Adaptive Tracking and Estimation for Nonlinear Control Systems

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Sponsored by NSF/DMS Grant 0708084

AMS-SIAM Special Session on Control and Inverse Problems for Partial Differential Equations
2011 Joint Mathematics Meetings, New Orleans
Adaptive Tracking and Estimation Problem

Consider a suitably regular nonlinear system

\[ \dot{\xi} = J(t, \xi, \Psi, u) \] (1)

with a smooth reference trajectory \( \dot{\xi}_R = J(t, \xi_R, \Psi, u_R) \).

Problem:
Design a dynamic feedback with estimator

\[ u = u(t, \xi, \hat{\Psi}), \hat{\Psi} = \tau(t, \xi, \hat{\Psi}) \] (2)

such that the error

\[ \gamma = (\tilde{\Psi}, \tilde{\xi}) = (\Psi - \hat{\Psi}, \xi - \xi_R) \to 0. \]

This is a central problem with numerous applications in flight control and electrical and mechanical engineering.

Persistent excitation.

Annaswamy, Astolfi, Narendra, Teel..
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with a smooth reference trajectory \( \xi_R \) and a vector \( \Psi \) of uncertain constant parameters.

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such that the error \( Y = (\hat{\Psi}, \hat{\xi}) = (\Psi - \hat{\Psi}, \xi - \xi_R) \rightarrow 0 \).

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In 2009, we gave a solution for the special case
\[
\dot{x} = \omega(x) \Psi + u.
\] (3)

We used adaptive controllers of the form
\[
u_s = \dot{x}_R(t) - \omega(x) \hat{\Psi} + K(x_R(t) - x),
\]
\[
\dot{\hat{\Psi}} = -\omega(x) ^\top (x_R(l) - x).
\]

We used the classical PE assumption:
\[
\exists \text{constant } \mu > 0 \text{ s.t. } \mu I_p \leq \int_{t-T}^{t} \omega(x_R(l)) ^\top \omega(x_R(l)) \, dl
\] for all \( t \in \mathbb{R} \). (4)

Novelty:
Our explicit global strict Lyapunov function for the \( Y = (\Psi - \hat{\Psi}, x - x_R) \) dynamics.

It gave input-to-state stability with respect to additive time-varying uncertainties \( \delta \) on \( \Psi \).
First-Order Case (FM-MdQ-MM-TAC’09)

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Input-to-State Stability (Sontag, TAC’89)

This generalization of uniform global asymptotic stability (UGAS) applies to systems with disturbances $\delta$, having the form

$$\dot{Y} = G(t, Y, \delta(t)).$$

(5)

It is the requirement that there exist functions $\gamma_i \in K_\infty$ such that the corresponding solutions of (5) all satisfy

$$|Y(t)| \leq \gamma_1(e^{t_0} - t \gamma_2(|Y(t_0)|)) + \gamma_3(|\delta|_{[0, t]}).$$

(6)

for all $t \geq t_0 \geq 0$. UGAS is the special case where $\delta \equiv 0$. Integral ISS is the same except with the decay condition

$$\gamma_0(|Y(t)|) \leq \gamma_1(e^{t_0} - t \gamma_2(|Y(t_0)|)) + \int_{t_0}^{t} \gamma_3(|\delta|_r) \, dr.$$

(7)

Both are shown by constructing specific kinds of strict Lyapunov functions for $\dot{Y} = G(t, Y, 0)$. 
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Both are shown by constructing specific kinds of strict Lyapunov functions for $\dot{Y} = G(t, Y, 0)$. 
We solved the adaptive tracking and estimation problem for
\[
\begin{align*}
\dot{x} &= f(\xi) \\
\dot{z}_i &= g_i(\xi) + k_i(\xi) \cdot \theta_i + \psi_i u_i, \quad i = 1, 2, \ldots, s.
\end{align*}
\]
Unknown constants \(\psi = (\psi_1, \ldots, \psi_s) \in \mathbb{R}^s\) and constants \(\theta = (\theta_1, \ldots, \theta_s) \in \mathbb{R}^{p_1 + \ldots + p_s}\). \(\xi = (x, z)\).

