Stability and Robustness Analysis for a Multi-Species Chemostat Model with Uncertainties

Frederic Mazenc
Michael Malisoff
Gonzalo Robledo

Background on Chemostats

Background on Chemostats

- Chemostat: Laboratory apparatus for continuous culture of microorganisms, with many biotechnological applications..

Background on Chemostats

- ▷ Chemostat: Laboratory apparatus for continuous culture of microorganisms, with many biotechnological applications..
- ▷ Models: Represent cell or microorganism growth, wastewater treatment, or natural environments like lakes..

Background on Chemostats

- Chemostat: Laboratory apparatus for continuous culture of microorganisms, with many biotechnological applications..
- Models: Represent cell or microorganism growth, wastewater treatment, or natural environments like lakes..
- States: Microorganism and substrate concentrations, prone to model uncertainties..

Background on Chemostats

- ▷ Chemostat: Laboratory apparatus for continuous culture of microorganisms, with many biotechnological applications..
- ▷ Models: Represent cell or microorganism growth, wastewater treatment, or natural environments like lakes..
- ▷ States: Microorganism and substrate concentrations, prone to model uncertainties..
- ▷ Our goals: Input-to-state stabilization of equilibria that allow co-existence of species with one limiting nutrient

Background on Chemostats

- ▷ Chemostat: Laboratory apparatus for continuous culture of microorganisms, with many biotechnological applications..
- ▷ Models: Represent cell or microorganism growth, wastewater treatment, or natural environments like lakes..
- ▷ States: Microorganism and substrate concentrations, prone to model uncertainties..
- ▷ Our goals: Input-to-state stabilization of equilibria that allow co-existence of species with one limiting nutrient
- ▷ O. Bernard, D. Dochain, J. Gouze, J. Monod, H. Smith, ...

Our Models and Theorem

Our Models and Theorem

$$\begin{cases} \dot{s}(t) = D[s_{\text{in}} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t))x_i(t) + \delta_0(t) \\ \dot{x}_i(t) = x_i(t)\mu_i(s(t)) + D[x_i^0 - x_i(t)] + \delta_i(t),\ 1 \le i \le n \end{cases} \quad \text{(M)}$$

$\mu_i(s) = \frac{m_i s}{a_i + s}$. Equilibria: $\mathcal{E}_* = (s_*, x_{1*}, \ldots, x_{n*}) \in (0, \infty) \times [0, \infty)^n$.

Our Models and Theorem

$$\left\{\begin{array}{rcl}\dot{s}(t) & = & D[s_{\rm in}-s(t)]-\sum_{i=1}^{n}\mu_i(s(t))x_i(t)+\delta_0(t)\\ \dot{x}_i(t) & = & x_i(t)\mu_i(s(t))+D[x_i^0-x_i(t)]+\delta_i(t),\ 1\le i\le n\end{array}\right. \qquad \text{(M)}$$

$\mu_i(s)=\frac{m_i s}{a_i+s}$. Equilibria: $\mathcal{E}_*=(s_*,x_{1*},\ldots,x_{n*})\in(0,\infty)\times[0,\infty)^n$.

Assumptions. The equilibria and disturbance bounds satisfy:

1) $\max_i \mu_i(s_*)<D$, $s_{\rm in}=s_*+\sum_{i=1}^{n}\frac{\mu_i(s_*)x_i^0}{D-\mu_i(s_*)}$, $x_{i*}=\frac{Dx_i^0}{D-\mu_i(s_*)}$

2) $\delta_i(t)\in[\underline{d}_i,\bar{d}_i]$ for all i where $Ds_{\rm in}+\underline{d}_0>0$, $\bar{d}_0<0.5Ds_*$, $Dx_i^0+\underline{d}_i>0$ for all indices $i\in\mathcal{P}$, and $\underline{d}_i=0$ for all indices $i\in\{1,2,\ldots,n\}\setminus\mathcal{P}$, where $\mathcal{P}=\{i\in\{1,2,\ldots,n\}:x_i^0>0\}$.

Our Models and Theorem

$$\begin{cases} \dot{s}(t) = D[s_{\rm in} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t))x_i(t) + \delta_0(t) \\ \dot{x}_i(t) = x_i(t)\mu_i(s(t)) + D[x_i^0 - x_i(t)] + \delta_i(t),\ 1 \le i \le n \end{cases} \quad \text{(M)}$$

$\mu_i(s) = \frac{m_i s}{a_i + s}$. Equilibria: $\mathcal{E}_* = (s_*, x_{1*}, \ldots, x_{n*}) \in (0,\infty)\times[0,\infty)^n$.

