On Strict Lyapunov Functions for Rapidly Time-Varying Nonlinear Systems

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Special case: \( \dot{x} = \bar{f}(x, t) \), a.k.a. \((\Sigma_l)\), called a limiting dynamics.
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\( t \mapsto \phi(t; x_o, t_o, u) \): max traj. for \( (\Sigma_\alpha) \) and \( u \in \mathcal{U} := \mathcal{MEB}([0, \infty), \mathbb{R}^m) \).
**STATEMENT and DISCUSSION of GOALS**

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**Goals:** Find reasonable conditions on \( f \) and \( g \) and on an appropriate dynamics \((\Sigma_l)\) so that, for large enough \( \alpha > 0 \), the system \((\Sigma_\alpha)\) is ISS.
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Motivation and Background: Ubiquity of rapidly time-varying systems: suspended pendulums (having vertical vibrations of small amplitude and high frequency), Raleigh’s and Duffing’s equations, and identification. See e.g. Peuteman-Aeyels and Solo MCSS papers. Essential to have Lyapunov functions in robustness analysis and controller design.
\**BASIC DEFINITIONS**

\( \mathcal{M} : \lim_{\eta \to +\infty} \eta N(\eta) = 0. \)

**PD**: \( \delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) cts. & zero only at 0.

\( \mathcal{K} : \delta \in \mathcal{PD} \) and strictly increases.  \( \mathcal{K}_\infty \): class \( \mathcal{K} \) and unbounded.

\( \mathcal{KL} \): cts. \( \beta : [0, \infty) \times [0, \infty) \to [0, \infty) \) s.t.

(a) \( \beta(\cdot, t) \in \mathcal{K}_\infty \) \( \forall t, \)

(b) \( \beta(s, \cdot) \) nonincreasing \( \forall s, \) and

(c) \( \forall s, \beta(s, t) \to 0 \) as \( t \to +\infty. \)
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\textit{KL}: \text{ cts. } \beta : [0, \infty) \times [0, \infty) \to [0, \infty) \text{ s.t. (a) } \beta(\cdot, t) \in \mathcal{K}_\infty \text{ } \forall t, \quad \text{(b) } \beta(s, \cdot) \text{ nonincreasing } \forall s, \text{ and (c) } \forall s, \beta(s, t) \to 0 \text{ as } t \to +\infty.

\textit{GAS}: \exists \beta \in \mathcal{KL} \text{ s.t. } |\phi(t; t_o, x_o)| \leq \beta(|x_o|, t - t_o) \forall t \geq t_o \geq 0, x_o \in \mathbb{R}^n. \text{ Called GES if } \exists \text{ constants } D > 1 \text{ and } \lambda > 0 \text{ such that } \beta(s, t) = Dse^{-\lambda t}. \)}
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\( KL \): cts. \( \beta : [0, \infty) \times [0, \infty) \to [0, \infty) \) s.t. (a) \( \beta(\cdot, t) \in K_\infty \forall t \), (b) \( \beta(s, \cdot) \) nonincreasing \( \forall s \), and (c) \( \forall s, \beta(s, t) \to 0 \) as \( t \to +\infty \).

\( GAS \): \( \exists \beta \in KL \) s.t. \( |\phi(t; t_0, x_0)| \leq \beta(|x_0|, t - t_0) \forall t \geq t_0 \geq 0, x_0 \in \mathbb{R}^n \).

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Lyapunov Function: \( (\Sigma_l) \) is GAS \( \iff \exists C^1 \ V : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) and \( \delta_1, \delta_2 \in K_\infty \) and \( \delta_3 \in K \) such that (L1) \( \delta_1(|\xi|) \leq V(\xi, t) \leq \delta_2(|\xi|) \) \& (L2) \( V_t(\xi, t) + V_\xi(\xi, t) \overline{f}(\xi, t) \leq -\delta_3(|\xi|) \) for all \( t \in \mathbb{R}_{\geq 0} \) and \( \xi \in \mathbb{R}^n \).
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**GAS:** \( \exists \beta \in \mathcal{KL} \) s.t.
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**Lyapunov Function:** \((\Sigma_l)\) is GAS \( \iff \exists C^1 \ V : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) and \( \delta_1, \delta_2 \in \mathcal{K}_\infty \) and \( \delta_3 \in \mathcal{K} \) such that \( (L_1) \) \( \delta_1(|\xi|) \leq V(\xi, t) \leq \delta_2(|\xi|) \) \& \( (L_2) V_t(\xi, t) + V_\xi(\xi, t) \bar{f}(\xi, t) \leq -\delta_3(|\xi|) \) for all \( t \in \mathbb{R}_{\geq 0} \) and \( \xi \in \mathbb{R}^n \).

