Viscosity Solutions of the Bellman Equation for Infinite Horizon Optimal Control Problems with Negative Instantaneous Costs [0]

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Abstract

In a series of papers, we characterized the value function in optimal control as the unique viscosity solution of the corresponding Bellman equation that satisfies appropriate side conditions. The novelty of our results was that they applied to exit time problems with general nonnegative instantaneous costs, including cases where the instantaneous cost is not uniformly bounded below by positive constants. This note will extend these results to control problems whose instantaneous costs are allowed to take both positive and negative values, including undiscounted examples. We apply our results to the generalized Zubov equation, which corresponds to the Bellman equation for a negative instantaneous cost. The unique solutions of the Zubov equations are maximum cost Lyapunov functions for perturbed asymptotically stable systems. We study the regularity of these Lyapunov functions, and we further extend Zubov's method for representing domains of attractions as sublevel sets of Lyapunov functions. We also illustrate some special properties of maximum cost Lyapunov functions that can occur when the instantaneous cost for the Lyapunov function is degenerate.

Key Words: viscosity solutions, optimal control, Lyapunov functions, stability, domains of attraction

1 Introduction

The theory of viscosity solutions[1] forms the basis for much of current research in control and optimization (cf. [1, 4, 5]). A fundamental issue in control and optimization theory is the characterization of minimum cost functions as unique viscosity solutions of (Hamilton-Jacobi-)Bellman equations (a.k.a. HJBE) that satisfy appropriate side conditions (cf. [1, 6, 7, 11]).[2] Starting from uniqueness results of this kind, one can estimate the rate of convergence of numerical schemes for approximating minimum cost functions, synthesize nearly-optimal controls, and study singular perturbations and much more (cf. [1, 4]).

In this note, we develop new uniqueness and regularity theory for viscosity solutions of infinite horizon HJBE (cf. [1]), and for the closely-related generalized Zubov equation (introduced in [3, 5]), and we study the regularity of robust Lyapunov functions. Infinite horizon HJBE have the form

$$\sup_{a\in A} B[f,\ell,h]\ (x,v(x),Dv(x),a)=0, \qquad (1)$$

where

$$B[f,\ell,h](x,v,p,a)=-f(x,a)\cdot p-\ell(x,a)+h(x,a)v$$

and $A\subset\mathbb{R}^M$ is a fixed nonempty compact set (called the **control set**). The control problem and assumptions corresponding to (1) are as follows. Set

$$\mathcal{A}:=\{\text{measurable functions } [0,+\infty)\to A\},$$

and define $\mathcal{A}^r$ (which is called the set of **relaxed controls**) by $\mathcal{A}^r:=\{\text{measurable functions } [0,+\infty)\to A^r\}$, where A^r is the set of all Radon probability measures on A topologized as a subset of the dual of $C(A):=\{\text{continuous functions } A\to\mathbb{R}\}$ with the weak-$\star$ topology (cf. [7]). We use the discount factor

$$\delta[h](x,s,\beta):=\int_0^s h^r(\phi(u,x,\beta),\beta(u))\mathrm{d}u$$

and define the running cost function

$$J[\ell,h](x,t,\beta)=\int_0^t e^{-\delta[h](x,s,\beta)}\ell^r(\phi(s,x,\beta),\beta(s))\mathrm{d}s$$

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[1] If $F:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is continuous, $S\subseteq\mathbb{R}^N$ is open, and $w:S\to\mathbb{R}$ is locally bounded, then w is called a **(viscosity) solution** of $F(Dw(x),w(x),x)=0$ on S provided: If $\phi:S\to\mathbb{R}$ is continuously differentiable, and if $w_\star-\phi$ (resp., $w^\star-\phi$) has a local minimum (resp., maximum) at $x_o\in S$, then $F(D\phi(x_o),w_\star(x_o),x_o)\geq 0$ (resp., $F(D\phi(x_o),w^\star(x_o),x_o)\leq 0$). Here $w_\star(x):=\liminf_{y\to x}w(y)$ and $w^\star(x):=\limsup_{y\to x}w(y)$.