Assumptions. The equilibria and disturbance bounds satisfy:

1) $\max_i \mu_i(s_*) < D,\ s_{\rm in} = s_* + \sum_{i=1}^{n} \frac{\mu_i(s_*)x_i^0}{D-\mu_i(s_*)},\ x_{i*} = \frac{Dx_i^0}{D-\mu_i(s_*)}$
2) $\delta_i(t) \in [\underline{d}_i, \bar{d}_i]$ for all i where $Ds_{\rm in} + \underline{d}_0 > 0$, $\bar{d}_0 < 0.5Ds_*$, $Dx_i^0 + \underline{d}_i > 0$ for all indices $i \in \mathcal{P}$, and $\underline{d}_i = 0$ for all indices $i \in \{1, 2, \ldots, n\} \setminus \mathcal{P}$, where $\mathcal{P} = \{i \in \{1, 2, \ldots, n\} : x_i^0 > 0\}$.

Assumption 2) maintains forward invariance of $(0,\infty)^{n+1}$ for (M).

Our Models and Theorem

$$\begin{cases} \dot{s}(t) = D[s_{\text{in}} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t))x_i(t) + \delta_0(t) \\ \dot{x}_i(t) = x_i(t)\mu_i(s(t)) + D[x_i^0 - x_i(t)] + \delta_i(t),\ 1 \le i \le n \end{cases} \quad \text{(M)}$$

$\mu_i(s) = \frac{m_i s}{a_i + s}$. Equilibria: $\mathcal{E}_* = (s_*, x_{1*}, \ldots, x_{n*}) \in (0,\infty)\times[0,\infty)^n$.

Reduces to Gouze-Robledo model when uncertainties δ_i are 0 and usual model when the constant inputs $x_i^0 \geq 0$ are also zero.

Our Models and Theorem

$$\begin{cases} \dot{s}(t) = D[s_{\rm in} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t))x_i(t) + \delta_0(t) \\ \dot{x}_i(t) = x_i(t)\mu_i(s(t)) + D[x_i^0 - x_i(t)] + \delta_i(t),\ 1 \le i \le n \end{cases} \quad \text{(M)}$$

$\mu_i(s) = \frac{m_i s}{a_i + s}$. Equilibria: $\mathcal{E}_* = (s_*, x_{1*}, \ldots, x_{n*}) \in (0, \infty) \times [0, \infty)^n$.

Reduces to Gouze-Robledo model when uncertainties δ_i are 0 and usual model when the constant inputs $x_i^0 \ge 0$ are also zero.

Theorem: Under our assumptions, for all constants $\underline{x} > 0$ and $\bar{s} \ge s_{\rm in}$, the dynamics for the error vector $\mathcal{E} = (s, x) - \mathcal{E}_*$ are ISS on the set $\mathcal{S}_{\bar{s},\underline{x}} = \{\mathcal{E} : \mathcal{E} + \mathcal{E}_* \in (0, \bar{s}] \times (0, \infty)^{n-1} \times (\underline{x}, \infty)\}$.

Our Models and Theorem

$$\begin{cases} \dot{s}(t) = D[s_{\rm in} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t))x_i(t) + \delta_0(t) \\ \dot{x}_i(t) = x_i(t)\mu_i(s(t)) + D[x_i^0 - x_i(t)] + \delta_i(t),\ 1 \le i \le n \end{cases} \qquad \text{(M)}$$

$\mu_i(s) = \frac{m_i s}{a_i + s}$. Equilibria: $\mathcal{E}_* = (s_*, x_{1*}, \ldots, x_{n*}) \in (0,\infty)\times[0,\infty)^n$.

Reduces to Gouze-Robledo model when uncertainties δ_i are 0 and usual model when the constant inputs $x_i^0 \ge 0$ are also zero.

Theorem: Under our assumptions, for all constants $\underline{x} > 0$ and $\bar{s} \ge s_{\rm in}$, the dynamics for the error vector $\mathcal{E} = (s, x) - \mathcal{E}_*$ are ISS on the set $\mathcal{S}_{\bar{s},\underline{x}} = \{\mathcal{E} : \mathcal{E} + \mathcal{E}_* \in (0, \bar{s}] \times (0,\infty)^{n-1} \times (\underline{x}, \infty)\}$.

