# On the Stability of Periodic Solutions in the Perturbed Chemostat

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## GENERAL $n$ SPECIES CHEMOSTAT MODEL

$$\dot{S} = D(S_0 - S) - \sum_{i=1}^{n} \mu_i(S)x_i/\gamma_i; \quad \dot{x}_i = x_i(\mu_i(S) - D) \qquad (\Sigma_n)$$

$x_i$ = concentration of $i$th species, $S$ = concentration of limiting nutrient,

$\mu_i$ = $i$th per-capita growth rate, $\gamma_i \in (0,1)$ = constant $i$th yield factor.

Controls: dilution rate $D$ and input nutrient concentration $S_0$.

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Competitive Exclusion: When $S_0$ and $D$ are constant and the $\mu_i$'s are increasing, at most one species survives. (There is a steady state with at most one nonzero species concentration, which attracts a.a. solutions.)

## OVERVIEW of LITERATURE

Coexistence: In real ecological systems, many species can coexist, so much of the literature aims at choosing $S_0$ and/or $D$ to force coexistence. "The Paradox of the plankton," Hutchinson, *American Naturalist*, 1961.

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Intra-Specific Competition: This can be modeled with growth rates $\mu_i(S, x_i)$ that decrease in $x_i$. See Mazenc-Lobry-Rapaport (*EJDE*’07), Grognard-Mazenc-Rapaport (*DCDS*’07).

## OUR WORK for ONE SPECIES CASE

Taking $S_0$ to be constant and rescaling gives

$$\dot{S} \;=\; D(1-S) - \mu(S)x, \qquad \dot{x} \;=\; x(\mu(S) - D) \tag{$\Sigma_1$}$$

evolving on $\mathcal{X} = (0,\infty)^2$. We assume a Monod growth rate

$$\mu(S) = \frac{mS}{a+S}\,, \quad m > 4a+1\,. \tag{G}$$

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- Instead of studying coexistence, we prove the stability of a prescribed periodic solution using a Lyapunov-type analysis; i.e., tracking.
- Lyapunov functions are useful for robustness analysis but have infrequently been used in chemostat research.
- Most chemostat/Lyapunov results use *nonstrict* Lyapunov functions and LaSalle invariance which are not suited to robustness analysis.

## MAIN TRACKING RESULT for $\Sigma_1$

Statement of Main Tracking Result: Given any componentwise positive trajectory $(S, x) : [0, \infty) \to \mathcal{X}$ for ($\Sigma_1$) and the dilution rate

$$D(t) \;=\; \frac{\sin(t)}{2+\cos(t)} \;+\; \frac{m(2-\cos(t))}{4a+2-\cos(t)} \tag{D}$$

and $\mu$ as in (G) with $m > 4a+1$, the corresponding deviation

$$(\tilde{S}(t), \tilde{x}(t)) := (S(t) - S_r(t), x(t) - x_r(t)) \tag{E}$$

of $(S, x)$ from the reference trajectory

$$(S_r(t), x_r(t)) := \left(\frac{1}{2} - \frac{1}{4}\cos(t), \frac{1}{2} + \frac{1}{4}\cos(t)\right) \tag{R}$$

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(Similar results hold if we instead pick any $x_r(t)$ s.t. $\exists \ell > 0$ s.t. $\forall t \geq 0$, $\max\{\ell, |\dot{x}_r(t)|\} \leq x_r(t) \leq \frac{3}{4}$ and $S_r = 1 - x_r$, for suitable $D$.)

## OUTLINE of PROOF of MAIN TRACKING RESULT for $\Sigma_1$

First transform the error dynamics for (E) into

$$\begin{cases} \dot{\tilde{z}} & = & -D(t)\tilde{z}, \\ \dot{\tilde{\xi}} & = & \mu(z - e^{\xi}) - \mu(1 - e^{\xi_r(t)}), \end{cases} \tag{TE}$$

where $\tilde{z} := z - 1$, $z = S + x$, $\tilde{\xi} := \xi - \xi_r$, $\xi := \ln(x)$, and $\xi_r := \ln(x_r)$.