The \(C^2\)T-periodic reference trajectory \(\xi_R = (x_R, z_R)\) to be tracked must satisfy
\[
\dot{x}_R(t) = f(\xi_R(t))
\]
everywhere.

New PE condition: positive definiteness of the matrices
\[
P_i \text{ def } = \int_0^T \lambda_i^\top(t) \lambda_i(t) \, dt \in \mathbb{R}^{(p_i+1) \times (p_i+1)},
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where \(\lambda_i(t) = (k_i(\xi_R(t)), \dot{z}_R, i(t) - g_i(\xi_R(t)))\) for each \(i\).
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P_i \overset{\text{def}}{=} \int_0^T \lambda_i^\top(t) \lambda_i(t) \, dt \in \mathbb{R}^{(p_i+1) \times (p_i+1)},
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where \( \lambda_i(t) = (k_i(\xi_R(t)), \dot{z}_{R,i}(t) - g_i(\xi_R(t))) \) for each \( i. \)
Two Other Key Assumptions

\[ F(t, \chi) = f(\chi + R(t)) - f(\Xi R(t)) \]

There is a feedback \( v \) and a global strict Lyapunov function \( V \) for the system:

\[
\begin{align*}
\dot{X} &= F(t, X, Z) \\
\dot{Z} &= v_f(t, X, Z)
\end{align*}
\]

so that \( -\dot{V} \) and \( V \) have positive definite quadratic lower bounds near 0 and \( V \), \( v \) are \( T \)-periodic.

Backstepping.

See Sontag text, Chap. 5.

There are known positive constants \( \theta_M, \psi \), and \( \psi \) such that:

\[
\psi < \psi_i < \psi
\]

and \( |\theta_i| < \theta_M \) for each \( i \in \{1, 2, \ldots, s\} \).

Known directions for the \( \psi_i \)’s.
Two Other Key Assumptions

- Set $\mathcal{F}(t, \chi) = f(\chi + \xi_R(t)) - f(\xi_R(t))$. 

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There are known positive constants $\theta_M, \psi$ and $\psi$ such that $\psi < \psi_i < \psi$ and $|\theta_i| < \theta_M$. Known directions for the $\psi_i$'s.
Two Other Key Assumptions

- Set $\mathcal{F}(t, \chi) = f(\chi + \xi_R(t)) - f(\xi_R(t))$. There is a feedback $\nu_f$ and a global strict Lyapunov function $V$ for

\[
\begin{align*}
\dot{X} &= \mathcal{F}(t, X, Z) \\
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so that $-\dot{V}$ and $V$ have positive definite quadratic lower bounds near 0.
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so that $-\dot{V}$ and $V$ have positive definite quadratic lower bounds near 0 and $V$, and $\nu_f$ are $T$-periodic.
Two Other Key Assumptions

- Set $\mathcal{F}(t, \chi) = f(\chi + \xi_R(t)) - f(\xi_R(t))$. There is a feedback $v_f$ and a global strict Lyapunov function $V$ for

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Backstepping. See Sontag text, Chap. 5.

- There are known positive constants $\theta_M$, $\underline{\psi}$ and $\overline{\psi}$ such that

$$
\underline{\psi} < \psi_i < \overline{\psi} \quad \text{and} \quad |\theta_i| < \theta_M
$$

for each $i \in \{1, 2, \ldots, s\}$. 

Two Other Key Assumptions

- Set $\mathcal{F}(t, \chi) = f(\chi + \xi_R(t)) - f(\xi_R(t))$. There is a feedback $\nu_f$ and a global strict Lyapunov function $V$ for

\[
\begin{align*}
\dot{X} &= \mathcal{F}(t, X, Z) \\
\dot{Z} &= \nu_f(t, X, Z)
\end{align*}
\]

so that $-\dot{V}$ and $V$ have positive definite quadratic lower bounds near 0 and $V$, and $\nu_f$ are $T$-periodic.