Significance: Persistence of all species for which $x_i^0 > 0$. ISS for arbitrarily large upper bounds $\bar{d}_i$ on the $\delta_i(t)$'s for $i \ge 1$.

Our Models and Theorem

$$\begin{cases} \dot{s}(t) = D[s_{\rm in} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t))x_i(t) + \delta_0(t) \\ \dot{x}_i(t) = x_i(t)\mu_i(s(t)) + D[x_i^0 - x_i(t)] + \delta_i(t),\ 1 \le i \le n \end{cases} \quad \text{(M)}$$

$\mu_i(s) = \frac{m_i s}{a_i + s}$. Equilibria: $\mathcal{E}_* = (s_*, x_{1*}, \ldots, x_{n*}) \in (0, \infty) \times [0, \infty)^n$.

Reduces to Gouze-Robledo model when uncertainties δ_i are 0 and usual model when the constant inputs $x_i^0 \ge 0$ are also zero.

Theorem: Under our assumptions, for all constants $\underline{x} > 0$ and $\bar{s} \ge s_{\rm in}$, the dynamics for the error vector $\mathcal{E} = (s, x) - \mathcal{E}_*$ are ISS on the set $\mathcal{S}_{\bar{s},\underline{x}} = \{\mathcal{E} : \mathcal{E} + \mathcal{E}_* \in (0, \bar{s}] \times (0, \infty)^{n-1} \times (\underline{x}, \infty)\}$.

Significance: Since $\underline{x} > 0$ and $\bar{s} \ge s_{\rm in}$ are arbitrary, we get ISS properties on all of $(0, \infty)^{n+1}$ under our disturbance bounds.

Main Idea of Proof

Main Idea of Proof

Construct a function $T \in \mathcal{K}_\infty$ and constants $c_i > 0$ and $k_i > 0$ such that the time derivative of

$$V(\mathcal{E}) = \tilde{s} - s_* \ln\left(\frac{\tilde{s}+s_*}{s_*}\right) + \sum_{i=1}^{n} \frac{1}{c_i}\Psi_i(\tilde{x}_i), \text{ where}$$
$$\Psi_i(\tilde{x}_i) = \tilde{x}_i - x_{i*} \ln\left(\frac{\tilde{x}_i+x_{i*}}{x_{i*}}\right) \text{ for all } i \in \mathcal{P}$$
$$\text{and } \Psi_i(\tilde{x}_i) = x_i \text{ for all } i \in \{1, 2, \ldots, n\} \setminus \mathcal{P}$$

along all solutions of (M) starting in $\mathcal{S}$ satisfies

$$\dot{V}(t) \leq -k_1 \left(\frac{\tilde{s}^2(t)}{s(t)} + \sum_{i=1}^{n} \frac{\tilde{x}_i^2(t)}{x_i(t)}\right) + k_2|\delta|_{[0,t]} \qquad (1)$$

for all $t \geq T(|\mathcal{E}(0)|)$, where $\tilde{x}_i = x_i - x_{i*}$ for all i and $\tilde{s} = s - s_*$.

Main Idea of Proof

Construct a function $T \in \mathcal{K}_\infty$ and constants $c_i > 0$ and $k_i > 0$ such that the time derivative of

$$V(\mathcal{E}) = \tilde{s} - s_* \ln\left(\frac{\tilde{s}+s_*}{s_*}\right) + \sum_{i=1}^{n} \frac{1}{c_i}\Psi_i(\tilde{x}_i), \text{ where}$$

$$\Psi_i(\tilde{x}_i) = \tilde{x}_i - x_{i*} \ln\left(\frac{\tilde{x}_i + x_{i*}}{x_{i*}}\right) \text{ for all } i \in \mathcal{P}$$

$$\text{and } \Psi_i(\tilde{x}_i) = x_i \text{ for all } i \in \{1, 2, \ldots, n\} \setminus \mathcal{P}$$

along all solutions of (M) starting in $\mathcal{S}$ satisfies

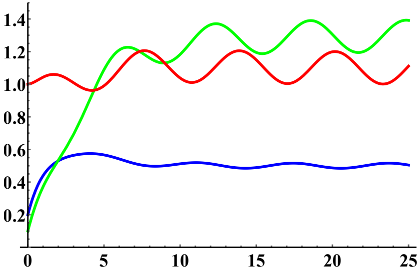
$$\dot{V}(t) \leq -k_1 \left(\frac{\tilde{s}^2(t)}{s(t)} + \sum_{i=1}^{n} \frac{\tilde{x}_i^2(t)}{x_i(t)}\right) + k_2|\delta|_{[0,t]} \qquad (1)$$

for all $t \geq T(|\mathcal{E}(0)|)$, where $\tilde{x}_i = x_i - x_{i*}$ for all i and $\tilde{s} = s - s_*$. Extend this to ISS estimate on $[0, \infty)$ by a trajectory analysis.

