STABILIZING A PERIODIC SOLUTION IN THE CHEMOSTAT:
A CASE STUDY IN TRACKING

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1. General Results

Three species model study

Taking $s_j$ to be constant and rescaling gives

$$S = D(S_0 - S) - \sum_{i=1}^{n} \mu_i(S)\frac{x_i}{S}$$  \hspace{1cm} (1)

for $x = (x_1, x_2, x_3)$. We assume a Monod growth rate

$$\mu(S) = \frac{K}{K + S}$$  \hspace{1cm} (2)

evolving on $x = (0, \infty)^3$. The system (1) is written

$$\dot{x} = x_j(\mu(S) - D)$$  \hspace{1cm} (3)

2. Review of Literature and Comparison with Our Work

Literature:

• Competitive Exclusion: When $s_j$ and $D$ in (1) are constant, at most one species survives.

• This means (1) has a steady state with at most one nonzero species concentration, which attracts almost all solutions.

• Much of the literature designs time-varying and state dependent $s_j$ so that force coexistence behaviors.

Our work:

• Instead of studying coexistence, we prove the stability of a pre-periodic solution using a Lyapunov-type analysis.

• Lyapunov functions are useful for robustness analysis but have infrequently been used in chemostat research.

• Most Lyapunov results for chemostats use nonstrict Lyapunov functions in conjunction with LaSalle’s invariance which do not tend themselves to robustness analysis.

3. One Species Model We Study

Taking $s_j$ to be constant and rescaling gives

$$\dot{x} = \frac{\mu(S) - D}{S}x$$  \hspace{1cm} (4)

4. Main Tracking Result for (2)

Statement of Main Tracking Result: Given any componentwise positive trajectory $(S, x) : [0, \infty) \to \mathbb{R}_+$ for (2) and the dilution rate

$$\dot{x}_i = \frac{\mu(S) - D}{S}x_i$$

1. concentration of $i$th species

$S$: concentration of limiting nutrient

$\mu_i$: per capita growth rate

$\gamma_i$: (0; 1]; constant yield factor

$D$: input nutrient concentration $S_0$.

The equations (1) are straightforwardly obtained from writing the mass-balance equations for the total amounts of the nutrient and each of the species, assuming the reactor content is well-mixed.

5. Outline of Proof of Main Tracking Result

First transform the error dynamics for (5) into

$$\begin{align*}
\dot{x} &= -x - D \\
\dot{z} &= \mu(S) - D - (M - D)z
\end{align*}$$

where $\dot{x} = x - 1, z = S - x, \dot{z} = \mu(S) - D, \mu(S) = \mu(S) - D$. Next show that (7) admits the Lyapunov function

$$L(z) = \frac{1}{2}\left(\frac{\mu(S) - D}{S} - 1\right) + \frac{1}{2}z^2$$

Along the trajectories of (7), we get

$$L(z) \leq \frac{1}{2}\left(\frac{\mu(S) - D}{S} - 1\right) \leq \frac{1}{2}z^2$$

The stability follows from a Barbalat’s Lemma argument which is fact that shows $(1, \dot{1}) \to 0$ exponentially.

6. Extension to Chemostats with Additional Species

Our stabilization result enjoys a number of highly desirable robustness properties. For example, consider the augmented model

$$\begin{align*}
\dot{S} &= D(S_0 - S) - \sum_{i=1}^{n} \mu_i(S)\frac{x_i}{S} \\
\dot{x}_i &= x_i(\mu(S) - D) - D
\end{align*}$$

and $\mu_i$ as in (3) with $m > \alpha + 1$, the corresponding deviation

$$(\tilde{S}(t), \tilde{x}_i(t)) = (S(t) - S_0, (S(t) - S_0) \cdot x_i)$$

of $(S, x)$ from the reference trajectory

$$(\bar{S}_0(t), \bar{x}_i(t)) = \left(\frac{1}{2}\cos(t) + 1, \frac{1}{2}\cos(t), 0, \ldots, 0\right)$$

for (2) asymptotically approaches $(0, 0)$ as $t \to \infty$.

Our choice (6) is motivated by commonly observed oscillations in biological applications e.g. waste water treatment plants. (Similar results hold if instead we choose any $\gamma_i$ that admits a constant $\Gamma > 0$ such that $\mu_i(S) < \Gamma$ for all $S > s_j$ and $S < \Gamma$ for suitable $D_j$). The condition $m > \alpha + 1$ is used to get a positive uniform bound $\Gamma$ on $S$ and so can be relaxed to $m > \alpha + 1$.

7. Robustness Result for Actuator Errors

Our robustness is maintained in the (integral) input-to-state stability sense if there are suitably small disturbances on the controllers i.e.

$$\begin{align*}
\dot{S} &= D(S_0 - S) - \sum_{i=1}^{n} \mu_i(S)\frac{x_i}{S} \\
\dot{x}_i &= x_i(\mu(S) - D) - D + D_i
\end{align*}$$

This means if $|\delta|$ stays below a computable bound, then there are $\beta \in \mathcal{C}$ and $\gamma \in \mathcal{C}$ such that the transformed error vector $\tilde{z}(t) = (S(t) - S_0, (S(t) - S_0) \cdot x_i)$ and $\tilde{S}(t) = \bar{S}_0(t)$.

Under the less stringent condition $|\delta| < m < 1$ in (10), there are functions $\alpha \in \mathcal{C}$ and $\beta, \gamma \in \mathcal{C}$ such that the error $\epsilon(t) \to 0$ exponentially.

$|\epsilon(t)| \leq |\epsilon(0)|e^{-\alpha t}$

8. Simulation for Perturbed Chemostat in Tracking

In Figures (a)-(b), we simulated the perturbed dynamics (12) with

$$\begin{align*}
\dot{S} &= D(S_0 - S) - \sum_{i=1}^{n} \mu_i(S)\frac{x_i}{S} + D_i \\
\dot{x}_i &= x_i(\mu(S) - D) - D + D_i
\end{align*}$$

$|\epsilon(t)| \leq |\epsilon(0)|e^{-\alpha t}$

Our results imply that the convergence of $(\tilde{S}_0(t), \tilde{x}_i(t))$ to $(\bar{S}_0(t), \bar{x}_i(t))$ is ISSS to disturbances $\nu$ that are valued in $(\epsilon, \epsilon^2)$ for any positive constant $\epsilon$. (Similar results hold in $|\epsilon(0)| = 1$. Our simulation illustrates how the state trajectory $(\tilde{S}_0(t), \tilde{x}_i(t))$ tracks the reference trajectory $(\bar{S}_0(t), \bar{x}_i(t))$ even in the presence of small disturbances and so validates our findings.

9. Conclusions and Future Work

• Chemostats are useful framework for modeling species competing for nutrients. They provide the foundation for much current research in bio-engineering, ecology, and population biology.

• For the case of one species competing for one nutrient and a suitable time-varying dilution rate, we proved the stability of an appropriate reference trajectory using Lyapunov methods.

• The stability is maintained when there are additional species that are being driven to extinction, or disturbances of small magnitude on the rate and input nutrient concentration.

• Extensions to chemostats with multiple competing species and more general perturbations remain open.

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11. Reference