On the Bellman equation for control problems with exit times and unbounded cost functionals [1]

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1 Introduction

This paper is devoted to the study of Hamilton-Jacobi-Bellman equations (HJB's) for a large class of unbounded optimal control problems for fully nonlinear systems having the form

$$\begin{cases} y'(t) \;=\; f(\,y(t),\alpha(t)\,), \quad t \geq 0, \quad \alpha(t) \in A \\ y(0) \;=\; x \end{cases} . \tag{1}$$

Our hypotheses will be such that (1) has a unique solution trajectory, defined on $[0,\infty)$, for each input. The optimal control problems are of the form

$$\text{Minimize} \;\; J(\,x, \alpha, t_x(\alpha)\,) \;\; \text{over} \;\; \alpha \in \mathcal{A}^f(x), \tag{2}$$

where $y_x(\cdot,\alpha)$ is the solution of (1) for the measurable input $\alpha : [0,\infty) \to A$, $t_x(\alpha)$ is the first time this solution reaches some target $\mathcal{T}$, $\mathcal{A}^f(x)$ is the set of measurable inputs for which $t_x(\alpha) < \infty$, and the infimand $J : \mathbb{R}^N \times \mathcal{A} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$J(x,\alpha,t) \;:=\; \int_0^t \ell(\,y_x(s,\alpha),\alpha(s)\,)\,ds \;+\; g\,(y_x(t,\alpha))\,,$$

where $\mathcal{A}$ is the set of measurable functions $[0,\infty) \to A$.

The value function of (2) will be denoted by v, and $\mathcal{R}$ is the set of all points that can be brought to $\mathcal{T}$ in finite time using the evolution (1) and some input α in $\mathcal{A}$. Thus, $v(x) = \inf_{\alpha \in \mathcal{A}^f(x)} J(x,\alpha,t_x(\alpha))$ for x in $\mathcal{R}$ and is $+\infty$ elsewhere. The class of problems we consider includes the Fuller Problem (FP) (cf. [6], [9], and [11]), as well as problems in which the control set A is noncompact, problems for which nonconstant trajectories can give zero running costs, and problems where ℓ can be negative. The HJB for (2) is

$$\sup_{a \in A} \{\, -f(x,a) \cdot Du(x) \,-\, \ell(x,a)\,\} \;=\; 0, \tag{3}$$

which we wish to satisfy on $\Omega \setminus \mathcal{T}$, where Ω is a suitable open subset of $\mathbb{R}^N$ containing $\mathcal{T}$. We will study (3) in the framework of the theory of viscosity solutions and relaxed controls (cf. [1]).

We characterize the value functions of these problems as the unique viscosity solutions of the associated HJB's among continuous functions with suitable boundary and growth conditions. As a consequence, we show that the FP value function is the unique radially unbounded viscosity solution of the corresponding HJB among functions which are zero at the origin. Value function characterizations of this kind have been studied and applied by many authors for a large number of stochastic and deterministic optimal control problems and for differential games, including problems for which the value function is discontinuous. Recent accounts of work in these areas may be found in [1], [3], and in the many references therein.

However, the FP is not covered by these results, since its cost functional ℓ vanishes at points outside $\mathcal{T}$. For example, see [1], where the main comparison results for exit time problems require $\ell \geq 1$, and [8], where this requirement is relaxed to requiring that for each $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that $\ell(x,a) \geq C_\varepsilon$ for all $a \in A$ and $x \in \mathbb{R}^n \setminus B(\mathcal{T},\varepsilon)$. In fact, one can find HJB's for optimal control problems with exit times that have several radially unbounded viscosity solutions when this lower bound requirement is violated. For example, consider the system $y'(t) = u(t)$, where $y(t) \in \mathbb{R}$ and $u(t) \in [-1,+1]$. Choose the running cost functional $\ell(x,a) = (x+2)^2\,(x-2)^2\,x^2\,(x+1)^2\,(x-1)^2$, and use the final cost $g \equiv 0$. Let v_1 and v_2 be the value function for the associated problem (2) with the targets $\mathcal{T}_1 = \{0\}$ and $\mathcal{T}_2 = \{0,2,-2\}$, respectively. One can easily check that v_1 and v_2 are both solutions of the associated HJB (3) with $\mathcal{T} := \mathcal{T}_1$ and that with this choice of $\mathcal{T}$, the problem satisfies all the hypotheses of the well-known theorems which characterize value functions of exit time control problems as the unique radially unbounded viscosity solutions of (3) which are zero on $\mathcal{T}$ save for the fact that the positive lower bound requirement on ℓ is not satisfied. One also checks that the ingredient missing from this problem is a property

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we will call 'strong compatibility'. This property will guarantee that there are no spurious solutions for the HJB of the FP.

The problems we study also satisfy an analog of the La Salle Invariance Principle. This condition will be called 'strong transience'. Roughly stated, the condition says that each trajectory of (1) eventually leaves any subset $\mathcal{N}$ of the state space outside the target in which the running cost is null along some trajectory. Our work is part of a larger research program which generalizes uniqueness results of the theory of viscosity solutions to versions that cover well-known optimal control problems whose dynamics do not necessarily admit unique solutions or whose running cost functionals violate the usual boundedness requirements. A continuation by the author appears elsewhere in this volume (cf. [5]).

