Stability and Robustness Analysis for a Multi-Species Chemostat Model with Uncertainties

Frederic Mazenc
Michael Malisoff
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- O. Bernard, D. Dochain, J. Gouze, J. Monod, H. Smith, ....
Our Models and Theorem

\[\dot{s}(t) = D[s_{in} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t))x_i(t) + \delta_0(t)\]

\[\dot{x}_i(t) = x_i(t)\mu_i(s(t)) + Dx_0i - x_i(t)\]

\[\mu_i(s) = m_isa_i + s_i.\]

Equilibria:

\[E^* = (s^*, x_1^*, ..., x_n^*) \in (0, \infty) \times (0, \infty)^n.\]

Assumptions. The equilibria and disturbance bounds satisfy:

1) \[\max_i \mu_i(s^*) < D,\]

\[s_{in} = s^* + \sum_{i=1}^{n} \mu_i(s^*)x_0i - \mu_i(s^*),\]

\[x_i^* = \frac{Dx_0i}{D - \mu_i(s^*)},\]

\[\delta_i(t) \in [d_i, \bar{d}_i] \text{ for all } i \in \mathbb{P},\]

where \(Ds_{in} + d_0 > 0, \bar{d}_0 < 0\).

5) \[Ds^*, Dx_0i + d_i > 0 \text{ for all indices } i \in P, \text{ and } d_i = 0 \text{ for all indices } i \in \{1, 2, ..., n\} \setminus P,\]

where \(P = \{i \in \{1, 2, ..., n\} : x_0i > 0\}\).
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\begin{align*}
\dot{s}(t) &= D[s_{\text{in}} - s(t)] - \sum_{i=1}^{n} \mu_i(s(t)) x_i(t) + \delta_0(t) \\
\dot{x}_i(t) &= x_i(t) \mu_i(s(t)) + D[x^0_i - x_i(t)] + \delta_i(t), \quad 1 \leq i \leq n
\end{align*}
\] (M)

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\mu_i(s) = \frac{m_i s}{a_i + s}.
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Equilibria: \( E_* = (s_*, x_{1*}, \ldots, x_{n*}) \in (0, \infty) \times [0, \infty)^n \).
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1) \( \max_i \mu_i(s_*) < D, \ s_{in} = s_* + \sum_{i=1}^{n} \frac{\mu_i(s_*)x_i^0}{D-\mu_i(s_*)}, \ x_{i*} = \frac{Dx_i^0}{D-\mu_i(s_*)} \)

2) \( \delta_i(t) \in [d_i, \bar{d}_i] \) for all \( i \) where \( Ds_{in} + d_0 > 0, \ \bar{d}_0 < 0.5Ds_*, \ Dx_i^0 + d_i > 0 \) for all indices \( i \in \mathcal{P} \), and \( d_i = 0 \) for all indices \( i \in \{1, 2, \ldots, n\} \setminus \mathcal{P} \), where \( \mathcal{P} = \{i \in \{1, 2, \ldots, n\} : x_i^0 > 0\} \).
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Assumption 2) maintains forward invariance of \((0, \infty)^{n+1}\) for (M).
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Reduces to Gouze-Robledo model when uncertainties \( \delta_i \) are 0 and usual model when the constant inputs \( x_i^0 \geq 0 \) are also zero.
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**Theorem:** Under our assumptions, for all constants \(\underline{x} > 0\) and \(\bar{s} \geq s_{in}\), the dynamics for the error vector \(E = (s, x) - E_*\) are ISS on the set \(S_{\bar{s}, \bar{x}} = \{E: E + E_* \in (0, \bar{s}] \times (0, \infty)^{n-1} \times (\underline{x}, \infty)\}\).
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**Theorem:** Under our assumptions, for all constants \(x > 0\) and \(\bar{s} \geq s_{\text{in}},\) the dynamics for the error vector \(E = (s, x) - E_*\) are ISS on the set \(S_{\bar{s}, \bar{x}} = \{E : E + E_* \in (0, \bar{s}] \times (0, \infty)^{n-1} \times (x, \infty)\}.

Significance: Persistence of all species for which \(x_i^0 > 0.\) ISS for arbitrarily large upper bounds \(\bar{d}_i\) on the \(\delta_i(t)\)'s for \(i \geq 1.\)
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Equilibria: \( E_* = (s_*, x_1*, \ldots, x_n*) \in (0, \infty)^{n+1} \times (0, \infty)^n \).