[2] See also the companion paper [10] to this note, which relaxes the 'quasi-stability' assumption ($\flat$) we impose below.

for all $x \in \mathbb{R}^N$, $t \geq 0$, and $\beta \in \mathcal{A}^r$, where $\phi(\cdot, x, \beta)$ is defined to be the solution of

$$\begin{aligned} &\frac{d}{ds}\phi(s,x,\beta) = f^r(\phi(s,x,\beta),\beta(s)) \text{ a.e. },\\ &\phi(0,x,\beta) = x, \quad \beta \in \mathcal{A}^r, \quad x \in \mathbb{R}^N \end{aligned} \tag{2}$$

on $[0, \infty)$ and

$$\Psi^r(x,m) := \int_A \Psi(x,a)dm(a)$$

for $\Psi = f, \ell, h$, and all $x \in \mathbb{R}^N$ and $m \in A^r$. We view $\mathcal{A}$ as the subset of $\mathcal{A}^r$ consisting of all Dirac probability measure valued relaxed controls. By the Filippov Selection Theorem, all of our results will remain true if $\mathcal{A}^r$ is replaced by $\mathcal{A}$ if $\{(f(x,a), \ell(x,a), h(x,a)) : a \in A\}$ is convex for all $x \in \mathbb{R}^N$. To account for the possibility of divergent integrals in our maximum cost Lyapunov functions, we use the smaller set of admissible controls

$$\mathcal{A}(x) := \left\{ \begin{array}{l} \alpha \in \mathcal{A}^r : J[\ell,h](x,+\infty,\alpha) \\ \text{converges in } \mathbb{R} \cup \{\pm\infty\} \end{array} \right\}.$$

The control problem corresponding to (1) is then the **(variable discount) infinite horizon problem**

$$\text{Infimize } J[\ell,h](x,+\infty,\alpha) \text{ over all } \alpha \in \mathcal{A}(x) \tag{3}$$

for all $x \in \mathbb{R}^N$. The relationship between (1) and (3) is that if the **value function** $V_\infty : \mathbb{R}^N \to [-\infty, +\infty]$ for (3), defined by

$$V_\infty(x) := \inf_{\alpha \in \mathcal{A}(x)} J[\ell,h](x,+\infty,\alpha), \tag{4}$$

is everywhere finite and differentiable, then it is a global solution of (1). Here and in the sequel, $\inf \emptyset = +\infty$, and $||\cdot||$ is the usual Euclidean norm. If V_∞ is not differentiable, then standard hypotheses imply that V_∞ is a viscosity solution of (1) on $\mathbb{R}^N$ (cf. [1]). In the context of (3), we refer to f as the **dynamics**, ℓ as the **instantaneous cost** (a.k.a. **Lagrangian**), and h as the **discount rate**. We set $B_R := \{x \in \mathbb{R}^N : ||x|| \leq R\}$ for all $R > 0$, and assume throughout this note that

- (A_1) $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ is continuous, $\exists L > 0$ for which $||f(x,a) - f(y,a)|| \leq L||x-y||$ for all $x, y \in \mathbb{R}^N$ and $a \in A$, and $f(0,a) \equiv 0$.
- (A_2) $\ell : \mathbb{R}^N \times A \to \mathbb{R}$ and $h : \mathbb{R}^N \times A \to [0,\infty)$ are continuous, and $\ell(0,a) \equiv 0$.

The novelty here is that ℓ can take both positive and negative values, and that we allow $h \equiv 0$, so the usual uniqueness results for nonnegative ℓ and strictly positive h (cf. [1, 7]) cannot be applied. It is natural to allow problems in which ℓ takes both positive and negative values, since this allows cases where a nonnegative instantaneous cost is minimized in one part of the state space and maximized in the rest of the state space. Also, many important examples have $h \equiv 0$ (cf. [7]). One of our objectives is to prove that V_∞ is the unique viscosity solution of the corresponding Bellman equation (1) that satisfies appropriate side conditions.

The main motivation for studying the problem (3) is that it plays an important role in stability, which we describe next. The theories of Lyapunov functions and domains of attraction form the basis for much of current work in stability theory (cf. [3, 5]). A well-known result in this area is the *Zubov method* (cf. [5]), which gives conditions under which the domain of attraction of an asymptotically stable fixed point of a given dynamics $\dot{x} = f(x)$ is $v^{-1}([0,1))$, where v is the solution of the *Zubov equation*

$$Dv(x) \cdot f(x) = -H(x)[1 - v(x)]\sqrt{1 + ||f(x)||^2},$$

for $x \in \mathbb{R}^N$, for suitable functions H. In [3, 5], Zubov's method was extended to the important case of perturbed asymptotically stable systems $\dot{x} = f(x,a)$ for which the fixed point 0 is stable under any perturbation a. The main results in [3, 5] are partial differential equations (PDE) characterizations for the *robust domain of attraction* $\mathcal{D}_o$ and for *robust Lyapunov functions* for perturbed dynamics f on $\mathcal{D}_o$.[3] While the *existence* of such Lyapunov functions followed from standard results using the stability of the dynamics, the papers [3, 5] gave new methods of *computing* the Lyapunov functions and $\mathcal{D}_o$.