Backstepping.. See Sontag text, Chap. 5.

- There are known positive constants $\theta_M$, $\underline{\psi}$ and $\overline{\psi}$ such that

\[
\underline{\psi} < \psi_i < \overline{\psi} \quad \text{and} \quad |\theta_i| < \theta_M
\]

for each $i \in \{1, 2, \ldots, s\}$. Known directions for the $\psi_i$'s.
Dynamic Feedback

The estimator evolves on \[ \prod_{s_i=1}^{\theta_M,\theta_M} p_i \times (\psi, \psi) s. \]

\[\begin{aligned}
\dot{\hat{\theta}}_{i,j} &= (\hat{\theta}_{2i,j} - \theta_{2M}) \varpi_{i,j}, \\
\dot{\hat{\psi}}_i &= (\hat{\psi}_i - \psi)(\hat{\psi}_i - \psi) \varphi_i, \\
\end{aligned}\]

Here \(\hat{\theta}_i = (\hat{\theta}_{1i}, \ldots, \hat{\theta}_{p_i})\) for \(i = 1, 2, \ldots, s\), \(\varpi_{i,j} = -\frac{\partial V}{\partial \tilde{z}_i(t, \tilde{\xi})} k_{i,j}(\tilde{\xi} + \xi R(t))\) and \(\varphi_i = -\frac{\partial V}{\partial \tilde{z}_i(t, \tilde{\xi})} u_i(t, \tilde{\xi}, \hat{\theta}, \hat{\psi}).\)

\[\begin{aligned}
u_i(t, \tilde{\xi}, \hat{\theta}, \hat{\psi}) &= \nu_f, i(t, \tilde{\xi}) - g_i(\xi) - k_i(\xi) \cdot \hat{\theta}_i + \dot{z}_R, i(t) \hat{\psi}_i.
\end{aligned}\]
Dynamic Feedback

The estimator evolves on \( \prod_{i=1}^{s} (-\theta M, \theta M)^{p_i} \times (\psi, \overline{\psi})^s \).
Dynamic Feedback

The estimator evolves on \( \prod_{i=1}^{s} (-\theta_M, \theta_M)^{p_i} \times (\psi, \overline{\psi})^s. \)

\[
\begin{align*}
\hat{\theta}_{i,j} &= (\hat{\theta}_{i,j}^2 - \theta_M^2) \varpi_{i,j}, \quad 1 \leq i \leq s, 1 \leq j \leq p_i \\
\hat{\psi}_i &= (\hat{\psi}_i - \psi)(\hat{\psi}_i - \overline{\psi}) \mathcal{U}_i, \quad 1 \leq i \leq s
\end{align*}
\] (12)
Dynamic Feedback

The estimator evolves on \( \prod_{i=1}^{s}(-\theta_M, \theta_M)^{p_i} \times (\psi, \overline{\psi})^s \).

\[
\begin{align*}
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\end{align*}
\]

(12)

Here \( \hat{\theta}_i = (\hat{\theta}_{i,1}, \ldots, \hat{\theta}_{i,p_i}) \) for \( i = 1, 2, \ldots, s \).
Dynamic Feedback

The estimator evolves on \( \prod_{i=1}^{s} (-\theta_M, \theta_M)^{p_i} \times (\psi, \bar{\psi})^s \).

\[
\begin{align*}
\dot{\theta}_{i,j} &= (\hat{\theta}_{i,j}^2 - \theta_M^2) \omega_{i,j}, \quad 1 \leq i \leq s, 1 \leq j \leq p_i \\
\dot{\psi}_i &= (\hat{\psi}_i - \psi) (\hat{\psi}_i - \bar{\psi}) U_i, \quad 1 \leq i \leq s
\end{align*}
\]

