On Strict Lyapunov Functions for Rapidly Time-Varying Nonlinear Systems

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Motivation and Background: Ubiquity of rapidly time-varying systems: suspended pendulums (having vertical vibrations of small amplitude and high frequency), Raleigh's and Duffing's equations, and identification. See e.g. Peuteman-Aeyels and Solo MCSS papers. *Essential* to have Lyapunov functions in robustness analysis and controller design.

BASIC DEFINITIONS

$\mathcal{M}$: $\lim_{\eta\to+\infty} \eta N(\eta) = 0$. $\mathcal{PD}$: $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ cts. & zero only at 0.

$\mathcal{K}$: $\delta \in \mathcal{PD}$ and strictly increases. $\mathcal{K}_\infty$: class $\mathcal{K}$ and unbounded.

$\mathcal{KL}$: cts. $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ s.t. (a) $\beta(\cdot, t) \in \mathcal{K}_\infty$ $\forall t$,

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GAS: $\exists \beta \in \mathcal{KL}$ s.t. $|\phi(t; t_o, x_o)| \leq \beta(|x_o|, t - t_o)$ $\forall t \geq t_o \geq 0$, $x_o \in \mathbb{R}^n$. Called GES if $\exists$ constants $D > 1$ and $\lambda > 0$ such that $\beta(s, t) = D s e^{-\lambda t}$.

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$\delta_1, \delta_2 \in \mathcal{K}_\infty$ and $\delta_3 \in \mathcal{K}$ such that (L_1) $\delta_1(|\xi|) \leq V(\xi, t) \leq \delta_2(|\xi|)$ &
(L_2) $V_t(\xi, t) + V_\xi(\xi, t)\, \overline{f}(\xi, t) \leq -\delta_3(|\xi|)$ for all $t \in \mathbb{R}_{\geq 0}$ and $\xi \in \mathbb{R}^n$.

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Compatibility: Given $\delta \in \mathcal{K}$, (Σ_l) is called δ-*compatible* provided $\exists$ Lyapunov function V and constants $\bar{c} \in (0, 1)$, $\bar{\bar{c}} > 0$ such that:
P1) $V_t(\xi, t) + V_\xi(\xi, t)\, \overline{f}(\xi, t) \leq -\bar{c}\, \delta^2(|\xi|)$, P2) $\delta(s) \leq \bar{\bar{c}}\, s$, and
P3) $|V_\xi(\xi, t)| \leq \delta(|\xi|)$ and $|\overline{f}(\xi, t)| \leq \delta(|\xi|/2)$. E.g. GES Lip. (Σ_l).

INPUT-TO-STATE STABILITY (ISS)

Consider a forward complete dynamic $(\Sigma_{\rm na})\ \ \dot{x} = F(x,t,u)$, continuous in all variables and C^1 in x with $F(0,t,0) \equiv 0$. $|\mathbf{u}|_I$=essential supremum of $\mathbf{u} \in \mathcal{U}$ restricted to any interval $I \subseteq [0,\infty)$. Includes (Σ_α) for fixed α.

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ISS: We call (Σ_{na}) ISS provided there exist $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ for which $|\phi(t;t_o,x_o,\mathbf{u})| \ \leq\ \beta(|x_o|, t-t_o) + \gamma\left(|\mathbf{u}|_{[t_o,t]}\right)$ holds when $t \geq t_o \geq 0$, $x_o \in \mathbb{R}^n$, and $\mathbf{u} \in \mathcal{U}$. If β has the form $\beta(s,t) \ =\ Dse^{-\lambda t}$, then we say that (Σ_{na}) is *input-to-state exponentially stable (ISES)*.

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ISS Lyapunov Function: Let $V : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be C^1 and admit $\delta_1, \delta_2 \in \mathcal{K}_\infty$ that satisfy (L1) above. We call V an *ISS Lyapunov function* for $(\Sigma_{\rm na})$ provided there exist $\chi, \delta_3 \in \mathcal{K}_\infty$ such that $\forall t \in [0,\infty)$, $\xi \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$: $|u| \leq \chi(|\xi|) \Rightarrow V_t(\xi,t) + V_\xi(\xi,t)\, F(\xi,t,u) \leq -\delta_3(|\xi|)$.