Under the conditions of [3], the robust domain of attraction $\mathcal{D}_o$ for the perturbed system $\dot{x} = f(x,a)$ is the sublevel set $v^{-1}([0,1))$, where v is the unique bounded continuous viscosity solution of the *generalized Zubov equation*

$$\inf_{a \in A} B[f,g,g]\,(x, v(x), Dv(x), a) = 0, \; x \in \mathbb{R}^N \tag{5}$$

that vanishes at the origin, under certain restrictions on g. In [3], the solution of (5) is a maximum cost type Lyapunov function for the dynamics f on $\mathcal{D}_o$ of the form

$$V_{\max}(x) := \sup_{\alpha \in \mathcal{A}^r} J[g,g](x,+\infty,\alpha)$$

We will usually make the following assumptions:

- (A_3) Condition (A_1) holds, f is (globally) bounded, $g : \mathbb{R}^N \times A \to [0,\infty)$ is continuous, $g(0,a) \equiv 0$.

[3] Recall that, under the assumption (A_4) below,

$$\mathcal{D}_o := \left\{ x \in \mathbb{R}^N : \sup\{t(x,\alpha) : \alpha \in \mathcal{A}\} < +\infty \right\},$$

where $t(x,\alpha) := \inf\{t \geq 0 : ||\phi(t,x,\alpha)|| \leq R\}$ for all $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{A}$ and R is the constant from (A_4). Recall that a function $V : \mathcal{O} \to \mathbb{R}$ on an open set $\mathcal{O} \subseteq \mathbb{R}^N$ containing the origin is called a **(robust) Lyapunov function** for f on $\mathcal{O}$ provided:

(i) V is positive definite

(ii) $V(x) > V(\phi(t,x,\alpha))$ for all $x \in \mathcal{O} \setminus \{0\}$, $t > 0$, and $\alpha \in \mathcal{A}$

where positive definiteness means that $V(x) \geq 0$ for all $x \in \mathcal{O}$ and $V(x) = 0$ iff $x = 0$.

(A_4) There exist constants $C, \sigma, R > 0$ such that $||\phi(t,x,\alpha)|| \le C||x||e^{-\sigma t}$ for all $x \in B_R$, $t \ge 0$, and $\alpha \in \mathcal{A}$.[4]

(A_5) $\int_0^t g(\phi(s,x,\alpha),\alpha(s))\,ds > 0\ \forall x \in \mathcal{D}_o \setminus \{0\}$, $t > 0$, and $\alpha \in \mathcal{A}$.

Sometimes we use the notation

$$V_{\max}[g,h](x) := \sup_{\alpha\in\mathcal{A}^r} J[g,h](x,+\infty,\alpha) \tag{6}$$

to emphasize the functions used to calculate the Lyapunov functions. Under our hypotheses, it will turn out that the supremum in (6) can be taken over all $\alpha \in \mathcal{A}$ without changing the values of $V_{\max}[g,h]$, and that (6) is continuous at the origin. Therefore, $V_{\max}[g,g]$ is the negative of (4) for $\ell \equiv -g$ and $h \equiv g$, and a function w is a viscosity solution of (5) exactly when $-w$ is a viscosity solution of (1) for $\ell \equiv -g$ and $h \equiv g$. Therefore, uniqueness and regularity of solutions for (5) will follow from the corresponding properties for (1) for negative ℓ. This motivates our study of (1)-(3). In [3], uniqueness results are given for solutions of (5), and additional assumptions are given that guarantee that $V_{\max}$ is a locally Lipschitz Lyapunov function for f. Furthermore, $V_{\max}[\delta g, \delta g]$ was shown to be globally Lipschitz for large enough positive constant $\delta > 0$. These results form the basis for discrete approximations of $\mathcal{D}_o$ (cf. [2]).

2 Motivating Example

In [2, 3, 5], g is assumed to be **nondegenerate**, meaning, for some positive constants g_o and r,

$$\begin{array}{l}(i)\ \inf\{g(x,a) : x \notin B_r, a \in A\} \ge g_o > 0\\ (ii)\ g(x,a) > 0\ \ \forall x \ne 0, \forall a \in A\end{array} \tag{7}$$

One can easily find equations (5) which admit several bounded solutions, all of which are null and continuous at the origin, when (7) is not satisfied. Here is an elementary example where this occurs. It illustrates how relaxing requirement (7), even at a single point, can give multiple bounded solutions of (5), which are all null at the origin, and also how these solutions are related to the problem (3). It also illustrates the special regularity properties that can sometimes occur for $V_{\max}[g,g]$ when g is degenerate.

