STABILIZING A PERIODIC SOLUTION IN THE CHEMOSTAT: A CASE STUDY IN TRACKING

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1. General n Species Chemostat Model

$$\dot{S} = D(S_0 - S) - \sum_{i=1}^{n} \mu_i(S)x_i/\gamma_i; \qquad \dot{x}_i = x_i(\mu_i(S) - D) \tag{1}$$

- x_i: concentration of ith species
- S: concentration of limiting nutrient
- μ_i: per capita growth rate
- $\gamma_i \in (0,1)$: constant yield factor
- controls: dilution rate D and input nutrient concentration S_0.

The equations (1) are straightforwardly obtained from writing the mass-balance equations for the total amounts of the nutrient and each of the species, assuming the reactor content is well-mixed.

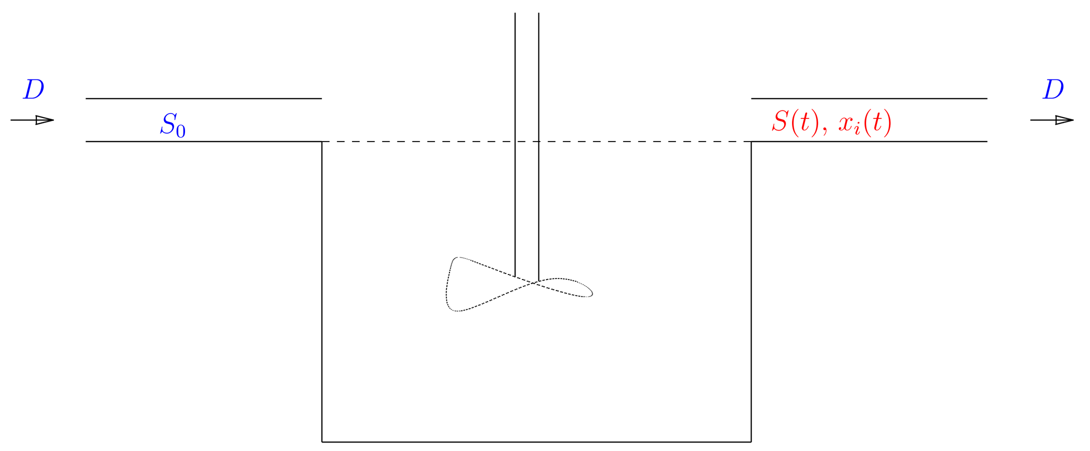


Figure 1: *Chemostat*

2. Review of Literature and Comparison with Our Work

Literature:

- Competitive Exclusion: When S_0 and D in (1) are constant, at most one species survives.
- This means (1) has a steady state with at most one nonzero species concentration, which attracts almost all solutions.
- This is at odds with observed coexistence behaviors in real ecological systems e.g. the "paradox of the plankton".
- Much of the literature designs time-varying and/or state dependent D and S_0 that force coexistence behaviors.

Our work:

- Instead of studying coexistence, we prove the stability of a prescribed periodic solution using a Lyapunov-type analysis.
- Lyapunov functions are useful for robustness analysis but have infrequently been used in chemostat research.
- Most Lyapunov results for chemostats use *nonstrict* Lyapunov functions in conjunction with LaSalle invariance and so do not lend themselves to robustness analysis.

3. One Species Model We Study

Taking S_0 to be constant and rescaling gives

$$\dot{S} = D(1-S) - \mu(S)x, \quad \dot{x} = x(\mu(S) - D) \tag{2}$$

evolving on $\mathcal{X} = (0,\infty)^2$. We assume a Monod growth rate

$$\mu(S) = \frac{mS}{a+S}, \quad m > 4a+1\,. \tag{3}$$

4. Main Tracking Result for (2)

Statement of Main Tracking Result: Given any componentwise positive trajectory $(S,x) : [0,\infty) \to \mathcal{X}$ for (2) and the dilution rate