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Lemma: If $(\Sigma_{\rm na})$ admits an ISS Lyapunov function, then it is ISS.

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Remark: Our results extend easily to integral ISS. Angeli-Sontag-Wang-...

MAIN THEOREM and CONSTRUCTION

Key Property: There exist $\delta \in \mathcal{K}$, a δ-compatible dynamic (Σ_l), and $N \in \mathcal{M}$ such that for all $x \in \mathbb{R}^n$, all $r \in \mathbb{R}$ and sufficiently large $\eta > 0$:

$$\left| \int_{r-\frac{1}{\eta}}^{r+\frac{1}{\eta}} \left\{ f(x, l, \eta^2 l) - \overline{f}(x, l) \right\} \mathrm{d}l \right| \leq \delta(|x|/2)\, N(\eta) \qquad \text{(KP)}$$

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Main Theorem: Assume there exist $\delta \in \mathcal{K}_\infty$, a δ-compatible GAS system (Σ_l), a constant $\eta_o > 0$ and $N \in \mathcal{M}$ such that (KP) holds whenever $\eta \geq \eta_o$, $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Assume there is a constant $K > 1$ such that:

$$\begin{array}{l} \left| \frac{\partial \overline{f}}{\partial x}(x, t) \right| \leq K \;\;,\;\; \left| \frac{\partial f}{\partial x}(x, t, \alpha t) \right| \leq K \;\;,\;\; \text{and} \\ |f(x, t, \alpha t)| \leq \delta(|x|/2) \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^n, \alpha > 0. \end{array} \qquad (1)$$

Then $\exists$ a constant $\underline{\alpha} > 0$ s.t. $\forall \alpha \geq \underline{\alpha}$, the system (Σ_u) is ISS for all $\alpha \geq \underline{\alpha}$. If in addition (Σ_l) is GES, then (Σ_u) is also ISES for all $\alpha \geq \underline{\alpha}$.

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$$V^{[\alpha]}(\xi, t) := V\left(\xi - \frac{\sqrt{\alpha}}{2} \int_{t-\frac{2}{\sqrt{\alpha}}}^{t} \int_{s}^{t} \{f(\xi, l, \alpha l) - \overline{f}(\xi, l)\} \, \mathrm{d}l \, \mathrm{d}s, t\right)$$

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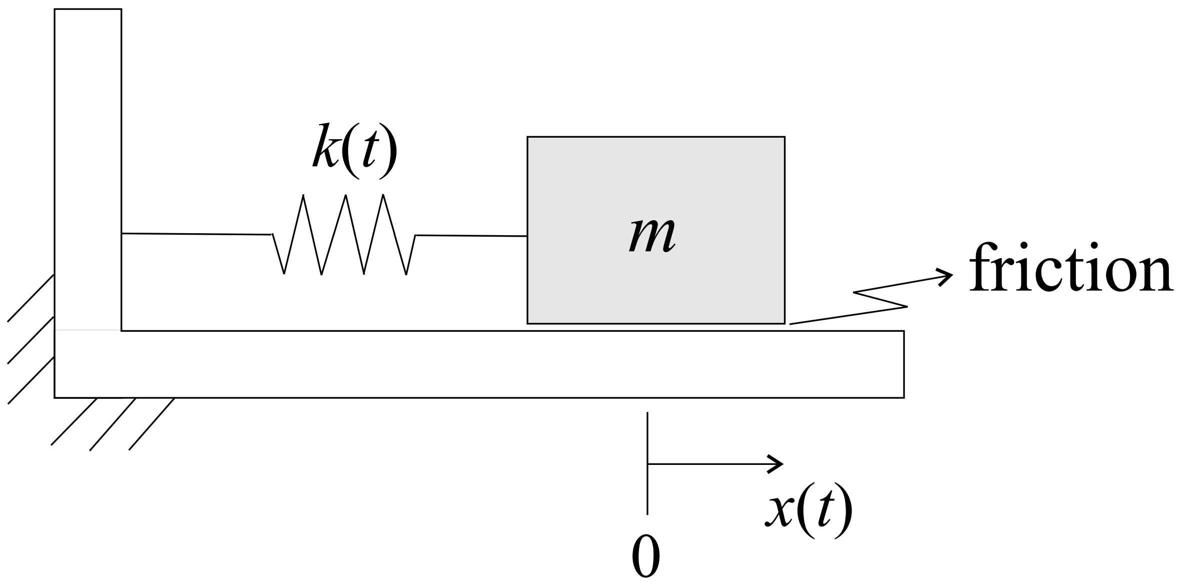
Extension to (Σ_α): Linear growth on g not enough: $\dot{x} = -x + xu$ is not ISS. Results go through for (Σ_α) if there is a constant $c_o > 1$ such that for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $\alpha > 0$, $||g(x, t, \alpha t)|| \leq c_o + \sqrt{\delta(|x|/2)}$.