[4]Our results will remain true if (A_4) is replaced by the following assumptions (cf. [5, 9] for the relevant definitions):

(i) $\exists\ \beta_f \in \mathcal{KL}$ and $R > 0$ such that $||\phi(t,x,\alpha)|| \le \beta_f(||x||,t)$ for all $x \in B_R$, $t \ge 0$, and $\alpha \in \mathcal{A}$

(ii) $\exists\ \delta, \varepsilon > 0$ such that $g(x,a) \le \delta\alpha_2^{-1}(||x||)$ for all $x \in B_\varepsilon$ and $a \in A$, where $\alpha_1, \alpha_2 \in \mathcal{K}^\infty$ are functions such that $\beta_f(r,t) \le \alpha_2(\alpha_1(r)e^{-t})$ for all $r, t \ge 0$

Given $\beta_f \in \mathcal{KL}$, the "$\mathcal{KL}$ Lemma" implies the existence of functions $\alpha_1, \alpha_2 \in \mathcal{K}^\infty$ satisfying $\beta_f(r,t) \le \alpha_2(\alpha_1(r)e^{-t})$ for all $r, t \ge 0$ (cf. [12], Proposition 7).

Example 2.1 Take $N = 1$ and $A = [-1,+1]$, and define $f, g : \mathbb{R} \times [-1,+1] \to \mathbb{R}$ by

$$\begin{array}{l} f(x,a) := \begin{cases} 1 + a, & x < -1 \\ -x + ax^2, & -1 \le x \le 1 \\ -1 + a/x, & x > 1 \end{cases} \\ g(x,a) \equiv \gamma(x) := x^2(x-1)^2 \end{array} \tag{8}$$

Then $\mathcal{D}_o = (-1,1)$. Set $W_o := V_{\max}[g,0]$. We first check that $W_o(1) < \infty$, which will follow once we check that $\sup\{W_o(x) : 0 < x < 1\}$ is finite, since the dynamics does not allow movement to the right from the initial value $\bar{x} = 1$. If $0 < x < 1$ and $\alpha \in \mathcal{A}$, and if we set $y(t) = \phi(t,x,\alpha)$, then the change of variables $u = y(t)$ and the fact that $\alpha(t) \le 1$ a.e. give

$$\int_0^{+\infty} \gamma(\phi(t,x,\alpha))\,\mathrm{d}t = \lim_{p\downarrow 0}\int_p^x \frac{u^2(1-u)^2}{u - a(y^{-1}(u))u^2}\,\mathrm{d}u$$
$$\le \lim_{p\downarrow 0}\int_p^x \frac{u^2(1-u)^2}{u-u^2}\,\mathrm{d}u \ \le\ \int_0^1 u(1-u)\,\mathrm{d}u < \infty$$

which proves $W_o(1) < \infty$.[5] A similar calculation shows $W_o(x)$ is finite for all $x > 1$ and all $x \in (-1,0]$. We will now construct an infinite collection of bounded solutions of the corresponding equation (5) which are null and continuous at the origin. Set $T := \{0,1\}$ and

$$\tau_x(\alpha) := \inf\{t \ge 0 : \phi(t,x,\alpha) \in \mathcal{T}\} \in [0,\infty]$$

for all $\alpha \in \mathcal{A}$ and $x \in \mathbb{R}$. Set

$$\mathcal{A}_\mathcal{T}^\infty(x) := \left\{\alpha \in \mathcal{A} : \lim_{t\to+\infty}\phi(t,x,\alpha) \in \mathcal{T}\right\}$$

for all $x \in \mathbb{R}$, i.e., the set of all inputs that bring x to $\mathcal{T}$ in finite time or asymptotically. For each function $\Phi : \mathbb{R} \to [0,\infty)$ satisfying $\Phi(0) = 0$, define $W[\Phi]$ by

$$W[\Phi](x) = \sup\left\{\int_0^{\tau_x(\alpha)} \gamma(\phi(t,x,\alpha))\,\mathrm{d}t \right.$$
$$\left. + \Phi\left(\lim_{s\to\tau_x(\alpha)^-}\phi(s,x,\alpha)\right) : \alpha \in \mathcal{A}_\mathcal{T}^\infty(x)\right\} \in [0,\infty],$$