$$D(t) = \frac{\sin(t)}{2+\cos(t)} + \frac{m(2-\cos(t))}{4a+2-\cos(t)} \tag{4}$$

and μ as in (3) with $m > 4a+1$, the corresponding deviation

$$(\tilde{S}(t), \tilde{x}(t)) := (S(t) - S_r(t), x(t) - x_r(t)) \tag{5}$$

of (S,x) from the reference trajectory

$$(S_r(t), x_r(t)) := \left(\frac{1}{2} - \frac{1}{4}\cos(t), \frac{1}{2} + \frac{1}{4}\cos(t)\right) \tag{6}$$

for (2) asymptotically approaches $(0,0)$ as $t \to +\infty$.

Our choice (6) is motivated by commonly observed oscillations in biological applications e.g. waste water treatment plants. (Similar results hold if we instead choose any $x_r(t)$ that admits a constant $\ell > 0$ such that $\max\{\ell, |\dot{x}_r(t)|\} \le x_r(t) \le \frac{3}{4}$ for all $t \ge 0$ and $S_r = 1 - x_r$, for suitable D.) The condition $m > 4a+1$ is used to get a positive uniform lower bound $\underline{D}$ on D and so can be relaxed to $m > \frac{2}{3}(4a+1)$.

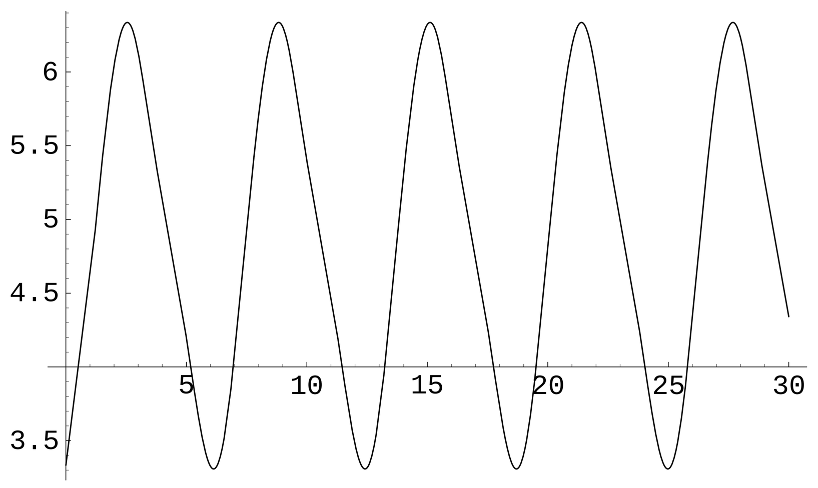


Figure 2: *Graph of Dilution Rate $D(t)$ with $m = 10$ and $a = \frac{1}{2}$ for the Chemostat From (4) Plotted Against Time t*

5. Outline of Proof of Main Tracking Result

First transform the error dynamics for (5) into

$$\begin{cases} \dot{\tilde{z}} = -D(t)\tilde{z}, \\ \dot{\tilde{\xi}} = \mu(z - e^{\xi}) - \mu(1 - e^{\xi_r(t)}), \end{cases} \tag{7}$$

where $\tilde{z} := z - 1$, $z = S + x$, $\tilde{\xi} := \xi - \xi_r$, $\xi := \ln(x)$, and $\xi_r := \ln(x_r)$. Next show that (7) admits the Lyapunov-like function

$$L_3(\tilde{z}, \tilde{\xi}) := e^{\tilde{\xi}} - 1 - \tilde{\xi} + \frac{4m}{a\underline{D}}\tilde{z}^2. \tag{8}$$

Along the trajectories of (7), we get

$$\dot{L}_3 \le -\frac{ma(e^{\tilde{\xi}}-1)^2}{16(a+2+\tilde{z}^2)(a+1)} - \frac{4m}{a}\tilde{z}^2\,. \tag{9}$$

The stability follows from a Barbalat's Lemma argument which in fact shows that $(\tilde{z}, \tilde{\xi}) \to 0$ exponentially.