2 Assumptions and Definitions

Let us make the following assumptions:

- (A_0) A is a nonempty topological space
- (A_1) $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ is continuous and bounded on $B_R(0) \times A$ for all $R > 0$, and there are $L > 0$ and a modulus ω_f such that for all $a \in A$, the following conditions hold:
  - (a) For all $R > 0$, and all $x, y \in B_R(0)$, $|| f(y,a) - f(x,a) || \leq \omega_f(|| x - y ||, R)$.
  - (b) $(f(x,a) - f(y,a)) \cdot (x - y) \leq L || x - y ||^2$ for all $x, y \in \mathbb{R}^N$.
- (A_2) $\mathcal{T} \subseteq \mathbb{R}^N$ is closed, $\mathcal{T} \neq \emptyset$, $g \in C(\mathcal{T})$, and g is bounded below.
- (A_3) $\ell : \mathbb{R}^N \times A \to \mathbb{R}$ is continuous and bounded below, and there is a modulus ω_ℓ such that $|\ell(y,a) - \ell(x,a)| \leq \omega_\ell(|| x - y ||)$ for all $a \in A$ and $x, y \in \mathbb{R}^N$.

In (A_1), a modulus is a function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $R > 0$, $\omega(\cdot, R)$ is continuous and nondecreasing and $\omega(0, R) = 0$; and ω_ℓ is a modulus of the same kind which is constant in its second variable. Also, $B_r(x) := \{p \in \mathbb{R}^N : ||x - p|| < r\}$ for all $r > 0$ and $x \in \mathbb{R}^N$. Condition (A_1) will guarantee uniqueness of solutions (cf. [1]). When A is a compact metric space, we view our controls $\alpha \in \mathcal{A}$ as members of the larger class $\mathcal{A}^r$ of relaxed controls (cf. [1]). Define $\ell^r : \mathbb{R}^N \times A^r \to \mathbb{R}$ and $f^r : \mathbb{R}^N \times A^r \to \mathbb{R}^N$ by $\ell^r(x,m) := \int_A \ell(x,a) \; dm(a)$ and $f^r(x,m) := \int_A f(x,a) \; dm(a)$, where A^r is the set of Radon probability measures on A. These relaxations satisfy analogs of (A_1) and (A_3), so we can define $y_x^r(\cdot, \alpha)$ to be the unique solution of $y'(s) = f^r(y(s), \alpha(s))$ starting at x for each $\alpha \in \mathcal{A}^r$.

Define $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$H(x,p) \; = \; \sup_{a \in A} \{ -f(x,a) \cdot p \; - \; \ell(x,a) \} \tag{4}$$

The set of $x \in \mathbb{R}^N$ such that $\int_0^t \ell^r(y_x^r(s,\alpha), \alpha(s)) ds > 0$ for all $t \in (0, \infty]$ and $\alpha \in \mathcal{A}^r$ will be denoted by P. We study cases where any $x \in \mathcal{R}$ is 'eventually' brought to P. More precisely, we have the following:

Definition 2.1 Let $S \subset \mathbb{R}^N$ be open, let A be a compact metric space, let $w : S \to \mathbb{R}$, and assume (A_o)-(A_3) are satisfied. We call (2) **strongly transient** with respect to $\mathcal{T}$, w, and S and write $\mathrm{S}\mathcal{T}(w, S)$ if the following conditions hold:

1. For each $x \in S \setminus [P \cup \mathcal{T}]$, there exist a bounded open set $B \subseteq S$ containing x so that $\partial B \subseteq S$ and a positive number L strictly less than
$$\inf_{\alpha \in \mathcal{A}} \{t > 0 : \mathrm{dist}(y_x(t,\alpha), \partial B) \leq \mathrm{dist}(x, \partial B)/2\}$$
such that, for all $\alpha \in \mathcal{A}$, $y_x(L, \alpha) \in P \cap S$ and $\int_0^L \ell(y_x(s,\alpha), \alpha(s)) \, ds \geq 0$.
2. For each $x \in S$ and $y \in \partial S$, there is an $\varepsilon > 0$ such that if $p \in S$ and $|| p - y || < \varepsilon$, then $w(x) < w(p)$.
3. If $x \in \mathbb{R}^N$, $\alpha \in \mathcal{A}^f(x)$, and
$$\{t \geq 0 : \; t \leq t_x(\alpha) \text{ and } y_x(t,\alpha) \notin S\} \neq \emptyset,$$
and if $\lambda = \sup\{t \geq 0 : t \leq t_x(\alpha) \;\&\; y_x(t,\alpha) \notin S\}$, then $\int_0^\lambda \ell(y_x(s,\alpha), \alpha(s)) \, ds \geq 0$.

When $S = \mathcal{R} = \mathbb{R}^N$, strong transience reduces to the much simpler Condition 1 on exits from $[P \cup \mathcal{T}]^c$. In that case, we write $\mathrm{S}\mathcal{T}$ instead of $\mathrm{S}\mathcal{T}(w, \mathbb{R}^N)$. We also need the following generalization of $\mathrm{STC}\mathcal{T}$ (cf. [7]):

Definition 2.2 Let $\mathcal{O} \subseteq \mathbb{R}^N$ and assume (A_0)-(A_3) are satisfied. We say that $\mathcal{O}$ satisfies the **strong small time control condition** with respect to $\mathcal{T}$ and write $\mathrm{SSTC}(\mathcal{O}, \mathcal{T})$ if there is an increasing sequence of bounded open sets $\{\Omega_j\}$ such that $\mathcal{O} = \cup_{j=1}^\infty \Omega_j$ and such that if $T_j : \Omega_j \setminus \mathcal{T} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$T_j(x) \; := \; \inf_\alpha \{ \, t : y_x(t,\alpha) \; \in \; \partial(\Omega_j \setminus \mathcal{T}\,) \}$$

for $j = 1, 2, \ldots$ and $\Omega_j \setminus \mathcal{T} \ni x \to x_0 \in \partial(\Omega_j \setminus \mathcal{T})$, then $T_j(x) \to 0$. When these conditions hold, $\{\Omega_j\}$ is called the **associated controllability sequence**.