MECHANICAL SYSTEM with FRICTION - SCHEMATICS



Examples: Disk drives and precision machines.

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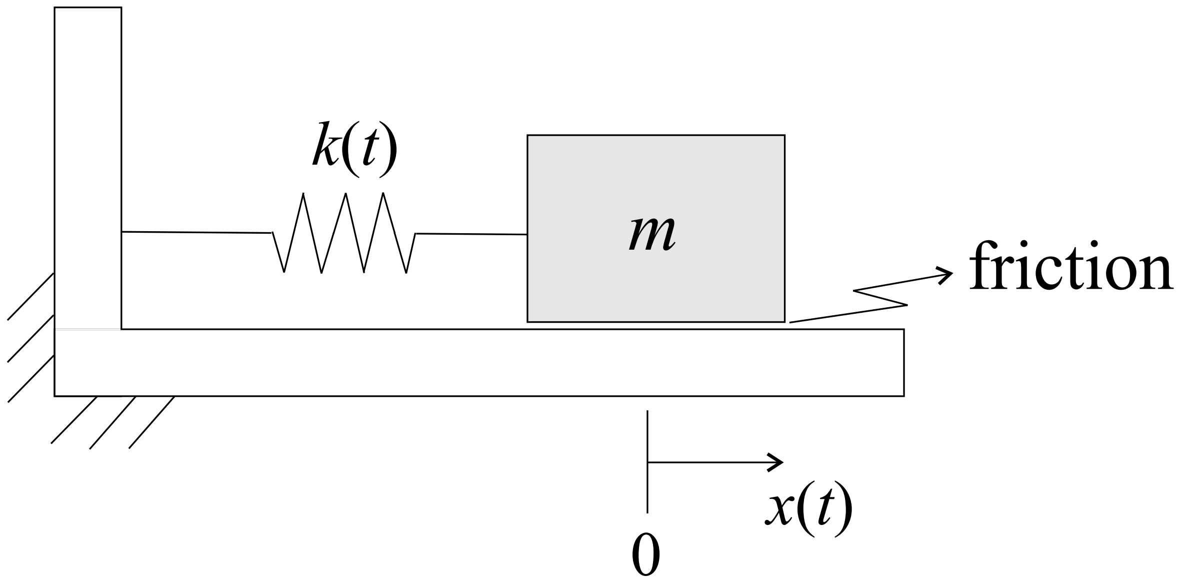

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Wear and Tear: Produces time variation in friction and spring (stiffness) coefficients. Affects friction properties more than spring. (Physical contact between mass and surface.) Hence, friction coefficients are more susceptible to variations over time, so use a rapidly time-varying model.