Here \( \hat{\theta}_i = (\hat{\theta}_{i,1}, \ldots, \hat{\theta}_{i,p_i}) \) for \( i = 1, 2, \ldots, s \),

\[
\omega_{i,j} = -\frac{\partial V}{\partial \xi_i}(t, \hat{\xi}) k_{i,j}(\tilde{\xi} + \xi_R(t))
\]
Dynamic Feedback

The estimator evolves on \( \prod_{i=1}^{s} (-\theta_M, \theta_M)^{p_i} \times (\psi, \overline{\psi})^s \).

\[
\begin{align*}
\dot{\hat{\theta}}_{i,j} &= (\hat{\theta}_{i,j}^2 - \theta_M^2) \varpi_{i,j}, \quad 1 \leq i \leq s, 1 \leq j \leq p_i \\
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\[
\begin{align*}
\varpi_{i,j} &= -\frac{\partial V}{\partial \xi_i}(t, \tilde{\xi}) k_{i,j}(\tilde{\xi} + \xi_R(t)) \quad \text{and} \\
\mathcal{U}_i &= -\frac{\partial V}{\partial \xi_i}(t, \tilde{\xi}) u_i(t, \tilde{\xi}, \hat{\theta}, \hat{\psi}) .
\end{align*}
\]

(13)
Dynamic Feedback

The estimator evolves on \( \prod_{i=1}^{s}(-\theta_M, \theta_M)^{p_i} \times (\psi, \overline{\psi})^s \). 

\[
\begin{align*}
\dot{\hat{\theta}}_{i,j} &= (\hat{\theta}^2_{i,j} - \theta^2_M) \varpi_{i,j}, \quad 1 \leq i \leq s, \ 1 \leq j \leq p_i \\
\dot{\hat{\psi}}_i &= (\hat{\psi}_i - \psi) (\hat{\psi}_i - \overline{\psi}) \mathcal{U}_i, \quad 1 \leq i \leq s
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(12)

Here \( \hat{\theta}_i = (\hat{\theta}_{i,1}, \ldots, \hat{\theta}_{i,p_i}) \) for \( i = 1, 2, \ldots, s \),

\[
\begin{align*}
\varpi_{i,j} &= -\frac{\partial V}{\partial \tilde{z}_i}(t, \tilde{\xi}) k_{i,j} (\tilde{\xi} + \xi_R(t)) \quad \text{and} \\
\mathcal{U}_i &= -\frac{\partial V}{\partial \tilde{z}_i}(t, \tilde{\xi}) u_i(t, \tilde{\xi}, \hat{\xi}, \hat{\psi}) \\
u_i(t, \tilde{\xi}, \hat{\xi}, \hat{\psi}) &= \frac{v_{f,i}(t, \tilde{\xi}) - g_i(\xi) - k_i(\xi) \cdot \hat{\theta}_i + \dot{\tilde{z}}_{R,i}(t)}{\hat{\psi}_i}
\end{align*}
\]

(13)
Dynamic Feedback

The estimator evolves on \( \{ \prod_{i=1}^{s}(-\theta_{M}, \theta_{M})^{p_{i}} \} \times (\psi, \overline{\psi})^{s} \).

\[
\begin{align*}
\dot{\hat{\theta}}_{i,j} &= (\hat{\theta}_{i,j}^{2} - \theta_{M}^{2})\varpi_{i,j}, \quad 1 \leq i \leq s, 1 \leq j \leq p_{i} \\
\dot{\hat{\psi}}_{i} &= (\hat{\psi}_{i} - \psi)(\hat{\psi}_{i} - \overline{\psi})\mathcal{U}_{i}, \quad 1 \leq i \leq s
\end{align*}
\]  \hspace{1cm} (12)