The limit condition in Definition 2.2 can be replaced by $\mathrm{STC}(\Omega_j^c) \wedge \mathrm{STC}\mathcal{T}$ for $j = 1, 2, \ldots$ (where $\mathrm{STC}\mathcal{U}$ is the condition that $\mathcal{U}$ i s interior to the set of points that can be brought to $\mathcal{U}$ in time $< \varepsilon$ for each $\varepsilon > 0$).

Definition 2.3 Assume (2) satisfies (A_o)-(A_3) and SSTC($\mathcal{O}$, $\mathcal{T}$). Let $\{\Omega_j\}$ be the associated controllability sequence, and let $w : \mathcal{O} \to \mathbb{R}$ and $\omega_o \in \mathbb{R} \cup \{+\infty\}$. We say w is **$(\mathcal{O}, \mathcal{T}, \omega_o)$-compatible** if $\partial(\Omega_j) \setminus \mathcal{T} \subseteq \mathcal{O}$ for all j and $\lim_{j\to\infty} \min\{w(p) : \ p \in \partial(\Omega_j) \setminus \mathcal{T}\} = \omega_o$.

3 Main Lemmas

Assume $\Omega \subseteq \mathbb{R}^N$ is open, $\Omega \subseteq S$, and $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $w : S \to \mathbb{R}$ are continuous. Call w a **viscosity solution** of $F(x, Dw(x)) = 0$ on Ω if the following conditions are satisfied:

(i) If $\gamma : \Omega \to \mathbb{R}$ is C^1 and x_o is a local minimum of $w - \gamma$, then $F(x_o, D\gamma(x_o)) \geq 0$.

(ii) If $\lambda : \Omega \to \mathbb{R}$ is C^1 and x_1 is a local maximum of $w - \lambda$, then $F(x_1, D\lambda(x_1)) \leq 0$.

When condition (i) (resp., (ii)) holds, we say that w is a **(viscosity) supersolution (**resp.**, subsolution)** of $F(x, Dw(x)) = 0$ on Ω. Recall the following results from [1], [4], and [10]:

Lemma 3.1 Assume (A_0)-(A_3) and define H by (4). Assume that $u \in C(\bar{\Omega})$ is a viscosity subsolution of $H(x, Du(x)) = 0$ on Ω, where $\Omega \subset \mathbb{R}^N$ is bounded and open. If [1] $\tau_x(\alpha) = \inf\{\, t \geq 0 : \ y_x(t, \alpha) \in \partial\Omega\}$ for each $\alpha \in \mathcal{A}$ and $x \in \Omega$, then, for all $\alpha \in \mathcal{A}$ and $x \in \Omega$,

$$u(x) \leq \int_0^t \ell(y_x(s, \alpha), \alpha(s))\, ds \ + \ u(y_x(t, \alpha))$$

for $0 \leq t < \tau_x(\alpha)$.

Lemma 3.2 Let the assumptions (A_0)-(A_3) be satisfied, let $B \subset \mathbb{R}^N$ be bounded and open, and assume $w \in C(\bar{B})$ is a viscosity supersolution of (3) on B. Set $T_\delta(p) := \inf_{\alpha\in\mathcal{A}}\{\, t : \ \text{dist}\,(y_p(t, \alpha), \ \partial B) \leq \delta\,\}$ for each $p \in B$ and $\delta > 0$. Then for any $\delta \in (0, \text{dist}\,(p, \partial B)/2)$,

$$w(p) \geq \inf_{\alpha\in\mathcal{A}} \left\{ \int_0^t \ell(y_p(s, \alpha), \alpha(s))\, ds + w(y_p(t, \alpha)) \right\} \quad (5)$$

for all $t \in (0, T_\delta(p))$ and $p \in B$.

Lemma 3.3 Let A be a compact metric space, let $\{\alpha_n\}$ be a sequence in $\mathcal{A}^r$, and let $c > 0$. Assume $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ satisfies (A_1) and is uniformly Lipschitz in A. [2] There is a subsequence of $\{\alpha_n\}$ (which we do not relabel) and an $\alpha \in \mathcal{A}^r$ such that $\alpha_n \to \alpha$ weak-star on $[0, c]$ and such that $y^r_{x_n}(\cdot, \alpha_n) \to y^r_x(\cdot, \alpha)$ uniformly on $[0, c]$ whenever $x_n \to x$ in $\mathbb{R}^N$.

[1] The glb of an empty set of real numbers is $+\infty$.

[2] This means there is a constant $L > 0$ such that $||f(x, a) - f(z, a)|| \leq L||x - z||$ for all $a \in A$ and $x, z \in \mathbb{R}^N$.

4 Main Results

Theorem 1 *Let (2) satisfy (A_0)-(A_3), let $\Omega \subseteq \mathbb{R}^N$ be an open set containing $\mathcal{T}$, let $\omega_o \in \mathbb{R} \cup \{+\infty\}$, let $w \in C(\Omega)$ be a viscosity solution of*

$$\begin{cases} H(x, Dw(x)) \ = \ 0, & x \in \Omega \setminus \mathcal{T} \\ w(x) \ = \ g(x), & x \in \mathcal{T} \end{cases} \quad (6)$$

which is bounded below, and let $w(x) < \omega_0$ on Ω.[3] *Assume either* (i) *there are positive constants m and M such that $m \leq \ell \leq M$ on $\mathbb{R}^N \times A$, and $\lim_{x\to x_0} w(x) = \omega_o$ for all $x_0 \in \partial\Omega$; or* (ii) *A is a compact metric space, f is uniformly Lipschitz in A, condition* $\mathrm{S}\mathcal{T}(w, \Omega)$ *holds, and w is $(\Omega \cap P, \ \mathcal{T}, \omega_o)$-compatible. Then $w = v$ on Ω.*