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where $\tilde{z} := z - 1$, $z = S + x$, $\tilde{\xi} := \xi - \xi_r$, $\xi := \ln(x)$, and $\xi_r := \ln(x_r)$. Next show that (TE) admits the Lyapunov-like function

$$L_3(\tilde{z}, \tilde{\xi}) := e^{\tilde{\xi}} - 1 - \tilde{\xi} + \frac{4m}{a\underline{D}}\tilde{z}^2 \tag{L}$$

where $D(t) \geq \underline{D} > 0\ \forall t$. Along the trajectories of (TE), we get

$$\dot{L}_3 \leq -\frac{ma(e^{\tilde{\xi}} - 1)^2}{16(a + 2 + \tilde{z}^2)(a + 1)} - \frac{4m}{a}\tilde{z}^2 \, . \tag{DK}$$

Using a Barbalat's Lemma argument, $(\tilde{z}, \tilde{\xi}) \to 0$ exponentially.

## ROBUSTNESS with respect to $n$ ADDITIONAL SPECIES

$$\begin{cases} \dot{S} = D(t)(1-S) - \mu(S)x - \sum_{i=1}^{n} \nu_i(S) y_i \,, \\ \dot{x} = x(\mu(S) - D(t)), \qquad \dot{y}_i = y_i(\nu_i(S) - D(t)) \end{cases} \tag{AS}$$

$D$ is from (D), $y_i =$ concentration of the $i$th additional species, each $\nu_i$ is continuous and increasing and satisfies $\nu_i(0) = 0$ and $\nu_i(1) < \underline{D}$.

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Multi-Species Result: The error between any componentwise positive solution $(S, x, y_1, y_2, \ldots, y_n)$ of (AS) and

$$(S_r, x_r, 0, \ldots, 0) \;=\; \left(\frac{1}{2} - \frac{1}{4}\cos(t), \frac{1}{2} + \frac{1}{4}\cos(t), 0, \ldots, 0\right)$$

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converges exponentially to the zero vector as $t \to +\infty$.

Significance: The stability of the reference trajectory (R) is robust with respect to additional species that are exponentially decaying to extinction.

## OUTLINE of PROOF of MULTI-SPECIES RESULT

Since $\nu_i(1) < \underline{D}$ for each $i$, the form of the dynamics for $S$ along our componentwise positive trajectories implies that there exist $\varepsilon > 0$ and $T \geq 0$ such that (i) $S(t) \leq 1 + \varepsilon$ for all $t \geq T$ and (ii) $\nu_i(1 + \varepsilon) < \underline{D}$ for all $i = 1, 2, \ldots, n$. We next choose

$$\delta := \underline{D} - \max_{i=1,\ldots,n} \nu_i(1 + \varepsilon) > 0.$$

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The result now follows using the Lyapunov-like function

$$L_4(\tilde{z}, \tilde{\xi}, y_1, ..., y_n) = L_3(\tilde{z}, \tilde{\xi}) + A \sum_{i=1}^{n} y_i^2, \quad \text{where} \quad A := \frac{16mn^2}{a\delta}.$$

in conjunction with Barbalat's Lemma. Along the relevant trajectories,

$$\dot{L}_4 \quad \leq \quad -\frac{ma(e^{\tilde{\xi}} - 1)^2}{16(a+1)(a+2+\tilde{z}^2)} - \frac{3m}{a}\tilde{z}^2 - \frac{16mn^2}{a}\sum_{i=1}^{n} y_i^2 \, .$$

## ROBUSTNESS with respect to ACTUATOR ERRORS

$$\left\{\begin{array}{rcl} \dot{S}(t) & = & [D(t)+u_1(t)](1+u_2(t)-S(t))-\mu(S(t))x(t)\,, \\ \dot{x}(t) & = & x(t)[\mu(S(t))-D(t)-u_1(t)]\,. \end{array}\right. \qquad (\Sigma_p)$$

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If $|u|$ stays below a computable prescribed bound, then there are functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the transformed error vector

$$y(t; t_o, y_o, \alpha) :=$$
$$(S(t; t_0, (S, x)(0), \alpha) - S_r(t), \ln(x(t; t_0, (S, x)(0), \alpha)) - \ln(x_r(t)))$$

for all disturbances $u = (u_1, u_2) = \alpha$ and initial conditions satisfies

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Under the less stringent condition $|u| < \frac{1}{2}\min\{1, \underline{D}\}$, there are functions $\delta_i \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ so that the trajectories everywhere satisfy

$$\delta_1(|y(t; t_o, y_o, \alpha)|) \;\leq\; \beta(|y_o|, t - t_o) + \int_{t_o}^{t+t_o} \delta_2(|\alpha(r)|)dr\,. \qquad \text{(iISS)}$$