Simulations

Simulations

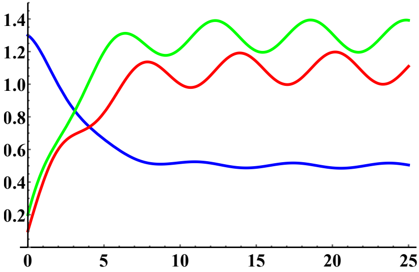
$n = 2,\ D = 0.4,\ s_* = 0.5,\ x_1^0 = 1,\ x_2^0 = 0.55,\ s_{\rm in} = 1.34412,$
$\mu_1(s) = \frac{s}{5+s},\ \mu_2(s) = \frac{s}{2+s},\ x_{1*} = 1.29412,\ x_{2*} = 1.1,$
$\delta(t) = (\delta_0(t), \delta_1(t), \delta_2(t)) = (0, -0.1\sin(t), 0.1\cos(t)).$



$x_1(t)$ and $x_2(t)$ are Green and Red Curves, Respectively. $s(t)$ is Blue Curve. Initial State $(s(0), x_1(0), x_2(0)) = (0.2, 0.1, 1)$.

Simulations

$n = 2,\ D = 0.4,\ s_* = 0.5,\ x_1^0 = 1,\ x_2^0 = 0.55,\ s_{\text{in}} = 1.34412,$
$\mu_1(s) = \frac{s}{5+s},\ \mu_2(s) = \frac{s}{2+s},\ x_{1*} = 1.29412,\ x_{2*} = 1.1,$
$\delta(t) = (\delta_0(t), \delta_1(t), \delta_2(t)) = (0, -0.1\sin(t), 0.1\cos(t)).$



$x_1(t)$ and $x_2(t)$ are Green and Red Curves, Respectively. $s(t)$ is Blue Curve. Initial State $(s(0), x_1(0), x_2(0)) = (1.3, 0.2, 0.1)$.

Conclusions

Conclusions

▷ Chemostats play a central role in microbial ecology.

Conclusions

- ▷ Chemostats play a central role in microbial ecology.
- ▷ Persistence and asymptotic stability of equilibria are desirable.

Conclusions

- ▷ Chemostats play a central role in microbial ecology.
- ▷ Persistence and asymptotic stability of equilibria are desirable.
- ▷ Gouze-Robledo showed coexistence by constant inputs x_i^0.

Conclusions

- ▷ Chemostats play a central role in microbial ecology.
- ▷ Persistence and asymptotic stability of equilibria are desirable.
- ▷ Gouze-Robledo showed coexistence by constant inputs x_i^0.
- ▷ We generalized their work to prove ISS under uncertainties.

Conclusions

- ▷ Chemostats play a central role in microbial ecology.
- ▷ Persistence and asymptotic stability of equilibria are desirable.
- ▷ Gouze-Robledo showed coexistence by constant inputs x_i^0.
- ▷ We generalized their work to prove ISS under uncertainties.
- ▷ Our novel method generalizes to cover gestation delays.

Conclusions

▷ Chemostats play a central role in microbial ecology.

▷ Persistence and asymptotic stability of equilibria are desirable.

▷ Gouze-Robledo showed coexistence by constant inputs x_i^0.

▷ We generalized their work to prove ISS under uncertainties.

▷ Our novel method generalizes to cover gestation delays.

Mazenc, F., G. Robledo, and M. Malisoff. Stability and robustness analysis for a multispecies chemostat model with delays in the growth rates and uncertainties. Discrete and Continuous Dynamical Systems Series B.

Conclusions

▷ Chemostats play a central role in microbial ecology.

▷ Persistence and asymptotic stability of equilibria are desirable.

▷ Gouze-Robledo showed coexistence by constant inputs x_i^0.

▷ We generalized their work to prove ISS under uncertainties.

▷ Our novel method generalizes to cover gestation delays.

Mazenc, F., G. Robledo, and M. Malisoff. Stability and robustness analysis for a multispecies chemostat model with delays in the growth rates and uncertainties. Discrete and Continuous Dynamical Systems Series B.

Thank you for your attention!