Theorem 2 *Assume (2) satisfies (A_0)-(A_3), $\mathcal{R}$ is open, and $v \in C(\mathcal{R})$. (i) If there are positive numbers m and M such that $m \leq \ell \leq M$ on $\mathbb{R}^N \times A$ and $STC\mathcal{T}$ holds, then v is the unique viscosity solution of*

$$\begin{cases} H(x, Dw(x)) \ = \ 0, & x \in \mathcal{R} \setminus \mathcal{T} \\ w(x) \ = \ g(x), & x \in \mathcal{T} \end{cases} \quad (7)$$

in the class of functions $w \in C(\mathcal{R})$ which are bounded below and satisfy $\lim_{x\to x_0} w(x) = +\infty$ for all x_0 in $\partial\mathcal{R}$. (ii) If A is a compact metric space, f is uniformly Lipschitz in A, condition $S\mathcal{T}(v, \mathcal{R})$ holds, and v is $(\mathcal{R} \cap P, \mathcal{T}, +\infty)$-compatible and bounded below, then v is the unique solution of (7) among $(\mathcal{R} \cap P, \mathcal{T}, +\infty)$-compatible functions $w \in C(\mathcal{R})$ that are bounded below and satisfy $S\mathcal{T}(w, \mathcal{R})$.

Before turning to the proofs, we indicate how our results give the uniqueness result for the FP. Recall that the FP is the minimization of $\int_0^{t_p(\alpha)} y^2_{1,p}(t, \alpha)\, dt$ over $\alpha \in \mathcal{A}^f(p)$ for each $p \in \mathbb{R}^2$, where $(y_{1,p}(t, u), \ y_{2,p}(t, u))'$ is the solution of $x'(t) = y(t)$, $y'(t) = u(t)$ with $(x(0), y(0))' = p$ for a given $[-1, +1]$-valued measurable control u, and $t_p(u)$ is the first time that trajectory reaches the target $\mathcal{T} := (0, 0)'$. Let $L > 0$ be given. The trajectory from $(0, L)'$ using the constant control $u \equiv -1/2$, $\phi_1(t) = \left(Lt - \frac{t^2}{4}, \ L - \frac{t}{2}\right)'$, reaches the point $(0, -L)'$ at time $4L$. The trajectory from $(0, -L)'$ using $u \equiv 1/2$, $\phi_2(t) = \left(-Lt + \frac{t^2}{4}, -L + \frac{t}{2}\right)'$, reaches $(0, L)'$ at time $4L$. Let ζ_L denote the concatenation of $\phi_1 : [0, 4L] \to \mathbb{R}^2$ followed by $\phi_2 : [0, 4L] \to \mathbb{R}^2$, and let Ω_L denote the open set bounded by this concatenation with the origin removed. By elementary calculations of trajectories, one sees that $STC(\Omega^c_L)$ holds for each $L > 0$. The calculation in based on the fact that ϕ_1 solves $x^2 = L^2 - y^2$ and ϕ_2 solves $x^2 = y^2 - L^2$. One easily checks that $STC(\{0\})$ holds also (cf. [7]). This establishes condition SSTC($\mathbb{R}^2 \setminus \{0\}, \{0\}$).

[3] By this, we mean a viscosity solution w of $H(x, Dw(x)) = 0$ on $\Omega \setminus \mathcal{T}$ which is also defined on $\mathcal{T}$ and agrees with g on $\mathcal{T}$.

For all $x \neq 0$, $\int_0^t \ell(y_x^r(s,\alpha),\alpha(s))\,ds > 0$ for all $\alpha \in \mathcal{A}^r$ and $t > 0$, since $\dot{x} \equiv y$ is continuous along any trajectory (which means that $|x(t)| > 0$ for a positive measure of small times whenever $y(0) \neq 0$). Also, $\mathbb{R}^2 \setminus (P \cup \mathcal{T}) = \emptyset$. By [11], the FP value function v is continuous and $\mathcal{R} = \mathbb{R}^2$, and one easily checks that the minimum norm of any ϕ_1 or ϕ_2 point $\to \infty$ as $L \to +\infty$. Theorem 2 therefore gives the following

Corollary: The FP value function is the unique viscosity solution of

$$-y(Dw((x,y)'))_1 + |(Dw((x,y)'))_2| - x^2 = 0$$

on $\mathbb{R}^2 \setminus \{0\}$ among functions $w \in C(\mathbb{R}^2)$ satisfying $w((0,0)') = 0$ and $\lim_{||x|| \to \infty} w(x) = +\infty$.

Since the FP Lagrangian ℓ vanishes outside $\mathcal{T}$, this result does not follow from the earlier uniqueness results for viscosity solutions of HJB's.

5 Proof of Main Results

This section proves Theorem 1 under the latter of the alternative hypotheses. The other case is covered in [1]. Theorem 2 follows from the fact that v is a viscosity solution of the associated HJB, which is shown by modifying standard arguments from [1].

Proposition 5.1 Assume that (2) satisfies (A_0)-(A_3), that $\Omega \subseteq \mathbb{R}^N$ is an open set containing $\mathcal{T}$, and that $w \in C(\Omega)$ is a viscosity subsolution of (6) such that $\mathcal{ST}(w,\Omega)$ holds. Then $w \leq v$ on Ω.