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Model: Dynamics for x_1=mass position and x_2=velocity:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sigma_1(\alpha t)x_2 - k(t)x_1 + u \\ &\quad - \left\{\sigma_2(\alpha t) + \sigma_3(\alpha t)e^{-\beta_1 \mu(x_2)}\right\} \mathrm{sat}(x_2) \end{aligned} \tag{MSF}$$

σ_i are positive friction-related coefficients; β_1 is a positive constant corresponding to Stribeck effect; $\mu \in \mathcal{PD}$ is related to Stribeck effect; k is a positive time-varying spring stiffness-related coefficient; and $\mathrm{sat}(x_2) = \tanh(\beta_2 x_2)$, where β_2 is a large positive constant. $\alpha > 1$.

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Assumptions: k and the σ_i's are bounded and C^1; μ has a globally bounded derivative; $\exists M : \mathbb{R} \to \mathbb{R}_{\geq 0} : s \mapsto M(s)$ that is $o(s)$ (i.e. $M(s)/s \to 0$ as $s \to +\infty$) and constants $\tilde{\sigma}_i$, with $\tilde{\sigma}_1 > 0$ and $\tilde{\sigma}_i \geq 0$ for $i = 2, 3$, s.t. $\left| \int_{t_1}^{t_2} (\sigma_i(t) - \tilde{\sigma}_i)\, \mathrm{d}t \right| \leq M(t_2 - t_1)$ $\forall i$ and $t_2 > t_1$. Also, $\exists k_o, \bar{k} > 0 \;\text{ s.t. }\; k_o \leq k(t) \leq \bar{k} \;\text{ and }\; k'(t) \leq 0 \;\; \forall t \geq 0$.

MECHANICAL SYSTEM with FRICTION (cont'd)

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Limiting Dynamics: We choose $(\Sigma_l)\ \ \dot{x} = \bar{f}(x,t)$ as follows:

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Compatibility: Holds with $\delta(s) = \bar{r}s$ for a suitable constant $\bar{r} > 0$: Take $V(x,t) = A(k(t)x_1^2 + x_2^2) + x_1 x_2$. $A := 1 + 1/k_o + [1 + S^2/k_o]/\tilde{\sigma}_1$ and $S := \tilde{\sigma}_1 + (\tilde{\sigma}_2 + \tilde{\sigma}_3)\beta_2$.

MECHANICAL SYSTEM with FRICTION (cont'd)

Limiting Dynamics: We choose $(\Sigma_l)\ \ \dot{x} = \bar{f}(x,t)$ as follows:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tilde{\sigma}_1 x_2 - \left\{\tilde{\sigma}_2 + \tilde{\sigma}_3 e^{-\beta_1 \mu(x_2)}\right\} \mathrm{sat}(x_2) - k(t) x_1\,,\end{aligned} \qquad \text{(LMSF)}$$

Compatibility: Holds with $\delta(s) = \bar{r}s$ for a suitable constant $\bar{r} > 0$: Take $V(x,t) = A(k(t)x_1^2 + x_2^2) + x_1 x_2$. $A := 1 + 1/k_o + [1 + S^2/k_o]/\tilde{\sigma}_1$ and $S := \tilde{\sigma}_1 + (\tilde{\sigma}_2 + \tilde{\sigma}_3)\beta_2$. Hence, for large $\alpha > 0$, (MSF) has ISS-CLF

$$V^{[\alpha]}(\xi,t) = V\left(\xi_1, \xi_2 + \frac{\sqrt{\alpha}}{2}\int_{t-\frac{2}{\sqrt{\alpha}}}^{t}\int_{s}^{t} \Gamma_\alpha(l,\xi)\,\mathrm{d}l\,\mathrm{d}s, t\right)$$

where $\Gamma_\alpha(l,\xi) := \{\sigma_1(\alpha l) - \tilde{\sigma}_1\}\xi_2 + \mu_\alpha(l,\xi)\tanh(\beta_2 \xi_2)$

and $\mu_\alpha(l,\xi) := \sigma_2(\alpha l) - \tilde{\sigma}_2 + (\sigma_3(\alpha l) - \tilde{\sigma}_3)e^{-\beta_1 \mu(\xi_2)}$

so the original friction dynamics (MSF) is ISS for large enough $\alpha > 0$.