**Compatibility:** Given \( \delta \in \mathcal{K} \), \((\Sigma_l)\) is called \( \delta\)-compatible provided \( \exists \)
Lyapunov function \( V \) and constants \( \bar{c} \in (0, 1) \), \( \bar{c} > 0 \) such that:
\( P1) \ V_t(\xi, t) + V_\xi(\xi, t) \bar{f}(\xi, t) \leq -\bar{c} \delta^2(|\xi|) \), \quad P2) \ \delta(s) \leq \bar{c} s, \) \( \text{and} \)
\( P3) \ |V_\xi(\xi, t)| \leq \delta(|\xi|) \) and \( |\bar{f}(\xi, t)| \leq \delta(|\xi|/2) \). E.g. GES Lip. \((\Sigma_l)\).
Consider a forward complete dynamic \((\Sigma_{na})\) \(\dot{x} = F(x, t, u)\), continuous in all variables and \(C^1\) in \(x\) with \(F(0, t, 0) \equiv 0\). \(|u|_I=\text{essential supremum of } u \in \mathcal{U}\) restricted to any interval \(I \subseteq [0, \infty)\). Includes \((\Sigma_\alpha)\) for fixed \(\alpha\).
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**ISS:** We call \((\Sigma_{na})\) ISS provided there exist \(\gamma \in \mathcal{K}_\infty\) and \(\beta \in \mathcal{KL}\) for which \(|\phi(t; t_o, x_o, u)| \leq \beta(|x_o|, t - t_o) + \gamma\left(|u|_{[t_o, t]}\right)\) holds when \(t \geq t_o \geq 0\), \(x_o \in \mathbb{R}^n\), and \(u \in U\). If \(\beta\) has the form \(\beta(s, t) = Dse^{-\lambda t}\), then we say that \((\Sigma_{na})\) is **input-to-state exponentially stable (ISES)**.
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**ISS Lyapunov Function:** Let $V : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ be $C^1$ and admit $\delta_1, \delta_2 \in \mathcal{K}_\infty$ that satisfy (L1) above. We call $V$ an **ISS Lyapunov function** for $(\Sigma_{na})$ provided there exist $\chi, \delta_3 \in \mathcal{K}_\infty$ such that $\forall t \in [0, \infty)$, $\xi \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$: $|u| \leq \chi(|\xi|) \Rightarrow V_t(\xi, t) + V_\xi(\xi, t) F(\xi, t, u) \leq -\delta_3(|\xi|).$
Consider a forward complete dynamic \((\Sigma_{na})\) \(\dot{x} = F(x, t, u)\), continuous in all variables and \(C^1\) in \(x\) with \(F(0, t, 0) \equiv 0\). \(|u|_I\) = essential supremum of \(u \in U\) restricted to any interval \(I \subseteq [0, \infty)\). Includes \((\Sigma_\alpha)\) for fixed \(\alpha\).

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**Lemma:** If \((\Sigma_{na})\) admits an ISS Lyapunov function, then it is ISS.
Consider a forward complete dynamic $(\Sigma_{na}) \quad \dot{x} = F(x, t, u)$, continuous in all variables and $C^1$ in $x$ with $F(0, t, 0) \equiv 0$. $|u|_I=$essential supremum of $u \in \mathcal{U}$ restricted to any interval $I \subseteq [0, \infty)$. Includes $(\Sigma_{\alpha})$ for fixed $\alpha$.

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**ISS Lyapunov Function:** Let $V : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty)$ be $C^1$ and admit $\delta_1, \delta_2 \in \mathcal{K}_\infty$ that satisfy (L1) above. We call $V$ an ISS Lyapunov function for $(\Sigma_{na})$ provided there exist $\chi, \delta_3 \in \mathcal{K}_\infty$ such that $\forall t \in [0, \infty), \xi \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$: $|u| \leq \chi(|\xi|) \Rightarrow V_t(\xi, t) + V_\xi(\xi, t) F(\xi, t, u) \leq -\delta_3(|\xi|)$.