Proof: Let $x \in \Omega \setminus \mathcal{T}$ be given, and let B be any bounded open set in Ω that contains x and is such that $\bar{B} \subseteq \Omega$. Then w is a viscosity subsolution of $\sup_{a \in A}\{-f(x,a)\cdot Dw(x) - \ell(x,a)\} = 0$ on $B \setminus \mathcal{T}$. Since $w \in C(\overline{B})$, it follows from Lemma 3.1 that

$$w(x) \leq \int_0^t \ell(y_x(s,\alpha),\alpha(s))\,ds + w(y_x(t,\alpha)) \qquad (8)$$

for all $\alpha \in \mathcal{A}$ and $t \in [0,\tau_x(\alpha))$, where the τ_x's are the exit times from $B \setminus \mathcal{T}$. Supposing that $w(x) > v(x)$, we get an $\tilde{\alpha} \in \mathcal{A}^f(x)$ such that

$$\int_0^{t_x(\tilde{\alpha})} \ell(y_x(s,\tilde{\alpha}),\tilde{\alpha}(s))\,ds + g(y_x(t_x(\tilde{\alpha}),\tilde{\alpha})) < w(x).$$

If $y_x(\cdot,\tilde{\alpha})$ remains in Ω, then $t_x(\tilde{\alpha})$ is a limit of exit times from sets $B = B_k$ as above. For example, take B_k to be a suitable tube around the restriction of the trace of $y_x(\cdot,\tilde{\alpha})$ to $[0, t_x(\tilde{\alpha}) - 1/k]$. In that case we arrive at a contradiction once we put $\alpha = \tilde{\alpha}$ and $t = t_x^k(\tilde{\alpha})$ in (8), where the t_x^k are exit times from the B_k's, and pass to the limit as $k \to \infty$. Otherwise, let $\hat{\tau}$ be the last time in $(0, t_x(\tilde{\alpha}))$ that $y_x(\cdot,\tilde{\alpha})$ is in $\partial\Omega$, and apply (8) to a sequence of points of the form $z_n = y_x(\hat{\tau} + 1/n, \tilde{\alpha})$, with $\alpha(\cdot) = \alpha_n(\cdot) := \tilde{\alpha}(\cdot + \hat{\tau} + 1/n)$, $t = t_{z_n}(\tilde{\alpha})$, and B chosen to be a tube in Ω around the trajectory from z_n to $\mathcal{T}$, to get $w(z_n) < w(x)$ for all n, contradicting the $\mathcal{ST}$ assumption. Indeed, there would be a $\delta > 0$ such that

$$\begin{aligned} w(x) - \delta \;\; &> \int_0^{\hat{\tau}+1/n} \ell(y_x(s,\tilde{\alpha}),\tilde{\alpha}(s))\,ds \\ &+ \int_0^{t_x(\tilde{\alpha})-\hat{\tau}-1/n} \ell(y_{z_n}(s,\alpha_n),\alpha_n(s))\,ds \\ &+ w(y_x(t_x(\tilde{\alpha}),\tilde{\alpha})) \\ &\geq \int_0^{\hat{\tau}+1/n} \ell(y_x(s,\tilde{\alpha}),\tilde{\alpha}(s))\,ds + w(z_n) \end{aligned}$$

By the $\mathcal{ST}$ condition, the last integral is $\geq -\delta$ for large n, so we get $w(x) > w(z_n)$ for large n. But $\Omega \ni z_n \to y_x(\hat{\tau},\tilde{\alpha}) \in \partial\Omega$, contradicting the $\mathcal{ST}$ assumption. ∎

Therefore, Theorem 1 will follow once we show that $w \geq v$. Fix $x \in [\Omega \cap P] \setminus \mathcal{T}$. Then $x \in \Omega_j \setminus \mathcal{T}$ for j large enough, and for such a j, we set $\mathcal{S} = \Omega_j$. We will later put some restrictions on the value of j. We now use (5) (with $B = \mathcal{S} \setminus \mathcal{T}$) to prove the existence of a trajectory starting at x and reaching $\mathcal{T}$ in finite time. This is possible since $\overline{\mathcal{S} \setminus \mathcal{T}} \subseteq \mathcal{S} \setminus \mathcal{T} \cup \partial\mathcal{S} \cup \partial\mathcal{T} \subseteq \Omega$ and w is continuous on Ω. We will first assume that $\omega_0 < \infty$. The proof is similar in spirit to the proof of Theorem IV.3.15 in [1] and arguments in [2], but we use a weak convergence and strong controllability argument to replace the positive lower bound requirement on ℓ. We will assume that all the δ's in Lemma 3.2 can be chosen to be 1. The general case then follows by replacing the corresponding $1/k$'s with δ_k/k's for a suitable sequence $\delta_k \searrow 0$. In what follows, $I(x,t,\alpha) = \int_0^t \ell(y_x(s,\alpha),\alpha(s))ds + w(y_x(t,\alpha))$.

Given $\varepsilon > 0$, let us begin by constructing an $\bar{\alpha} \in \mathcal{A}$ such that $\tau_x(\bar{\alpha}) < +\infty$ and such that

$$w(x) \;\; \geq \int_0^{\tau_x(\bar{\alpha})} \ell(y_x(s,\bar{\alpha}),\bar{\alpha}(s))\,ds + \lambda_x(\bar{\alpha}) - \varepsilon, \; (9)$$

where $\tau_x(\alpha)$ is the first time $y_x(\cdot,\alpha)$ exits $\mathcal{S} \setminus \mathcal{T}$ and

$$\lambda_x(\alpha) := \begin{cases} \dfrac{w(x)+\omega_0}{2}, & t_x(\alpha) \neq \tau_x(\alpha) \\ w\left(y_x(\tau_x(\alpha),\alpha)\right), & t_x(\alpha) = \tau_x(\alpha) \end{cases}.$$