where we use the limit from the left in Φ to allow $\tau_x(\alpha) = +\infty$. Then $W[\Phi]$ is the usual asymptotic exit time value function for the final cost Φ, except with a sup instead of an inf. Moreover, the restrictions $W[\Phi]\lceil\mathcal{D}_o$ coincide with the negative of $V_\infty\lceil\mathcal{D}_o$ for the data $\ell \equiv -g$ and $h \equiv 0$. The preceding argument shows that, for each choice of Φ, $W[\Phi](x) < \infty$ exactly when $x \in (-1,+\infty)$. Define the functions

$$\Gamma_k(x) = k|x|W_o(1), \quad x \in \mathbb{R} \tag{9}$$

[5]Notice that W_o is a robust Lyapunov function for f on $\mathcal{D}_o$, and that $W_o(x) \not\to +\infty$ as $x \to 1^-$. This is different from the case of Lyapunov functions on $\mathcal{D}_o$ for nondegenerate g (cf. [3]), which grow without bound near $\partial(\mathcal{D}_o)$. On the other hand, W_o is proper, meaning $W_o(x) \to +\infty$ as $|x| \to +\infty$.

for each constant $k \geq 1$. The usual dynamic programming methods (cf. [3]) now imply that

$$\check{V}_k(x) := 1 - e^{-W[\Gamma_k](x)}, \quad k \geq 1 \text{ constant}$$

where we define $e^{-\infty} := 0$, are all viscosity solutions of (5) on $\mathbb{R} \setminus \mathcal{T}$. Moreover, the argument of [9], §1 (based on the semidifferentials of the $\check{V}_k$'s at $+1$ and the fact that the constants k in (9) are all ≥ 1) shows that the $\check{V}_k$'s are all bounded viscosity solutions of (5) on all of $\mathbb{R}$ that satisfy $\check{V}_k(0) = 0$. Furthermore,

$$\check{V}_k^{-1}([0,1)) = (-1, +\infty) \neq \mathcal{D}_o$$

for all $k \geq 1$. ∎

The preceding example illustrates how the degeneracy of g at any point outside the origin can give multiple bounded solutions of (5) which are null and continuous at the origin, none of which satisfy the sublevel set characterization $v^{-1}([0,1)) = \mathcal{D}_o$ from the generalized Zubov method.

Remark 2.2 Our uniqueness theory will imply that $v^{-1}([0,1)) = \mathcal{D}_o$, where v is the unique bounded solution of the corresponding generalized Zubov equation that satisfies $v(0) = 0$ and that is continuous at the origin. One of our hypotheses will be g **quasi-stability** of f, which is the following condition:

($\flat$) If $x \in \mathbb{R}^N$, $\alpha \in \mathcal{A}$, and $J[g, 0](x, +\infty, \alpha) < +\infty$, then $\phi(s, x, \alpha) \to 0$ as $s \to +\infty$

(cf. [9]). In Example 2.1, ($\flat$) is not satisfied, since the constant trajectory at 1 gives 0 total costs. On the other hand, the example will satisfy all the other hypotheses of our uniqueness theorem. For uniqueness results for HJBE solutions under weaker asymptotic conditions, see [10]. ∎

3 Statement of Uniqueness Results

Recall (cf. [9]) that if $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ satisfies (A_1) and $\mathcal{G} \subseteq \mathbb{R}^N$, then we say that $\mathcal{G}$ is **asymptotically null** for f provided the following conditions hold:

(i) $0 \in \mathcal{G}$, and $\mathcal{G}$ is open

(ii) $\mathcal{G}$ is **relaxed strongly invariant**, meaning, $\phi(t, x, \beta) \in \mathcal{G}$ for all $t \geq 0$, $x \in \mathcal{G}$, and $\beta \in \mathcal{A}^r$

(iii) $\forall x \in \mathcal{G}$ and $\forall \beta \in \mathcal{A}^r$, $\phi(t, x, \beta) \to 0$ as $t \to +\infty$

For example, our hypotheses will imply that $\mathcal{D}_o$ is asymptotically null for f, by a variant of the Filippov-Ważewski Relaxation Theorem (cf. [9]). The following theorem is shown in [9] and forms the basis for our uniqueness characterizations for solutions of (5) (cf. §5 below for a discussion of how our results extend [3, 5]):

Theorem 1 *Assume the following:*

1) (A_1)-(A_2) hold.

2) $\mathcal{G}$ is asymptotically null for f.

3) $w : \mathcal{G} \to \mathbb{R}$ is a continuous viscosity solution of (1) on $\mathcal{G} \setminus \{0\}$ satisfying $w(0) = 0$.

Then $w \equiv V_\infty$ on $\mathcal{G}$.

