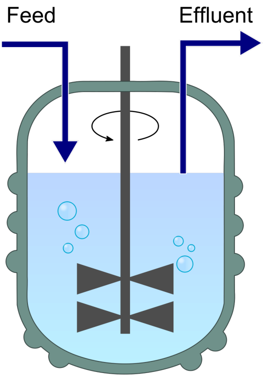

Constant volume. Substrate pumped in and substrate/biomass mixture pumped out at same rate

Uncertain Controlled Chemostat with Sampling

$$\begin{cases} \dot{s}(t) = D(s(t-\tau(t))[s_{\text{in}} - s(t)] - (1+\delta(t))\mu(s(t))x(t) \\ \dot{x}(t) = [(1+\delta(t))\mu(s(t)) - D(s(t-\tau(t))]x(t) \end{cases} \tag{C}$$

$$\tau(t) = \begin{cases} \tau_f, & t \in [0, \tau_f) \\ \tau_f + t - t_j, & t \in [t_j + \tau_f, t_{j+1} + \tau_f) \text{ and } j \geq 0 \end{cases}$$

$0 < \epsilon_1 \leq t_{i+1} - t_i \leq \epsilon_2$. $\delta : [0, \infty) \to [\underline{d}, \infty)$, with $\underline{d} \in (-1, 0]$.

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Goal: Under suitable conditions involving an upper bound τ_M for $\tau(t)$, and for constants $s_* \in (0, s_{\rm in})$, design the control D to render the dynamics for $Y(t) = (s(t), x(t)) - (s_*, s_{\rm in} - s_*)$ ISS.

Main Result for Unperturbed Case

$$\varpi_s = \inf_{s\in[0,s_{\rm in}]} \mu_1'(s)\ , \quad \varpi_I = \sup_{s\in[0,s_{\rm in}]} \mu_1'(s)\ , \quad \rho_I = \sup_{s\in[0,s_{\rm in}]} \gamma'(s),$$

$$\rho_m = \frac{\rho_I^2}{2\varpi_s} \max_{l\in[0,s_{\rm in}]} \frac{\mu_1^2(l+1.1\mu_1(s_*)s_{\rm in}\tau_M)}{1+\gamma(l)}, \quad \text{where} \quad \mu(s) = \frac{\mu_1(s)}{1+\gamma(s)}$$

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Assume that $\frac{\mu_1(s_{\rm in})}{1+\gamma(s_{\rm in})} - \frac{\mu_1(s_*)}{1+\gamma(s_{\rm in}-\mu_1(s_*)s_{\rm in}\tau_M)} > 0$

and $\tau_M < \max\left\{\frac{1}{2\sqrt{2\rho_m\varpi_l}s_{\rm in}}, \frac{1}{2\rho_l s_{\rm in}\mu_1(s_{\rm in})}\right\}$, with $s_* < s_{\rm in}$.

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Theorem: For all componentwise positive initial conditions, all solutions of the chemostat system (C) with $\delta(t) = 0$ and

$$D(s(t-\tau(t))) = \frac{\mu_1(s_*)}{1+\gamma(s(t-\tau(t)))} \tag{1}$$

remain in $(0,\infty)^2$ and converge to $(s_*, s_{\rm in} - s_*)$. □

Extensions and Applications

ISS with respect to $\delta(t)$ without upper bounds on $|\delta|_\infty$...

$$\frac{(1+\underline{d})\mu_1(s_{\rm in})}{1+\gamma(s_{\rm in})} - \frac{\mu_1(s_*)}{1+\gamma(s_{\rm in}-\mu_1(s_*)s_{\rm in}\tau_M)} > 0 \quad ...(1+\delta(t))\mu(s(t))...$$

$$\mathcal{U}_2(s_t) = \int_0^{s(t)-s_*} \frac{m}{s_{\rm in}-s_*-m}\mathrm{d}m + 2\rho_m\tau_M\int_{t-\tau_M}^{t}\int_{\ell}^{t}(\dot{s}(m))^2\mathrm{d}m\,\mathrm{d}\ell.$$

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Mazenc, F., J. Harmand, and M. Malisoff, "Stabilization in a chemostat with sampled and delayed measurements and uncertain growth functions," *Automatica*, 75, 2017, to appear.

Conclusions and Future Work

Chemostats model substrate and species interactions.

They have uncertainties, delays, and discrete measurements.

Discretization of continuous time controls can produce errors.

Our control only needs discrete delayed substrate values.

Our general growth functions are not monotone.

Our barrier functions gave ISS with uncertain growth functions.

We plan to generalize this work to multispecies chemostats.

Thank you for your attention!

Backup Slides to Use if Time Allows or Questions Warrant

Review of Controlled Chemostat with Sampling

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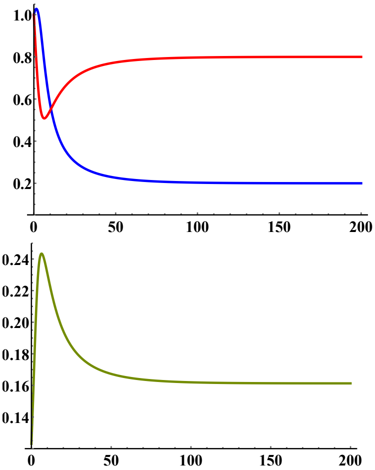
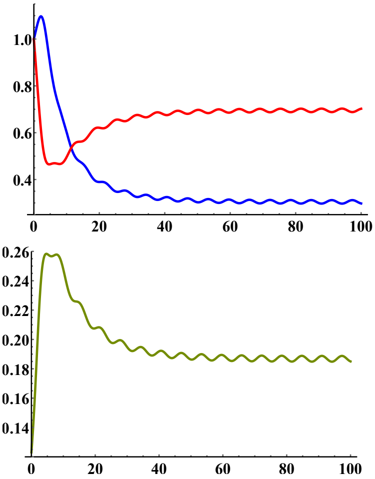


$$s_{\text{in}} = 1,\ \mu(s) = \frac{0.5s}{1+0.25s+2s^2},\ t_j = 0.24j,\ \delta(t) = 0.$$

$s(t)$ in Red, $x(t)$ in Blue, $D(t)$ in Green.



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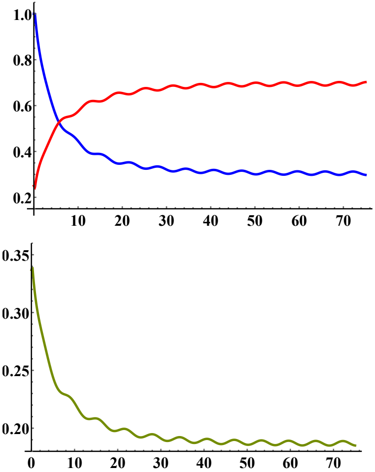

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