Letting the T_δ's be as in Lemma 3.2 with $B = \mathcal{S} \setminus \mathcal{T}$, we define $x_1 := x$, $\tau_1 := T_1(x_1)$ when $T_1(x_1) < +\infty$, and $\tau_1{:=}10$ when $T_1(x_1) = +\infty$, and use (5) to get an α_1 such that $w(x_1) \geq I(x_1,\tau_1,\alpha_1) - \varepsilon/4$. Note that $y_{x_1}(\tau_1,\alpha_1) \in \mathcal{S} \setminus \mathcal{T}$. By induction, we define $x_k :=$

$y_{x_{k-1}}(\tau_{k-1}, \alpha_{k-1})$, where $\tau_k := T_{1/k}(x_k)$ if $T_{1/k}(x_k) < +\infty$ and 10^k otherwise. Since $x_k \in \mathcal{S} \setminus \mathcal{T}$ we can use (5) to get an $\alpha_k \in \mathcal{A}$ such that

$$w(x_k) \;\geq\; I(x_k, \tau_k, \alpha_k) - 2^{-(k+1)}\varepsilon. \tag{10}$$

We also set $\sigma_k := \tau_1 + \ldots + \tau_k$, $\bar{\sigma} = \limsup_k \sigma_k$, and, for an arbitrary $\bar{a} \in A$,

$$\bar{\alpha}(s) := \begin{cases} \alpha_1(s) & \text{if } 0 \leq s < \sigma_1, \\ \alpha_2(s - \sigma_1) & \text{if } \sigma_1 \leq s < \sigma_2, \\ \ldots & \\ \alpha_k(s - \sigma_{k-1}) & \text{if } \sigma_{k-1} \leq s < \sigma_k, \\ \ldots & \\ \bar{a} & \text{if } \bar{\sigma} \leq s, \end{cases}$$

with the last line making sense for $\bar{\sigma} < +\infty$. From the definitions of x_k, P, and $\bar{\alpha}$, we know that, when $s < \bar{\sigma}$,

$$y_x(s, \bar{\alpha}) = y_{x_k}(s - \sigma_{k-1}, \alpha_k) \in \mathcal{S} \setminus \mathcal{T}, \text{ and} \tag{11}$$

$\int_0^{\tau_k} \ell(y_{x_k}(s, \alpha_k), \alpha_k(s))\; ds \;=\; \int_{\sigma_{k-1}}^{\sigma_k} \ell(y_x(s, \bar{\alpha}), \bar{\alpha}(s)) ds$ is nonnegative for all k. Via (10),

$$\begin{aligned} w(x) \;&\geq\; \int_0^{\tau_1} \ell(y_x(s, \bar{\alpha}), \bar{\alpha}(s))\; ds + w(x_2) - \varepsilon/4 \\ &\geq\; \int_0^{\sigma_2} \ell(y_x(s, \bar{\alpha}), \bar{\alpha}(s))\; ds + w(x_3) - \varepsilon\left(\frac{1}{4} + \frac{1}{8}\right) \\ &\geq\; \cdots \\ &\geq\; I(x, \sigma_k, \bar{\alpha}) - \varepsilon/2\; \left(1 - 2^{-k}\right) \;\; \forall k. \end{aligned} \tag{12}$$

By (11) and the boundedness of the Ω_j's, the x_k's are bounded and therefore cluster. Let $\bar{x}$ be a cluster point of the x_k's, so $\bar{x} \in \overline{\mathcal{S} \setminus \mathcal{T}}$. We will need the following minimality property of $\bar{x}$:

Proposition 5.2 With the previous notation, we have $\limsup_k \tau_k \;\geq\; \inf_\alpha\{t : y_{\bar{x}}(t, \alpha) \in \partial(\mathcal{S} \setminus \mathcal{T})\} =: \bar{\tau}$.

Proof: For $\delta > 0$ arbitrary, suppose that $\bar{\tau} < \infty$ and that, for k as large as desired, we had $\tau_k < \bar{\tau} - \delta$. If $\tau_k \to 0$, then by passing to a suitable weak-$\star$ limit of relaxed controls, $\bar{\alpha}$, we would get

$$\text{dist}\;(\bar{x},\; \partial(\mathcal{S}\setminus\mathcal{T})) \leftarrow\; \text{dist}\;(y_{x_k}(\tilde{\tau}_k, \alpha_k),\; \partial(\mathcal{S}\setminus\mathcal{T})) \;\leq\; \frac{1}{k}$$

along some $\tilde{\tau}_k \searrow 0$, so $\bar{\tau} = 0$, which is impossible. The details are as follows. Assuming $\tau_k \searrow 0$, the definition of the infimum gives a $\tilde{\tau}_k \searrow 0$ and a sequence $\{\alpha_k\}$ in $\mathcal{A}$ satisfying the inequality. We apply Lemma 3.3 with $c := \tilde{\tau}_1$ to conclude that for any $\delta > 0$, the RHS of

$$\begin{aligned} ||\, \bar{x} - y_{x_k}(\tilde{\tau}_k, \alpha_k)\, || \;\;&\leq\;\; ||\, \bar{x} - y^r_{\bar{x}}(\tilde{\tau}_k, \bar{\alpha})\, || \\ &+\;\; ||\, y^r_{\bar{x}}(\tilde{\tau}_k, \bar{\alpha}) - y_{x_k}(\tilde{\tau}_k, \alpha_k)\, || \end{aligned}$$

is $< \delta$ for large enough k, which establishes the left arrow, since $\text{dist}(\cdot, \partial(\mathcal{S} \setminus \mathcal{T}))$ is continuous.

