Robustness of Nonlinear Systems with Respect to Delay and Sampling of the Controls

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Given a constant $\nu > 0$, we want to compute upper bounds on $\delta + \tau$ such that the input delayed sampled system defined by

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Assumption 1: There are a C^1 positive definite radially unbounded function V and a continuous positive definite function W such that $\dot{V}(t,x) \leq -W(x)$ along all trajectories of (CL), u_s is C^1, and f and g are locally Lipschitz in x and piecewise continuous in t.

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Assumption 2: There are constants $c_i > 0$ such that

$$\left|\frac{\partial u_s}{\partial x}(t,x)g(t,x)\right|^2 \le c_1 \ , \ \left|\frac{\partial V}{\partial x}(t,x)g(t,x)\right|^2 \le c_2 W(x) \ , \qquad (1)$$

$$|\dot{u}_s(t,x)|^2 \le c_3 W(x) \ , \ \text{and} \qquad (2)$$

$$\left|\frac{\partial V}{\partial x}(t,x)g(t,x)u_s(t,x)\right| \le c_4[V(t,x)+1] \qquad (3)$$

hold for all $t \ge 0$ and $x \in \mathbb{R}^n$.

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Here $\dot{u}_s(t,x) = \frac{\partial u_s}{\partial t}(t,x) + \frac{\partial u_s}{\partial x}(t,x)\big(f(t,x) + g(t,x)u_s(t,x)\big)$.

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Theorem : Let Assumptions 1-2 hold. If δ and τ_* are any two positive constants such that

$$\delta + \tau_* \leq \frac{1}{\sqrt{4c_1 + 8c_2c_3}} \tag{4}$$

and if $\tau \in (0, \tau_*]$, then the system

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u_s\big(t_i - \tau, x(t_i - \tau)\big), \quad t_i \leq t < t_{i+1}$$

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Remarks : Covers dynamics that are not necessarily globally Lipschitz or locally exponentially stable. The proof is based on a functional of a new type.

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Choose $U(t, x_t) = V(t, x(t)) + \Gamma(t, x_t)$, where V is the Lyapunov function for the undelayed nonsampled system as before,

$$\Gamma(t, x_t) = \frac{1}{4c_3(\delta+\tau_*)} \int_{t-\delta-\tau_*}^{t} \left[\int_{\ell}^{t} |\psi(m, x_m)|^2 \, \mathrm{d}m\right] \mathrm{d}\ell,$$

$$\psi(t, x_t) = \frac{\partial u_s}{\partial t}(t, x(t)) + \frac{\partial u_s}{\partial x}(t, x(t))\dot{x}(t), \text{ and}$$

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for $t \in [t_i, t_{i+1})$ and all i as before. Then

$$\dot{U} \le -\frac{1}{4} W(x(t)) - \frac{1}{8c_3(\delta+\tau_*)} \int_{t-\delta-\tau_*}^{t} |\psi(m, x_m)|^2 \,\mathrm{d}m$$

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along (SD). We can find C^1 $\mathcal{K}_\infty$ functions κ and γ such that $U_\kappa = \kappa(U)$ satisfies $\dot{U}_\kappa \leq -\gamma(U_\kappa)$ along all trajectories of (SD).

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Proposition : If a system $\dot{x} = f(x) + u$ with f bounded is rendered GAS on $\mathbb{R}^n$ by a bounded feedback $u_s(x)$, then for each Lyapunov function $V(t, x)$ of the closed loop system, the requirements from [MML08] on the delayed system

$$\dot{x}(t) = f(x(t)) + u_s(x(t - \tau)) \tag{5}$$

fail to hold.

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$u_s(t, x) = -x_1 - x_2$, and $V(t, x) = 0.1x_1^{10} + 0.5x_1^2 + 2x_2^2$.

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$$\delta + \tau_* < \frac{1}{2\zeta} \min\left\{ \frac{1}{\sqrt{3}\mho}, \frac{1}{53(\mho + 1)} \right\} \tag{10}$$

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We hope to extend our work to time varying or state dependent delays, and systems that are not control affine.