Asymptotic Stabilization for Feedforward Systems with Delayed Feedbacks

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Fridman, Jankovic, Karafyllis, Krstic, Lin, Teel, ...

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Find γ_i's by building certain LKFs for $Y'(t) = \mathcal{G}(t, Y_t, 0)$.

Linear Feedforward Systems

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Consider the set of all systems having the feedforward form

$$\left\{ \begin{array}{rcl} \dot{x} & = & h_1(z) + h_2(z)v(t-\tau) \\ \dot{z} & = & f(z) + g(z)v(t-\tau) \; . \end{array} \right. \tag{3}$$

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The state space is $\mathbb{R}^n \times \mathbb{R}^p$. Linearizing (3) around period τ reference trajectories produces a system of the form

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where A, B, C, and D are C^1 matrix valued functions of period τ.

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We focus on (4), and cases where uncertainties δ are added to u.

Our Two Assumptions

Assumption 1. *The system*

$$\dot{\theta}(t) = A(t)\theta(t) \tag{5}$$

is UGAS. The matrices A, B, C, and D are C^1 and have period τ.

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$$\left\{ \begin{array}{rcl} \frac{\partial \psi_a}{\partial t}(t,m) & = & -\psi_a(t,m)A(t) \\ \psi_a(m,m) & = & \mathrm{I} \end{array} \right. \tag{6}$$

for all $t \in \mathbb{R}$ and $m \in \mathbb{R}$.

Our Two Assumptions

Lemma

Let Assumption 1 hold. Then the function $\mathrm{I} - \psi_a(\ell, \ell - \tau)$ *is invertible for all* $\ell \in \mathbb{R}$*. Also, the function* $q : \mathbb{R} \to \mathbb{R}^{n \times p}$ *defined by*

$$q(t) = -\int_{t-\tau}^{t} C(\ell)[\mathrm{I} - \psi_a(\ell, \ell - \tau)]^{-1}\psi_a(t, \ell)\mathrm{d}\ell \tag{7}$$

has period τ*, and* $\dot{q}(t) + q(t)A(t) + C(t) = 0$ *for all* $t \in \mathbb{R}$*.* □

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Assumption 2. *There exists a constant* $c > 0$ *such that the matrix* $R(t) = q(t)B(t) + D(t)$ *satisfies*

$$\int_{t-\tau}^{t} R(m)R(m)^{\top} \mathrm{d}m \ \geq \ c\mathrm{I} \tag{8}$$

for all $t \in \mathbb{R}$*. (That means* I *is the* $n \times n$ *identity matrix.)*

Main Result

Our coordinate change $\xi(t) = x(t) + q(t)z(t)$ gives

$$\begin{cases} \dot{\xi}(t) &= R(t)u(t-\tau) \\ \dot{z}(t) &= A(t)z(t) + B(t)u(t-\tau) \end{cases} \tag{9}$$

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Theorem

Let Assumptions 1 and 2 hold. Then for all constants $\tau > 0$ and $\epsilon \in (0, \frac{1}{1+4\tau||R||^2})$, the controller

$$u(t-\tau) = -\epsilon \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \tag{10}$$

renders (9) UGAS.

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Allowing additive uncertainties on the control gives

$$\begin{cases} \dot{\xi}(t) &= R(t)\big[u(t-\tau)+\delta(t)\big] \\ \dot{z}(t) &= A(t)z(t)+B(t)\big[u(t-\tau)+\delta(t)\big] \,. \end{cases} \tag{11}$$

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$$\overline{\delta} = \frac{c}{9k||R||(1+2\overline{u})^{1/2}}, \quad \text{where} \quad k = \frac{4\sqrt{2}}{3\epsilon}\left(\tau + \frac{1}{2c}||R||^6\tau^4\epsilon^2\right)$$
$$\text{and} \quad \overline{u} = \max\left\{\frac{1}{2} + \frac{\epsilon||R||^2\tau}{4\sqrt{2}}, \frac{\epsilon||R||^2\tau}{4\sqrt{2}}\left(1+2\epsilon||R||^2\tau\right)\right\} . \tag{12}$$

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Theorem

Under the preceding assumptions, (11) in closed loop with

$$u(t-\tau) = -\epsilon\frac{R(t-\tau)^\top\xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \quad (13)$$

is ISS with respect to the set of all disturbances δ bounded by $\overline{\delta}$.