Therefore, we can assume by passing to a further subsequence that for some positive number μ, $\tau_k > \mu > 0$ for all k and $\tau_k \to z \in [\mu, \bar{\tau} - \delta]$. That gives, for a sequence $\tilde{\tau}_k \to z$ as above,

$$\begin{aligned} \text{dist}\;(y^r_{\bar{x}}(z, u), \partial(\mathcal{S} \setminus \mathcal{T})) &\leftarrow \text{dist}\;(y_{x_k}(\tilde{\tau}_k, u_k), \partial(\mathcal{S} \setminus \mathcal{T})) \\ &\leq\; \frac{1}{k} \to 0 \;\text{ as }\; k \to +\infty, \end{aligned}$$

where now u is a relaxed control which is a weak-$\star$ limit of some subsequence of the u_j's. This follows from the same compactness theorem with c chosen to be some upper bound of the $\tilde{\tau}_k$'s with k large enough.

The standard trajectories $y_{\bar{x}}(\cdot, u_k)$ approximate $y^r_{\bar{x}}(\cdot, u)$ uniformly well on $[0, z+1]$ (by Lemma 3.3), so for large k, $y_{\bar{x}}(\tilde{\tau}_k, u_k)$ lies in $\mathcal{S}\setminus\mathcal{T}$ (since $\tau_k < \bar{\tau}$ for all k) and, by the preceding argument, can be brought to $\partial(\mathcal{S}\setminus\mathcal{T})$ by a standard control $\tilde{u}$ in time less than $\delta/2$. If we now assemble the control for such an approximating standard trajectory and $\tilde{u}$, we get a (standard) trajectory which brings $\bar{x}$ to $\partial(\mathcal{S} \setminus \mathcal{T})$ in time $\leq \bar{\tau} - \delta/2$, which is impossible. We conclude that if $\bar{\tau} < \infty$, then $\tau_k \geq \bar{\tau} - \delta$ for k large enough. Since $\delta > 0$ was arbitrary, we can assume wlog that $\tau_k \to l \geq \bar{\tau}$ in this case (allowing $l = +\infty$). If we had $\bar{\tau} = +\infty$, then replace $\bar{\tau} - \delta$ in the argument above with any fixed positive number. ■

One easily sees that $\bar{\tau}$ is zero iff $\bar{x} \in \partial(\mathcal{S} \setminus \mathcal{T})$. The following is a consequence of Proposition 5.2.

Corollary 5.3 With the above notation, any cluster point of $\{x_k\}$ lies in $\partial(\mathcal{S} \setminus \mathcal{T})$.

Proof: Let $\tilde{\alpha}$ be a weak-$\star$ limit of a subsequence of the α_k's in $\mathcal{A}^r$ (which we assume to be the sequence itself for brevity). Let $M > 0$ be given. We then get

$$0 \leftarrow \int_{\sigma_{k-1}}^{\sigma_k \wedge \{\sigma_{k-1}+M\}} \ell(y_x(s, \bar{\alpha}), \bar{\alpha}(s))\; ds =$$

$$\int_0^{\tau_k \wedge M} \ell(y_{x_k}(s, \alpha_k), \alpha_k(s))\; ds \to$$

$$\int_0^{l \wedge M} \ell^r(y^r_{\bar{x}}(s, \tilde{\alpha}), \tilde{\alpha}(s))\; ds.$$

The left arrow is by the divergence test applied to the last integral in (12), using the facts that w is bounded below and $y_x(\cdot, \bar{\alpha})$ stays in $\mathcal{S} \setminus \mathcal{T} \subseteq P$. To justify the right arrow, apply Lemma 3.3 on $[0, M]$ to get

$$y^r_{x_k}(s, \alpha_k) \to y^r_{\bar{x}}(s, \tilde{\alpha}) \;\text{ uniformly on } [0, M]. \tag{13}$$

Since A is compact, it therefore suffices to show that $\ell^r(y^r_{x_k}(s, \alpha_k), \alpha_k(s)) \;\to\; \ell^r(y^r_{\bar{x}}(s, \tilde{\alpha}), \tilde{\alpha}(s))$ on $[0, M]$, (and then the result follows from the Dominated Convergence Theorem). Fix $s \in [0, M]$, and let $\alpha_{k,s}(\cdot)$

(resp., $\tilde{\alpha}_s(\cdot)$) denote the Radon measures $\alpha_k(s)$ (resp., $\tilde{\alpha}(s)$) for each k. Then,

$$|\ell^r(y^r_{\bar{x}}(s,\tilde{\alpha}),\tilde{\alpha}(s)) - \ell^r(y^r_{x_k}(s,\alpha_k),\alpha_k(s))| \leq$$

$$\left|\int_A \ell(y^r_{\bar{x}}(s,\tilde{\alpha}),a)\, d\tilde{\alpha}_s(a) - \int_A \ell(y^r_{\bar{x}}(s,\tilde{\alpha}),a)\, d\alpha_{k,s}(a)\right| +$$

$$\left|\int_A \left[\ell(y^r_{\bar{x}}(s,\tilde{\alpha}),a) - \ell(y^r_{x_k}(s,\alpha_k),a)\right]\, d\alpha_{k,s}(a)\right|.$$

The first RHS term $\to 0$ because $\ell(y^r_{\bar{x}}(s,\tilde{\alpha}),\cdot)$ is continuous and $\alpha_k \to \tilde{\alpha}$ weak-* on $[0,M]$. The second RHS term $\to 0$ by (13) and (A_3).