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Theorem: Under our assumptions, for all constants \( \underline{x} > 0 \) and \( \bar{s} \geq s_{\text{in}} \), the dynamics for the error vector \( E = (s, x) - E_* \) are ISS on the set \( S_{\bar{s}, \underline{x}} = \{ E : E + E_* \in (0, \bar{s}] \times (0, \infty)^{n-1} \times (\underline{x}, \infty) \} \).

Significance: Since \( \underline{x} > 0 \) and \( \bar{s} \geq s_{\text{in}} \) are arbitrary, we get ISS properties on all of \( (0, \infty)^{n+1} \) under our disturbance bounds.
Main Idea of Proof
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Construct a function $T \in \mathcal{K}_\infty$ and constants $c_i > 0$ and $k_i > 0$ such that the time derivative of

$$V(E) = \dot{s} - s_\ast \ln \left( \frac{s + s_\ast}{s_\ast} \right) + \sum_{i=1}^{n} \frac{1}{c_i} \psi_i(\tilde{x}_i), \quad \text{where}$$

$$\psi_i(\tilde{x}_i) = \tilde{x}_i - x_i \ast \ln \left( \frac{\tilde{x}_i + x_i \ast}{x_i \ast} \right) \quad \text{for all} \quad i \in \mathcal{P}$$

and $\psi_i(\tilde{x}_i) = x_i$ for all $i \in \{1, 2, \ldots, n\} \setminus \mathcal{P}$

along all solutions of (M) starting in $S$ satisfies

$$\dot{V}(t) \leq -k_1 \left( \frac{\tilde{s}^2(t)}{s(t)} + \sum_{i=1}^{n} \frac{\tilde{x}_i^2(t)}{x_i(t)} \right) + k_2 |\delta|_{[0,t]} \quad (1)$$

for all $t \geq T(|E(0)|)$, where $\tilde{x}_i = x_i - x_i \ast$ for all $i$ and $\tilde{s} = s - s_\ast$. 

Extend this to ISS estimate on $[0, \infty)$ by a trajectory analysis.
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Construct a function $T \in \mathcal{K}_\infty$ and constants $c_i > 0$ and $k_i > 0$ such that the time derivative of

$$V(\mathcal{E}) = \tilde{s} - s_* \ln \left( \frac{\tilde{s} + s_*}{s_*} \right) + \sum_{i=1}^{n} \frac{1}{c_i} \Psi_i(\tilde{x}_i), \text{ where}$$

$$\Psi_i(\tilde{x}_i) = \tilde{x}_i - x_i^* \ln \left( \frac{\tilde{x}_i + x_i^*}{x_i^*} \right) \text{ for all } i \in \mathcal{P}$$

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for all $t \geq T(|\mathcal{E}(0)|)$, where $\tilde{x}_i = x_i - x_i^*$ for all $i$ and $\tilde{s} = s - s_*$. Extend this to ISS estimate on $[0, \infty)$ by a trajectory analysis.
Simulations

\[ n = 2, \quad D = 0.4, \quad s^* = 0.5, \quad x_0^1 = 1, \quad x_0^2 = 0.55, \quad s_{in} = 1.34412, \mu_1(s) = s_5 + s, \mu_2(s) = s_2 + s, \]

\[ x_1^* = 1.29412, \quad x_2^* = 1.1 \]

\[ \delta(t) = (\delta_0(t), \delta_1(t), \delta_2(t)) = (0, -0.1 \sin(t), 0.1 \cos(t)). \]
Simulations

\[ n = 2, \; D = 0.4, \; s_* = 0.5, \; x_1^0 = 1, \; x_2^0 = 0.55, \; s_{\text{in}} = 1.34412, \]
\[ \mu_1(s) = \frac{s}{5+s}, \; \mu_2(s) = \frac{s}{2+s}, \; x_{1*} = 1.29412, \; x_{2*} = 1.1, \]
\[ \delta(t) = (\delta_0(t), \delta_1(t), \delta_2(t)) = (0, -0.1 \sin(t), 0.1 \cos(t)). \]

\( x_1(t) \) and \( x_2(t) \) are Green and Red Curves, Respectively. \( s(t) \) is Blue Curve. Initial State \((s(0), x_1(0), x_2(0)) = (0.2, 0.1, 1)\).
Simulations

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\( x_1(t) \) and \( x_2(t) \) are Green and Red Curves, Respectively. \( s(t) \) is Blue Curve. Initial State \( (s(0), x_1(0), x_2(0)) = (1.3, 0.2, 0.1). \)
Conclusions

Chemostats play a central role in microbial ecology.

Persistence and asymptotic stability of equilibria are desirable.

Gouze-Robledo showed coexistence by constant inputs $x_0$.

We generalized their work to prove ISS under uncertainties.

Our novel method generalizes to cover gestation delays.

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