Bounded Backstepping Control Designs for Time-Varying Systems

Frederic Mazenc

Michael Malisoff

Jerome Weston
Background on Backstepping

Motivation: Need controls to give global asymptotic stability for systems of nonlinear ODEs with robustness under uncertainties.

Strategy: Recursively build stabilizing feedbacks for subsystems into globally asymptotically stabilizing controls for full system.

Structure: Many engineering models admit cascade forms that lend themselves to backstepping after changes of variables.

Challenges: Unknown current states give input delays, need for explicit control formulas, input constraints, uncertainties,...

ZP. Jiang, M. Krstic, F. Mazenc, H. Nijmeijer, J. Tsinias,..
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Our Systems

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\begin{aligned}
\dot{x}(t) &= \mathcal{F}(t, x(t), z_1(t)) \\
\dot{z}_i(t) &= z_{i+1}(t), \quad i \in \{1, 2, \ldots, k - 1\} \\
\dot{z}_k(t) &= u(t) + h(t, x(t), z(t))
\end{aligned}
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Assumption 1: \(\mathcal{F}\) and \(h\) are uniformly bounded in \(t\) and uniformly locally Lipschitz in \((x, z)\).
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\(|\mathcal{F}(t, x, z_1)| + |h(t, x, z)| \leq \alpha(||(x, z)||)

After changes of variables and feedback for any constant \(q > 0\):

\[
\dot{x}(t) = \mathcal{F}(t, x(t), y_k(t)) \\
\dot{y}_i(t) = -q y_i(t) + y_{i-1}(t), \quad i \in \{2, \ldots, k\} \\
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\]

(2)
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\] (2)

We can also allow \(\mathcal{F}(t, x, z)\), and actuator errors added to \(u\).
Converging-Input-Converging-State Assumption

Assumption 2: There are a locally Lipschitz bounded $\omega$, and constants $q > 0$, $\tau > 0$, and $T \geq 0$, such that for all continuous $\delta$’s that exponentially converge to zero, all solutions $x(t)$ of

$$\dot{x}(t) = \mathcal{F}(t, x(t), G(x_t) + \delta(t))$$

where

$$(CICS)\quad G(x_t) = \int_{t-\tau}^{t} \cdots \int_{m_{k-1}-\tau}^{m_{k-1}} \cdots \int_{m_1-\tau}^{m_1} e^{q(m_0-t)} \omega(x(m_0 - T)) dm_0 \cdots dm_{k-1}$$

satisfy $\lim_{t \to \infty} x(t) = 0$. Also, $\omega(0) = 0$. 

Sufficient conditions using Lyapunov functions...

$T$ will be smallest delay in $\nu$.

Optimization problem related to this...

Ex: $\dot{x}(t) = Ax(t) + \delta(t)$ with $A$ Hurwitz. Many nonlinear cases.
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Our Theorem

Under Assumptions 1-2, all solutions of

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\dot{y}_i(t) &= -qy_i(t) + y_{i-1}(t), \quad i \in \{2, \ldots, k\} \\
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\end{align*}
\]

in closed loop with the bounded control

\[
v(t) = \sum_{j=0}^{k} \frac{k!(-1)^j e^{-jq\tau}}{j!(k-j)!} \omega(x(t - j\tau - T)),
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Extensions: If \( \dot{x}(t) = \mathcal{F}(t, x(t), G(x_t) + \delta(t)) \) is ISS, then we can prove ISS of (3) with respect to additive uncertainty on \( v \).
Ex: Khalil’s 2002 Nonlinear Systems, 3rd Edition

\[
\dot{x} = x^2 - x^3 + z_1, \quad \dot{z}_1 = z_2, \quad \dot{z}_2 = u. \quad (5)
\]
We compare Khalil’s control $u_K$ with our control $u_J$.

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\[
u_K(x, z) = -\frac{\partial V_0}{\partial z_1}(x, z_1) + \frac{\partial \phi}{\partial z_1}(x, z_1)z_2 - z_2
\]
\[
+ \frac{\partial \phi}{\partial x}(x, z_1)(x^2 - x^3 + z_1) + \phi(x, z_1),
\]

where \(V_0(x, z_1) = \frac{1}{2}x^2 + \frac{1}{2}(z_1 + x + x^2)^2\) and \(\phi(x, z_1) = -2x - (1 + 2x)(x^2 - x^3 + z_1) - z_1 - x^2\),
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\[
u_J(t) = \frac{q^2}{(1 - e^{-q\tau})^2} \left\{ \omega(x(t - T)) - 2e^{-q\tau}\omega(x(t - \tau - T)) + e^{-2q\tau}\omega(x(t - 2\tau - T)) \right\} - 2qz_2(t) - q^2z_1(t),
\]

where \( \omega(x) = -\sin \left( \frac{\pi x}{2} \right) 1_{[-2,2]}(x), \ T = .055, \ q = 1/\tau, \ \tau = .001. \)
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Problem: Given a function $\mathcal{F}$ that satisfies Assumption 1, find the largest $T$ such that: There exist a function $\omega$ and constants $q > 0$ and $\tau > 0$ such that Assumption 2 is satisfied.
Conclusions

Backstepping applies to many systems e.g., normal forms.
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Converging-input-converging-state conditions can be helpful.
Our Lyapunov sufficient conditions have delay bounds.
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Abstract

We will present fundamental results pertaining to ordinary differential equations, discrete-time systems and nonlinear control theory. In particular, we will review the notion of Lyapunov function, the LaSalle Invariance Principle, the Jurdjevic-Quinn’s theorem and the techniques called backstepping and forwarding. We will perform construction of strict Lyapunov functions. We will study the notion of positive systems. We will study several applied problems (chemostats, PVTOL, cart-pendulum system).

The module is partially based on the research monograph: M. Malisoff, F. Mazenc, Constructions of Strict Lyapunov Functions, Springer-Verlag, serie : Communications and Control Engineering, 2009

Outline:

1) **Introduction to dynamical systems:** Ordinary Differential Equations, discrete-time systems, time-varying systems, basic notions (existence and uniqueness of solutions, finite escape time phenomenon). Notions of stability (local, global, basin of attraction), notion of input-to-state stability.

2) **Fundamental results. Linear systems:** stability analysis, linearization. Hartman-Grobman Theorem, Two dimentional systems : Poincaré–Bendixson theorem. Dulac’s criterion, properties of $\omega$-limit sets.


4) **Control design:** Lyapunov design, Jurdjevic-Quinn theorem, classical backstepping, bounded backstepping, backstepping for time-varying systems, strabilization and tracking though forwarding, Sontag’s formula.

5) **Positive systems:** Cooperative nonlinear systems, linear positive systems, linear Lyapunov function. Notion of interval observer.
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