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- It would be interesting to extend the analysis to

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  Nonlinear analogs involving PDEs would also be interesting.

What is a Lyapunov-Krasovskii Functional (LKF) ?

Definition: We call $V^\sharp$ an ISS-LKF for $Y'(t) = \mathcal{G}(t, Y_t, \delta(t))$ provided there exist functions $\gamma_i \in \mathcal{K}_\infty$ such that:

1. $\gamma_1(|\phi(0)|) \leq V^\sharp(t,\phi) \leq \gamma_2(|\phi|_{[-\tau,0]})$ for all $(t,\phi) \in [0,+\infty) \times \mathcal{C}([-\tau,0],\mathbb{R}^n)$ and
2. $\frac{d}{dt}\left[V^\sharp(t, Y_t)\right] \leq -\gamma_3(V^\sharp(t, Y_t)) + \gamma_4(|\delta(t)|)$ along all trajectories of the system

Example: The function $V(Y) = \frac{1}{2}|Y|^2$ is an ISS-LKF for $Y'(t) = -Y(t) + \frac{1}{4}Y(t) + \delta(t)$ for any $\mathcal{D}$. Fix $\tau > 0$.

$$V^\sharp(Y_t) = V(Y(t)) + \frac{1}{4}\int_{t-\tau}^{t} |Y(\ell)|^2 \mathrm{d}\ell + \frac{1}{8\tau}\int_{t-\tau}^{t} \left[\int_s^t |Y(r)|^2 \mathrm{d}r\right] \mathrm{d}s$$

is an ISS-LKF for $Y'(t) = -Y(t) + \frac{1}{4}Y(t-\tau) + \delta(t)$.

Main Result

Our coordinate change $\xi(t) = x(t) + q(t)z(t)$ gave

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where $R(t) = q(t)B(t) + D(t)$ and q is from the lemma.

Theorem

Let Assumptions 1 and 2 hold. Then for all constants $\tau > 0$ and $\epsilon \in (0, \frac{1}{1+4\tau||R||^2})$, the controller

$$u(t-\tau) = -\epsilon \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \qquad (16)$$

renders (15) UGAS.

Proof of Theorem

Show that the closed loop system (15) admits the LKF

$$V^\sharp(t,\xi_t,z(t)) \ = \ z^\top(t)P(t)z(t) + 21\beta_1 W_3(t,\xi_t), \ \text{ where}$$

$$W_3(t,\xi_t) = W_2(t,\xi_t) + k\left[(1+2U(\xi_t))^{3/2}-1\right],$$

$$W_2(t,\xi_t) = W_1(t,\xi_t) + \beta_0 \int_{t-\tau}^{t}\left|\frac{R(m)^\top\xi(m)}{\sqrt{1+|\xi(m)|^2}}\right|^2 \mathrm{d}m,$$

$$W_1(t,\xi_t) = \xi(t)^\top\left[\int_{t-\tau}^{t}\int_m^t R(\ell)R(\ell)^\top \mathrm{d}\ell\mathrm{d}m\right]\xi(t),$$

$$U(\xi_t) = \tfrac{1}{2}|\xi|^2 + \tfrac{1}{4\tau}\int_{t-2\tau}^{t}\int_m^t \frac{\epsilon|R(\ell)^\top\xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}}\,\mathrm{d}\ell\mathrm{d}m,$$

$$\beta_0 = \tfrac{1}{2c}||R||^6\tau^4\epsilon^2, \quad k = \tfrac{4\sqrt{2}}{3\epsilon}(\tau+\beta_0),$$

$$\beta_1 = \max\{v_1,v_2\}, \quad v_1 = \tfrac{2}{c}[4||P||^2\,||B||^2\,||R||^2+1],$$

$$\text{and } \ v_2 = \tfrac{16\sqrt{2}\tau}{3\epsilon k}(1+8\tau||P||^2\,||B||^2\,||R||^4).$$