The hypotheses in Theorem 1 that w is continuous and $\mathcal{G}$ is relaxed strongly invariant can be replaced by the hypothesis that (i) w is continuous at the origin, (ii) either ℓ is everywhere nonnegative or ℓ is everywhere nonpositive, and (iii) $\mathcal{G}$ is strongly invariant, in which case the conclusion is changed (cf. [9]) to

$$w(x) \equiv \inf\{J(x, +\infty, \alpha) : \alpha \in \mathcal{A} \cap \mathcal{A}(x)\}$$

For other uniqueness results for (1), see [9].

An analogous result for the generalized Zubov equation (5) is as follows.[6] We say that a real-valued function w defined on an open set containing the origin is **origin regular** if w is continuous at the origin and null at the origin.

Theorem 2 *Assume (A_3)-(A_5), f is g quasi-stable, and g is uniformly local Lipschitz. Then:*

(1) $V_{\max}$ is a robust Lyapunov function for f on $\mathcal{D}_o$ for which $\mathcal{D}_o = V_{\max}^{-1}([0,1))$.

(2) If w is an origin regular solution of (5) on $\mathcal{D}_o$, then $w \equiv V_{\max}$ on $\mathcal{D}_o$.

(3) If w is a bounded origin regular solution of (5) on $\mathbb{R}^N$, then $w \equiv V_{\max}$ on $\mathbb{R}^N$.

In particular,

(4) $V_{\max}$ is the unique origin regular bounded solution of (5) on $\mathbb{R}^N$; and

(5) $\mathcal{D}_o = w^{-1}([0,1))$ for any origin regular bounded solution w of (5) on $\mathbb{R}^N$.

Remark In Example 2.1, all the hypotheses of Theorem 2 hold except for the quasi-stability, and there are *infinitely many* bounded origin regular solutions of (5) on $\mathbb{R}^N$. Therefore, the quasi-stability hypothesis in Theorem 2 cannot be omitted. For uniqueness of solutions of (1) and (5) on general open sets $S \subseteq \mathbb{R}^N$, see [9]. By allowing degenerate g, we obtain Lyapunov functions $V_{\max}[g, g]$ which cannot be written as $V_{\max}[\tilde{g}, \tilde{g}]$ if $\tilde{g}$ is nondegenerate (cf. [9]). ∎

[6] By **uniform local Lipschitzness** of g, we will mean the requirement that for each $R > 0$, there is an $L_R > 0$ for which $|g(x, a) - g(y, a)| \leq L_R ||x - y||$ for all $x, y \in B_R$ and $a \in A$.

4 Regularity Results

Under the hypotheses of [3, 9], $V_{\max}$ is a continuous Lyapunov function for f on $\mathcal{D}_o$. In applications (e.g., characterizations of stability), one sometimes also wishes to construct locally or globally Lipschitz Lyapunov functions for f on $\mathcal{D}_o$. In [3, 5], conditions are imposed on f and g under which

$$v_\delta(x) := V_{\max}[\delta g, \delta g] \tag{10}$$

is globally Lipschitz on $\mathbb{R}^N$ for large enough constant $\delta > 0$. These conditions include the nondegeneracy of g. The following example from [9] shows that if all the hypotheses in [3, 5] hold *except* the nondegeneracy of g, then it could be that *none* of the Lyapunov functions (10) are globally Lipschitz. This example will show that by allowing degenerate g, we can obtain Lyapunov functions with special properties not encountered under the hypotheses of [3, 5] (cf. [9] for other special properties of Lyapunov functions for degenerate g and more regularity results for $V_{\max}$).

Example For each $k \in \mathbb{N}_o := \{0, 1, 2, \ldots\}$, set

$$t_k^- = 10^k - \frac{1}{10^{2k+1}} \quad \text{and} \quad t_k^+ = 10^k + \frac{1}{10^{2k+1}}.$$

Define I_Δ and $\Delta : I_\Delta \to \mathbb{R}$ by

$$I_\Delta = \bigcup_{k \in \mathbb{N}_o} \left[t_k^-, t_k^+\right],$$
$$\Delta(x) = \begin{cases} 10^{3k+1}\left(x - t_k^-\right), & t_k^- \le x \le 10^k, \, k \in \mathbb{N}_o \\ 10^{3k+1}\left(t_k^+ - x\right), & 10^k \le x \le t_k^+, \, k \in \mathbb{N}_o \end{cases}$$