Here \( \hat{\theta}_{i} = (\hat{\theta}_{i,1}, \ldots, \hat{\theta}_{i,p_{i}}) \) for \( i = 1, 2, \ldots, s \),

\[
\varpi_{i,j} = -\frac{\partial V}{\partial \tilde{z}_{i}}(t, \tilde{\xi})k_{i,j}(\tilde{\xi} + \xi_{R}(t)) \quad \text{and} \quad \mathcal{U}_{i} = -\frac{\partial V}{\partial \tilde{z}_{i}}(t, \tilde{\xi})u_{i}(t, \tilde{\xi}, \hat{\theta}, \hat{\psi}).
\]  \hspace{1cm} (13)

\[
u_{i}(t, \tilde{\xi}, \hat{\theta}, \hat{\psi}) = \frac{v_{f,i}(t, \tilde{\xi}) - g_{i}(\xi) - k_{i}(\xi) \cdot \hat{\theta}_{i} + \dot{z}_{R,i}(t)}{\hat{\psi}_{i}}
\]  \hspace{1cm} (14)

The estimator and feedback can only depend on things we know.
Stabilization Analysis

We build a global strict Lyapunov function for the $Y = (\tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = (\xi - \xi_R, \theta - \hat{\theta}, \psi - \hat{\psi})$ dynamics to prove the UGAS condition $|Y(t)| \leq \gamma_1(e^{t_0} - t \gamma_2(|Y(t_0)|))$.

We start with the nonstrict Lyapunov function $V_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = V(t, \tilde{\xi}) + s \sum_{i=1}^{p} \sum_{j=1}^{m} \int_{\theta_i, j}^{0} (m - \theta_i, j)^2 d m + s \sum_{i=1}^{p} \int_{\psi_i}^{0} (\psi_i - m - \psi)(\psi - \psi_i + m) d m$.

It gives $\dot{V}_1 \leq -W(\tilde{\xi})$ for some positive definite function $W$.

This is insufficient for robustness analysis because $\dot{V}_1$ could be zero outside 0. Therefore, we transform $V_1$. 


Stabilization Analysis

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- We start with the nonstrict Lyapunov function $V_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = V(t, \tilde{\xi}) + s\sum_{i=1}^p \sum_{j=1}^{M} \int_{0}^{\theta_i,j} m \theta^2 M - (m - \theta_i,j)^2 dm + s\sum_{i=1}^p \int_{0}^{\psi_i} m (\psi_i - m - \psi)(\psi - \psi_i + m) dm$.

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Stabilization Analysis

- We build a global strict Lyapunov function for the dynamics $Y = (\tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = (\xi - \xi_R, \theta - \hat{\theta}, \psi - \hat{\psi})$ to prove the UGAS condition $|Y(t)| \leq \gamma_1(e^{t_0-t} \gamma_2(|Y(t_0)|))$.

- We start with the nonstrict Lyapunov function

\[
V_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = V(t, \tilde{\xi}) + \sum_{i=1}^s \sum_{j=1}^{p_i} \int_0^{\tilde{\theta}_{i,j}} \frac{m}{\theta^2_M - (m - \theta_{i,j})^2} \, dm \\
+ \sum_{i=1}^s \int_0^{\tilde{\psi}_i} \frac{m}{(\psi_i - m - \underline{\psi})(\overline{\psi} - \psi_i + m)} \, dm.
\]
Stabilization Analysis

- We build a global strict Lyapunov function for the dynamics to prove the UGAS condition \( |Y(t)| \leq \gamma_1 (e^{t_0 - t} \gamma_2 (|Y(t_0)|)) \).

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Stabilization Analysis

- We build a global strict Lyapunov function for the dynamics to prove the UGAS condition
  \[ |Y(t)| \leq \gamma_1 (e^{t_0 - t} \gamma_2(|Y(t_0)|)) \].

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- It gives \( \dot{V}_1 \leq -W(\tilde{\xi}) \) for some positive definite function \( W \).