**Lemma:** If $(\Sigma_{na})$ admits an ISS Lyapunov function, then it is ISS.

**Remark:** Our results extend easily to integral ISS. Angeli-Sontag-Wang-...
**MAIN THEOREM and CONSTRUCTION**

**Key Property:** There exist \( \delta \in \mathcal{K} \), a \( \delta \)-compatible dynamic \((\Sigma_l)\), and \( N \in \mathcal{M} \) such that for all \( x \in \mathbb{R}^n \), all \( r \in \mathbb{R} \) and sufficiently large \( \eta > 0 \):

\[
\left| \int_{r-\frac{1}{\eta}}^{r+\frac{1}{\eta}} \{ f(x, l, \eta^2 l) - \bar{f}(x, l) \} \, dl \right| \leq \delta(|x|/2) \, N(\eta) \quad \text{(KP)}
\]

First consider a system \((\Sigma_u)\) \( \dot{x} = f(x, t, \alpha t) + u \) with \( f \) as above.
Key Property: There exist $\delta \in \mathcal{K}$, a $\delta$-compatible dynamic $(\Sigma_l)$, and $N \in \mathcal{M}$ such that for all $x \in \mathbb{R}^n$, all $r \in \mathbb{R}$ and sufficiently large $\eta > 0$:

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First consider a system $(\Sigma_u)$ $\dot{x} = f(x, t, \alpha t) + u$ with $f$ as above.

Main Theorem: Assume there exist $\delta \in \mathcal{K}_\infty$, a $\delta$-compatible GAS system $(\Sigma_l)$, a constant $\eta_o > 0$ and $N \in \mathcal{M}$ such that (KP) holds whenever $\eta \geq \eta_o$, $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Assume there is a constant $K > 1$ such that:

$$\left| \frac{\partial \bar{f}}{\partial x}(x, t) \right| \leq K, \quad \left| \frac{\partial f}{\partial x}(x, t, \alpha t) \right| \leq K, \quad \text{and}$$

$$|f(x, t, \alpha t)| \leq \delta(|x|/2) \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^n, \alpha > 0. \quad (1)$$

Then $\exists$ a constant $\alpha > 0$ s.t. $\forall \alpha \geq \alpha$, the system $(\Sigma_u)$ is ISS for all $\alpha \geq \alpha$. If in addition $(\Sigma_l)$ is GES, then $(\Sigma_u)$ is also ISES for all $\alpha \geq \alpha$. 
Novelty:

- Allows cases where $\Sigma_l$ is GAS but not exponentially stable
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Corollary: Let the hypotheses of the theorem hold for some $\delta \in \mathcal{K}$, and let $V \in C^1$ be a Lyapunov function for $(\Sigma_l)$ satisfying the $\delta$-compatibility requirements. Then there exists a constant $\alpha > 0$ such that for all $\alpha > \alpha$,

$$V^{[\alpha]}(\xi, t) := V \left( \xi - \frac{\sqrt{\alpha}}{2} \int_{t-\frac{2}{\sqrt{\alpha}}}^{t} \int_{s}^{t} \{ f(\xi, l, \alpha l) - \bar{f}(\xi, l) \} \, dl \, ds, t \right)$$

is a Lyapunov function for $\dot{x} = f(x, t, \alpha t)$. If in addition $\delta \in \mathcal{K}_\infty$, then $V^{[\alpha]}$ is also an ISS Lyapunov function for $(\Sigma_u)$. 
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- Allows cases where $(\Sigma_l)$ is GAS but not exponentially stable
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is a Lyapunov function for $\dot{x} = f(x, t, \alpha t)$. If in addition $\delta \in \mathcal{K}_\infty$, then $V^{[\alpha]}$ is also an ISS Lyapunov function for $(\Sigma_u)$.

Extension to $(\Sigma_\alpha)$: Linear growth on $g$ not enough: $\dot{x} = -x + xu$ is not ISS. Results go through for $(\Sigma_\alpha)$ if there is a constant $c_o > 1$ such that for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $\alpha > 0$, $\|g(x, t, \alpha t)\| \leq c_o + \sqrt{\delta(|x|/2)}$. 
Examples: Disk drives and precision machines.
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Wear and Tear: Produces time variation in friction and spring (stiffness) coefficients. Affects friction properties more than spring. (Physical contact between mass and surface.) Hence, friction coefficients are more susceptible to variations over time, so use a rapidly time-varying model.
Model: Dynamics for $x_1=$ mass position and $x_2=$ velocity:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\sigma_1(\alpha t)x_2 - k(t)x_1 + u \\
&\quad - \left\{ \sigma_2(\alpha t) + \sigma_3(\alpha t)e^{-\beta_1\mu(x_2)} \right\} \text{sat}(x_2)
\end{align*}
\]  

(\text{MSF})

$\sigma_i$ are positive friction-related coefficients; $\beta_1$ is a positive constant corresponding to Střibeck effect; $\mu \in \mathcal{PD}$ is related to Střibeck effect; $k$ is a positive time-varying spring stiffness-related coefficient; and $\text{sat}(x_2) = \tanh(\beta_2 x_2)$, where $\beta_2$ is a large positive constant. $\alpha > 1$. 