Then the graph of Δ is a sequence of nonoverlapping triangles centered at the points 10^k for $k \in \mathbb{N}_o$ which become taller and thinner as $k \to +\infty$. In fact, while

$$\Delta(10^k) = 10^k \tag{11}$$

for all $k \in \mathbb{N}_o$, we have

$$\int_{I_\Delta} \Delta(x)\,\mathrm{d}x = \frac{1}{10}\sum_{k=0}^{\infty} 10^{-k} < \infty$$

Let $Q : \mathbb{R} \to \mathbb{R}$ be any continuous function that satisfies the following conditions:

(a) $Q(x) \ge 0$ for all $x \ge 0$, $Q(x) = \Delta(x)$ for all $x \in I_\Delta$, $Q(x) = (\frac{9}{10})^6 x^6$ for all $x \in [0, \frac{9}{20}]$

(b) $|Q(x)| \le 1$ for all $x \in \mathbb{R} \setminus I_\Delta$, and $Q(x) = 0$ iff $x \in \{0, \pm t_1^-, \pm t_1^+, \pm t_2^-, \pm t_2^+, \pm t_3^-, \pm t_3^+, \ldots\}$

(c) $Q\lceil(\mathbb{R} \setminus I_\Delta)$ is Lipschitz with Lipschitz constant $\mathcal{L} \le 1$, Q is odd, and $\int_{[0,\infty)} Q(x)\,\mathrm{d}x < \infty$

We leave the easy construction of Q to the reader. The functions

$$g(x) = -Q(x)f(x),$$
$$f(x) = -\begin{cases} \left(\frac{10}{9}\right)^6 x, & -\frac{9}{10} \le x \le \frac{9}{10} \\ \frac{1}{x^5}, & x \ge \frac{9}{10} \ \text{ or } \ x \le -\frac{9}{10} \end{cases}$$

are globally Lipschitz. Take f as the dynamics, with no effective controls, and g as the cost function. Then all the hypotheses of [3] needed for global Lipschitzness of $v_\delta(x)$ for large δ are satisfied, except that g is degenerate. Moreover, $V_{\max}[g, 0]$ is locally Lipschitz on $\mathcal{D}_o$. Indeed, let $\phi(t, x)$ denote the trajectory for f and the initial value x. For each fixed $x \ne 0$, it follows from the change of variables $u = \phi(t, x)$ that

$$V_{\max}[g, 0](x) = \textstyle\int_0^\infty g(\phi(t, x))\,\mathrm{d}t$$
$$= \int_0^\infty \frac{g(\phi(t,x))}{f(\phi(t,x))}\,\frac{\partial \phi}{\partial t}(t, x)\,\mathrm{d}t = \int_0^x Q(u)\,\mathrm{d}u.$$

Therefore, if $\delta > 0$ is given, then since

$$v_\delta(x) = 1 - e^{-\delta V_{\max}[g,0](x)}$$

for all x, conditions (11), (a), and (c) give

$$|Dv_\delta(10^k)| = \delta\Delta(10^k)\exp\left(-\delta \textstyle\int_0^{10^k} Q(s)\,\mathrm{d}s\right)$$
$$\to +\infty \quad \text{as} \quad k \to +\infty.$$

Therefore, while $V_{\max}[g, 0]$ is locally Lipschitz, there cannot exist $\delta > 0$ such that v_δ is globally Lipschitz. This shows that the nondegeneracy condition (7) cannot be omitted from the statement of the Lipschitzness results in [3]. For conditions guaranteeing Lipschitzness of $V_{\max}$, see [3, 9]. ∎

5 Discussion of Findings

The preceding examples motivate our study of the solutions of (1) and (5) under more general conditions on g, ℓ, and h. We will develop *uniqueness* theory for solutions of these equations which includes the PDE characterizations in [3, 5], and which also applies to cases which are not tractable by the known results. We also study the *regularity* of Zubov equation solutions. Our results have the following novel features.

1. Our results extend those of [7, 11] on uniqueness of solutions for the infinite horizon Hamilton-Jacobi-Bellman equation. The infinite horizon equation is the same as the exit time equation. Whereas [11] assumes that the undiscounted infinite horizon Lagrangian is nonnegative, our results apply for Lagrangians which could also take negative values. It is natural to consider optimization with Lagrangians that take both positive and negative values, to allow cost minimization in one part of the state space and

maximization elsewhere. Uniqueness results for undiscounted exit problem HJBE with negative Lagrangians were given in [7], which requires controllability to the so-called positivity set of the Lagrangian. This controllability condition is not needed in our results, nor do we need uniform positive lower bounds on the interest rates. Our uniqueness theorem for (1) does not put any growth or lower bound assumptions on solutions, nor does it require any controllability at the origin. On the other hand, the earlier uniqueness characterizations for HJBE (cf. [1, 7, 11]) prove uniqueness of solutions in classes of proper or bounded-from-below functions that satisfy an asymptotic condition at the boundary of the domain. Therefore, our results extend previous work by allowing more general comparison functions, including functions which take negative values, and which are neither bounded-from-above nor bounded-from-below.