If we had $\int_0^{\bar{\tau}} \ell^r(y^r_{\bar{x}}(s,\tilde{\alpha}),\tilde{\alpha}(s))\, ds > 0$, then we would have $\int_0^G \ell^r(y^r_{\bar{x}}(s,\tilde{\alpha}),\tilde{\alpha}(s))\, ds > 0$ for some $G \in (0,\bar{\tau})$, and then, since $l \geq \bar{\tau}$, we would reach a contradiction by putting $M = G$ above. Therefore, $\bar{\tau} = 0$, or $\bar{x}$ is not in P. But, $\bar{x} \in \overline{\mathcal{S} \setminus \mathcal{T}}$, and $\mathcal{S} \setminus \mathcal{T} \subset P$, so we conclude that $\bar{x} \in \partial(\mathcal{S} \setminus \mathcal{T})$ and $\bar{\tau} = 0$, as needed. ■

We can therefore find a K such that if $k > K$, then $\int_0^{\tau_{x_k}(\beta_k)} \ell(y_{x_k}(s,\beta_k),\beta_k(s))\, ds < \varepsilon/4$ for some $\beta_k \in \mathcal{A}$ driving x_k to $\tilde{x} \in \partial(\mathcal{S} \setminus \mathcal{T})$, where $\tau_p(\gamma) = \inf\{t \geq 0 : y_p(t,\gamma) \notin \mathcal{S} \setminus \mathcal{T}\}$ for $\gamma \in \mathcal{A}$ and $p \in \mathcal{S} \setminus \mathcal{T}$. This is possible since ℓ is continuous, $x_k \to \bar{x}$, and $T_j \to 0$ near $\partial(\mathcal{S} \setminus \mathcal{T})$. Since $y_{x_k}(\tau_k,\alpha_k) = y_x(\sigma_k,\bar{\alpha})$, we get

$$I(x,\sigma_k,\bar{\alpha}) = \int_0^{\sigma_k} \ell(y_x(s,\bar{\alpha}),\bar{\alpha}(s))\, ds + w(y_{x_k}(\tau_k,\alpha_k))$$

For such k, the construction (12) therefore gives

$$\begin{aligned} w(x) \quad &\geq \quad I(x,\sigma_{k-1},\bar{\alpha}) \\ &+ \quad \int_0^{\tau_{x_k}(\beta_k)} \ell(y_{x_k}(s,\beta_k),\beta_k(s))\, ds \\ &- \quad \varepsilon/2\,(1 - 2^{-(k-1)} + 1/2). \qquad (14) \end{aligned}$$

For large k and $\bar{x} \in \mathcal{T}$, $w(x_k) + \varepsilon/4 > w(\tilde{x})$, since $w \in C(\Omega)$. Otherwise, $\bar{x} \in \partial\mathcal{S} \setminus \mathcal{T}$, so $w(y_{x_{k-1}}(\tau_{k-1},\alpha_{k-1})) \geq 1/2(w(x) + \omega_0)$, since, by the compatibility condition, we can assume that j is so large that $w(p) > \frac{1}{4}w(x) + \frac{3}{4}w_0$ for $p \in \mathcal{S} \setminus \mathcal{T}$ near $\partial\mathcal{S} \setminus \mathcal{T}$. In the former case, we add and subtract $\varepsilon/4$ in (14). We complete the construction by assembling $\bar{\alpha}$ up to time σ_{k-1} and β_k for k large enough.

Thus, when $\omega_0 \in \mathbb{R}$, our construction always gives us a control $\bar{\alpha}$ for which $\tau_x(\bar{\alpha}) < +\infty$ satisfying (9). Since $\varepsilon > 0$ was arbitrary, we conclude that $w(x)$ majorizes

$$\inf_\alpha \left\{ \int_0^{\tau_x(\alpha)} \ell(y_x(s,\alpha),\alpha(s))ds + \lambda_x(\alpha) : \tau_x(\alpha) < +\infty \right\}.$$

If $\tau_x(\alpha) \neq t_x(\alpha)$, then the corresponding infimand is $\int_0^{\tau_x(\alpha)} \ell(y_x(s,\alpha),\alpha(s))\ ds + \frac{1}{2}\ (\omega_0 + w(x)) > w(x)$, since $w(x) < w_0$ for all $x \in \Omega$ and $x \in P$, so such a control is irrelevant for the infimum. Since $g = w$ on $\mathcal{T}$, $w(x) \geq v(x)$ on $\Omega \cap P$. If $x \in \Omega \setminus [P \cup \mathcal{T}]$, $w(x) = [w(x) - w(y_x(L,\alpha'))] + w(y_x(L,\alpha'))$ for some $\alpha' \in \mathcal{A}$ and $L > 0$ such that $y_x(L,\alpha') \in P \cap \Omega$ and $w(x) - w(y_x(L,\alpha')) \geq \int_0^L \ell(y_x(s,\alpha'),\alpha'(s))\, ds - \varepsilon$ (via the first $\mathrm{S}\mathcal{T}(w,\Omega)$ condition and Lemma 3.2), then assemble $\alpha' \lceil [0,L]$ and the $\bar{\alpha}$ constructed above to get $w(x) \geq v(x)$ as above. Thus, $w \geq v$ on Ω.

The proof for the $w_0 = +\infty$ case is similar to this. We view a fixed $x \in [P \cap \Omega] \setminus \mathcal{T}$ as a member of $\Omega_j \setminus \mathcal{T}$ where j is large enough so that values of w on $\partial\Omega_j \setminus \mathcal{T}$ majorize $w(x)$, and then we construct a trajectory that reaches $\partial(\Omega_j)$ or $\mathcal{T}$ in finite time. The controls whose exit times for $\Omega_j \setminus \mathcal{T}$ are not exit times for $\mathcal{T}$ are irrelevant for the calculation of the appropriate infimum, as above, so $w \geq v$ as before.

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