MECHANICAL SYSTEM with FRICTION

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σ_i are positive friction-related coefficients; β_1 is a positive constant corresponding to Stribeck effect; μ ∈ PD is related to Stribeck effect; k is a positive time-varying spring stiffness-related coefficient; and sat(x_2) = tanh(β_2 x_2), where β_2 is a large positive constant. α > 1.

Assumptions: k and the σ_i’s are bounded and $C^1$; μ has a globally bounded derivative; \( \exists M : \mathbb{R} \to \mathbb{R}_{\geq 0} : s \mapsto M(s) \) that is o(s) (i.e. \( M(s)/s \to 0 \) as \( s \to +\infty \)) and constants \( \tilde{\sigma}_i \), with \( \tilde{\sigma}_1 > 0 \) and \( \tilde{\sigma}_i \geq 0 \) for \( i = 2, 3 \), s.t. \( |\int_{t_1}^{t_2} (\sigma_i(t) - \tilde{\sigma}_i) \, dt| \leq M(t_2 - t_1) \forall i \) and \( t_2 > t_1 \).

Also, \( \exists k_o, \bar{k} > 0 \) s.t. \( k_o \leq k(t) \leq \bar{k} \) and \( k'(t) \leq 0 \ \forall t \geq 0 \).
MECHANICAL SYSTEM with FRICTION (cont’d)

Limiting Dynamics: We choose \((\Sigma_1)\) \(\dot{x} = \bar{f}(x, t)\) as follows:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\tilde{\sigma}_1 x_2 - \{\tilde{\sigma}_2 + \tilde{\sigma}_3 e^{-\beta_1 \mu(x_2)}\} \text{sat}(x_2) - k(t)x_1, \\
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(LMSF)
Limiting Dynamics: We choose \((\Sigma_l) \dot{x} = \bar{f}(x, t)\) as follows:

\[
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\]

(LEMSF)

Compatibility: Holds with \(\delta(s) = \bar{r}s\) for a suitable constant \(\bar{r} > 0\): Take \(V(x, t) = A(k(t)x_1^2 + x_2^2) + x_1 x_2\). \(A := 1 + 1/k_o + [1 + S^2/k_o]/\tilde{\sigma}_1\) and \(S := \tilde{\sigma}_1 + (\tilde{\sigma}_2 + \tilde{\sigma}_3)\beta_2\).
MECHANICAL SYSTEM with FRICTION (cont’d)

Limiting Dynamics: We choose \( (\Sigma_l) \quad \dot{x} = \bar{f}(x, t) \) as follows:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\tilde{\sigma}_1 x_2 - \{\tilde{\sigma}_2 + \tilde{\sigma}_3 e^{-\beta_1 \mu(x_2)}\} \text{sat}(x_2) - k(t)x_1,
\end{align*}
\]

(LMSF)

Compatibility: Holds with \( \delta(s) = \bar{r}s \) for a suitable constant \( \bar{r} > 0 \): Take

\[
V(x, t) = A(k(t)x_1^2 + x_2^2) + x_1x_2.
\]

\( A := 1 + 1/k_o + [1 + S^2/k_o]/\tilde{\sigma}_1 \) and \( S := \tilde{\sigma}_1 + (\tilde{\sigma}_2 + \tilde{\sigma}_3)\beta_2 \). Hence, for large \( \alpha > 0 \), (MSF) has ISS-CLF

\[
V^{[\alpha]}(\xi, t) = V\left(\xi_1, \xi_2 + \frac{\sqrt{\alpha}}{2} \int_{t-\frac{2}{\sqrt{\alpha}}}^{t} \int_{s}^{t} \Gamma_\alpha(l, \xi) \, dl \, ds, t\right)
\]

where \( \Gamma_\alpha(l, \xi) := \{\sigma_1(\alpha l) - \bar{\sigma}_1\}\xi_2 + \mu_\alpha(l, \xi) \tanh(\beta_2 \xi_2) \)

and \( \mu_\alpha(l, \xi) := \sigma_2(\alpha l) - \bar{\sigma}_2 + (\sigma_3(\alpha l) - \bar{\sigma}_3)e^{-\beta_1 \mu(\xi_2)} \)

so the original friction dynamics (MSF) is ISS for large enough \( \alpha > 0 \).