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$$V_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = V(t, \tilde{\xi}) + \sum_{i=1}^{s} \sum_{j=1}^{p_i} \int_{0}^{\tilde{\theta}_{i,j}} \frac{m}{\theta_i^2 - (m - \theta_{i,j})^2} \, dm$$

$$+ \sum_{i=1}^{s} \int_{0}^{\tilde{\psi}_i} \frac{m}{(\psi_i - m - \psi)(\tilde{\psi} - \psi_i + m)} \, dm .$$

- It gives $\dot{V}_1 \leq -W(\tilde{\xi})$ for some positive definite function $W$.

- This is insufficient for robustness analysis because $\dot{V}_1$ could be zero outside 0. Therefore, we transform $V_1$. 
Transformation (FM-MM-MdQ, NATMA’11)

Theorem: We can construct $K \in \mathbb{K}^\infty \cap \mathcal{C}_1$ such that

$$V^\#(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = K(V_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi})) + s \sum_{i=1}^{\Upsilon} \Upsilon_i(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}), (15)$$

where

$$\Upsilon_i(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = -\tilde{z}_i \lambda_i(t) \alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) + 1 T \psi \alpha_i^\top(\tilde{\theta}_i, \tilde{\psi}_i) \Omega_i(t) \alpha_i(\tilde{\theta}_i, \tilde{\psi}_i), (16)$$

$$\lambda_i(t) = (k_i(\xi R(t)), \dot{z}_R, i(t) - g_i(\xi R(t))), (17)$$

$$\alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) = [\tilde{\theta}_i \psi_i - \theta_i \tilde{\psi}_i - \tilde{\psi}_i],$$

and

$$\Omega_i(t) = \int_t^{t-T} \int_s^t m \lambda_i^\top(s) \lambda_i(s) ds ms, (18)$$

is a global strict Lyapunov function for the $Y = (\tilde{\xi}, \tilde{\theta}, \tilde{\psi})$ dynamics. Hence, the dynamics are UGAS to 0.
Theorem: We can construct $K \in \mathcal{K}_\infty \cap C^1$ such that

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(15)

where

$$\mathcal{T}_i(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = -\tilde{z}_i \lambda_i(t) \alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) + \frac{1}{T^\psi} \alpha_i^\top(\tilde{\theta}_i, \tilde{\psi}_i) \Omega_i(t) \alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) ,$$

(16)

$$\lambda_i(t) = (k_i(\xi_R(t)), \dot{Z}_{R,i}(t) - g_i(\xi_R(t))) ,$$

(17)

$$\alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) = \begin{bmatrix} \tilde{\theta}_i \psi_i - \theta_i \psi_i \\ \tilde{\psi}_i \end{bmatrix} ,$$

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Theorem: We can construct $K \in \mathcal{K}_\infty \cap C^1$ such that

$$V^\#(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = K(V_1(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi})) + \sum_{i=1}^{S} \overline{\Upsilon}_i(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) \ ,$$

where

$$\overline{\Upsilon}_i(t, \tilde{\xi}, \tilde{\theta}, \tilde{\psi}) = -\tilde{z}_i \lambda_i(t) \alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) + \frac{1}{T\psi} \alpha_i^\top(\tilde{\theta}_i, \tilde{\psi}_i) \Omega_i(t) \alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) \ ,$$

$$\lambda_i(t) = (k_i(\xi_R(t)), \dot{z}_{R,i}(t) - g_i(\xi_R(t))) \ ,$$

$$\alpha_i(\tilde{\theta}_i, \tilde{\psi}_i) = \begin{bmatrix} \tilde{\theta}_i \psi_i - \theta_i \tilde{\psi}_i \\ \tilde{\psi}_i \end{bmatrix} \ ,$$

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Conclusions

Adaptive tracking and estimation is a central problem with applications in many branches of engineering.

Standard adaptive control treatments based on nonstrict Lyapunov functions only give tracking and are not robust.

Our strict Lyapunov functions gave robustness to additive uncertainties on the parameters using the ISS paradigm.

We covered systems with unknown control gains including brushless DC motors turning mechanical loads.

It would be useful to extend to cover models that are not affine in $\theta$, feedback delays, and output feedbacks.
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