2. Our uniqueness theory for (5) applies for general functions g, and gives stronger conclusions than the uniqueness theory of [2, 3, 5]. Clearly, a function w is a solution of (5) exactly when $-w$ is a solution of (1) with interest rate $h = g$ and Lagrangian $\ell = -g$. However, since the usual uniqueness results for (1) require nonnegative ℓ or strictly positive h (cf. [1]), these earlier results cannot in general be applied to (5) when g is nonnegative. Moreover, the known results on (5) (cf. [2, 3, 5]) require the nondegeneracy condition (7). By allowing degenerate costs g, we obtain solutions of (5) with special properties not found in the Zubov equation solutions of [3] (cf. [9]). The unique solutions of (5) are robust Lyapunov functions for f, and are the Kružkov transformations of zero discount maximal cost type robust Lyapunov functions V_L for f. The functions V_L are in turn unique solutions of

$$\inf_{a\in A}\{-f(x,a)\cdot Dv(x) - g(x,a)\} = 0 \qquad (12)$$

on $\mathcal{D}_o$. Our results generalize the PDE characterizations for (5) and (12) in [3] to cover more general dynamics f and costs g. On the other hand, the PDE characterizations for (5) and (12) in [3, 5] all follow from our results. Therefore, we obtain new classes of 'flat' maximal cost type robust Lyapunov functions $V_{\max}$, corresponding to degenerate g, which can be characterized as unique PDE solutions. This leads to new characterizations of $\mathcal{D}_o$ as sublevel sets of Lyapunov functions for degenerate instantaneous costs.

For precise statements of our findings, and for their detailed proofs, see [9].

References

[1] Bardi, M., and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, Boston, 1997.

[2] Camilli, F., L. Grüne, and F. Wirth, "A regularization of Zubov's equation for robust domains of attraction," in *Nonlinear Control in the Year 2000, Volume 1*, A. Isidori, F. Lamnabhi-Lagarrigue, W. Respondek, Eds., Springer, London, 2000, pp. 277-290.

[3] Camilli, F., L. Grüne, and F. Wirth, "A generalization of Zubov's method to perturbed systems," *SIAM J. Control Optim.*, **40**(2001), pp. 496-515.

[4] Falcone, M., and C. Makridakis, Eds., *Numerical methods for viscosity solutions and applications*, World Scientific, Singapore, 2001.

[5] Grüne, L., and F. Wirth, "Computing control Lyapunov functions via a Zubov type algorithm," in *Proceedings of the 39th IEEE Conference on Decision and Control (Sydney, Australia, 2000)*, pp. 2129-2134.

[6] Malisoff, M., "Viscosity solutions of the Bellman equation for exit time optimal control problems with non-Lipschitz dynamics," *ESAIM: Control, Optimisation and Calculus of Variations*, **6**(2001), pp. 415-441.

[7] Malisoff, M., "Viscosity solutions of the Bellman equation for exit time optimal control problems with vanishing Lagrangians," *SIAM J. Control Optim.*, **40**(2002), pp. 1358-1383.

[8] Malisoff, M., "Bounded-from-below solutions of the Hamilton-Jacobi equation for optimal control problems with exit times: vanishing Lagrangians, eikonal equations, and shape-from-shading," *NoDEA Nonlinear Differential Equations and Applications*, to appear.[7]

[9] Malisoff, M.,"Further results on Lyapunov functions and domains of attraction for perturbed asymptotically stable systems," *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, to appear.[7]

[10] Malisoff, M., "Viscosity solutions of the Bellman equation for perturbed optimal control problems with exit times," in *Proc. 41st IEEE Conference on Decision and Control (Las Vegas, NV, December 2002)*, to appear.[7]

[11] Malisoff, M., and H. J. Sussmann, "Further results on the Bellman equation for optimal control problems with exit times and nonnegative Lagrangians: The case of Fuller's Problem," in *Proc. 39th IEEE Conference on Decision and Control (Sydney, Australia, December 2000)*, pp. 2308-2310.

[12] Sontag, E.D., "Comments on integral variants of ISS," *Systems and Control Letters*, **34**(1998), pp. 93-100.

[7]Preprint at `www.math.lsu.edu/~malisoff/research.html`