6. Extension to Chemostats with Additional Species

Our stabilization result enjoys a number of highly desirable robustness properties. For example, consider the *augmented model*

$$\begin{cases} \dot{S} = D(t)(1-S) - \mu(S)x - \sum_{i=1}^{n} \nu_i(S)y_i\,, \\ \dot{x} = x(\mu(S) - D(t)), \quad \dot{y}_i = y_i(\nu_i(S) - D(t)), \quad i = 1,\ldots,n \end{cases} \tag{10}$$

where D is from (4), y_i is the concentration of the ith additional species, and ν_i is continuous and increasing and satisfies $\nu_i(0) = 0$ and $\nu_i(1) < \underline{D}$ for $i = 1,2,\ldots,n$.

Multi-species Result: The error between any componentwise positive solution $(S, x, y_1, y_2, \ldots, y_n)$ of (10) and

$$(S_r, x_r, 0, \ldots, 0) = \left(\frac{1}{2} - \frac{1}{4}\cos(t), \frac{1}{2} + \frac{1}{4}\cos(t), 0, \ldots, 0\right)$$

converges exponentially to the zero vector as $t \to +\infty$.

Proof (Sketch): Since $\nu_i(1) < \underline{D}$ for each i, the form of the dynamics for S along our componentwise positive trajectories implies that there exist $\varepsilon > 0$ and $T \ge 0$ such that (i) $S(t) \le 1 + \varepsilon$ for all $t \ge T$ and (ii) $\nu_i(1+\varepsilon) < \underline{D}$ for all $i = 1,2,\ldots,n$. We next choose

$$\delta := \underline{D} - \max_{i=1,\ldots,n} \nu_i(1+\varepsilon) > 0.$$

The result now follows using the Lyapunov-like function

$$L_4(\tilde{z}, \tilde{\xi}, y_1, \ldots, y_n) = L_3(\tilde{z}, \tilde{\xi}) + A\sum_{i=1}^{n} y_i^2, \quad \text{where} \quad A := \frac{16mn^2}{a\delta}. \tag{11}$$

using Barbalat's Lemma. Along the error dynamics trajectories,

$$\dot{L}_4 \le -\frac{ma(e^{\tilde{\xi}}-1)^2}{16(a+1)(a+2+\tilde{z}^2)} - \frac{3m}{a}\tilde{z}^2 - \frac{16mn^2}{a}\sum_{i=1}^{n} y_i^2\,.$$

7. Robustness Result for Actuator Errors

Our robustness is maintained in the (integral) input-to-state stability sense if there are suitably small disturbances on the controllers i.e.

$$\begin{cases} \dot{S}(t) = [D(t) + u_1(t)](1 + u_2(t) - S(t)) - \mu(S(t))x(t)\,, \\ \dot{x}(t) = x(t)[\mu(S(t)) - D(t) - u_1(t)]\,. \end{cases} \tag{12}$$

This means that if $|u|$ stays below a computable bound, then there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the transformed error vector

$$\begin{aligned} &y(t; t_o, y_o, \alpha) := \\ &(S(t; t_0, (S,x)(0), \alpha) - S_r(t), \ln(x(t; t_0, (S,x)(0), \alpha)) - \ln(x_r(t))) \end{aligned} \tag{13}$$

for all disturbances $u = (u_1, u_2) = \alpha$ and initial conditions satisfies

$$|y(t; t_o, y_o, \alpha)| \le \beta(|y_o|, t - t_o) + \gamma(|\alpha|_\infty)\,. \tag{ISS}$$

Under the less stringent condition $|u| < \frac{1}{2}\min\{1, \underline{D}\}$, there are functions $\delta_i \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ so that the trajectories everywhere satisfy

$$\delta_1(|y(t; t_o, y_o, \alpha)|) \le \beta(|y_o|, t - t_o) + \int_{t_o}^{t+t_o} \delta_2(|\alpha(r)|)dr\,. \tag{iISS}$$