## SIMULATIONS

We simulated ($\Sigma_p$) with $m = 10$, $a = \frac{1}{2}$, $u_1(t) = 0.5e^{-t}$, $u_2(t) \equiv 0$, $t_o = 0$, $x(0) = 2$, and $S(0) = 1$. Our theory implies that the convergence of $(S(t), x(t))$ to $(S_r(t), x_r(t))$ satisfies iISS for disturbances $u$ that are valued in $[-\bar{u}, \bar{u}]^2$ for any positive constant $\bar{u} < \min\{1, \underline{D}\} = 1$.

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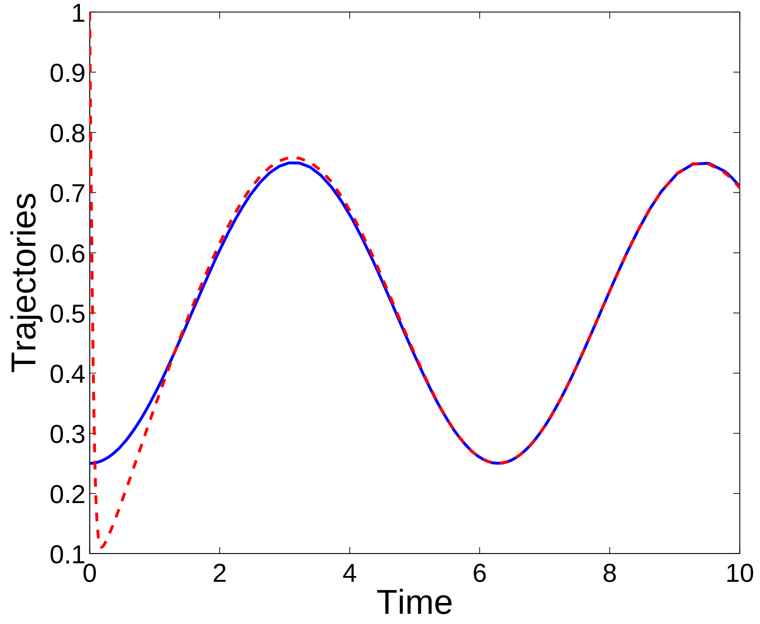
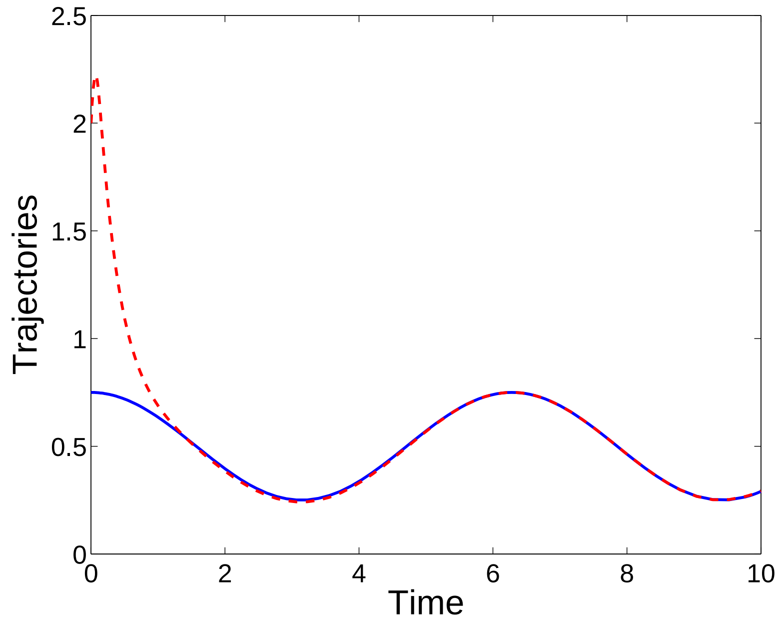



(a) $S(t)$ Tracking $S_r(t)$




(b) $x(t)$ Tracking $x_r(t)$

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- The stability is maintained when there are additional species that are being driven to extinction, or disturbances of small magnitude on the dilution rate and input nutrient concentration.
- Extensions to chemostats with multiple competing species, time delays, limited information about the current state, and measurement uncertainty would be desirable and are being studied.

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