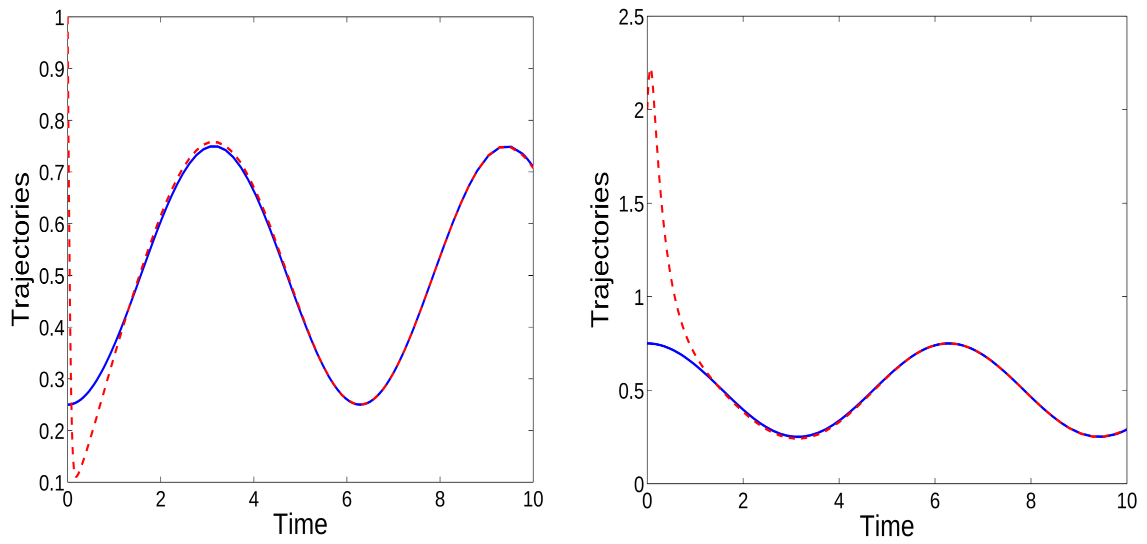
This is significant e.g. because D is proportional to the speed of the pump that supplies the nutrient which is prone to small errors [1]. No ISS estimate is possible for (13) without input constraints.

8. Simulation for Perturbed Chemostat (12)

In Figures (a)-(b), we simulated the perturbed dynamics (12) with

- $D(t)$ from (4) with $m = 10$, $a = \frac{1}{2}$;
- $u_1(t) = 0.5e^{-t}$, $u_2(t) \equiv 0$; and
- $t_o = 0$, $x(0) = 2$, $S(0) = 1$.

Our results imply that the convergence of $(S(t), x(t))$ to $(S_r(t), x_r(t))$ is iISS to disturbances u that are valued in $[-\bar{u}, \bar{u}]^2$ for any positive constant $\bar{u} < \min\{1, \underline{D}\} = 1$. Estimate (iISS) holds with $\delta_2(r) = Cr$ for some constant $C > 0$. Our simulation illustrates how the state trajectory $(S(t), x(t))$ tracks the reference trajectory $(S_r(t), x_r(t))$ even in the presence of small disturbances and so validates our findings.



(a) $S(t)$ Tracking $S_r(t)$ (b) $x(t)$ Tracking $x_r(t)$

9. Conclusions and Future Work

- Chemostats are a useful framework for modeling species competing for nutrients. They provide the foundation for much current research in bio-engineering, ecology, and population biology.
- For the case of one species competing for one nutrient and a suitable time-varying dilution rate, we proved the stability of an appropriate reference trajectory using Lyapunov methods.
- The stability is maintained when there are additional species that are being driven to extinction, or disturbances of small magnitude on the dilution rate and input nutrient concentration.
- Extensions to chemostats with multiple competing species and more general disturbances would be desirable.

10. Acknowledgements

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11. Reference

[1] F. Mazenc, M. Malisoff, and P. De Leenheer, "On the stability of periodic solutions in the perturbed chemostat," *Mathematical Biosciences and Engineering*, to be published.