Asymptotic Stabilization for Feedforward Systems with Delayed Feedbacks

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$$Y'(t) = \mathcal{G}(t, Y_t, \delta(t)), \quad Y(t) \in \mathcal{Y}, \qquad (2)$$

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Fridman, Jankovic, Karafyllis, Krstic, Lin, Teel, ...

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Find γ_i's by building certain LKFs for $Y'(t) = \mathcal{G}(t, Y_t, 0)$.

Linear Feedforward Systems

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Consider the set of all systems having the feedforward form

$$\left\{ \begin{array}{rcl} \dot{x} & = & h_1(z) + h_2(z)v(t-\tau) \\ \dot{z} & = & f(z) + g(z)v(t-\tau) \, . \end{array} \right. \tag{3}$$

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The state space is $\mathbb{R}^n \times \mathbb{R}^p$. Linearizing (3) around period τ reference trajectories produces a system of the form

$$\begin{cases} \dot{x}(t) & = & C(t)z(t) + D(t)u(t-\tau) \\ \dot{z}(t) & = & A(t)z(t) + B(t)u(t-\tau)\,, \end{cases} \tag{4}$$

where A, B, C, and D are C^1 matrix valued functions of period τ.

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We focus on (4), and cases where uncertainties δ are added to u.

Our Two Assumptions

Assumption 1. *The system*

$$\dot{\theta}(t) = A(t)\theta(t) \tag{5}$$

is UGAS. The matrices A, B, C, and D are C^1 and have period τ.

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$$\left\{ \begin{array}{rcl} \frac{\partial \psi_a}{\partial t}(t,m) & = & -\psi_a(t,m)A(t) \\ \psi_a(m,m) & = & \mathrm{I} \end{array} \right. \qquad (6)$$

for all $t \in \mathbb{R}$ and $m \in \mathbb{R}$.

Our Two Assumptions

Lemma

Let Assumption 1 hold. Then the function $\mathrm{I} - \psi_a(\ell, \ell - \tau)$ *is invertible for all* $\ell \in \mathbb{R}$*. Also, the function* $q : \mathbb{R} \to \mathbb{R}^{n\times p}$ *defined by*

$$q(t) = -\int_{t-\tau}^{t} C(\ell)[\mathrm{I} - \psi_a(\ell, \ell - \tau)]^{-1}\psi_a(t, \ell)\mathrm{d}\ell \qquad (7)$$

has period τ*, and* $\dot{q}(t) + q(t)A(t) + C(t) = 0$ *for all* $t \in \mathbb{R}$*.* □

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Assumption 2. *There exists a constant* $c > 0$ *such that the matrix* $R(t) = q(t)B(t) + D(t)$ *satisfies*

$$\int_{t-\tau}^{t} R(m)R(m)^{\top}\mathrm{d}m \;\geq\; c\mathrm{I} \qquad (8)$$

for all $t \in \mathbb{R}$*. (That means I is the* $n \times n$ *identity matrix.)*

Main Result

Our coordinate change $\xi(t) = x(t) + q(t)z(t)$ gives

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & R(t)u(t-\tau) \\ \dot{z}(t) & = & A(t)z(t) + B(t)u(t-\tau) \end{array} \right. \quad (9)$$

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Theorem

Let Assumptions 1 and 2 hold. Then for all constants $\tau > 0$ and $\epsilon \in (0, \frac{1}{1+4\tau||R||^2})$, the controller

$$u(t-\tau) = -\epsilon \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \quad (10)$$

renders (9) UGAS.

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$$\begin{cases} \dot{\xi}(t) &= R(t)\big[u(t-\tau)+\delta(t)\big] \\ \dot{z}(t) &= A(t)z(t)+B(t)\big[u(t-\tau)+\delta(t)\big]\ . \end{cases} \tag{11}$$

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$$\overline{\delta} = \frac{c}{9k||R||(1+2\overline{u})^{1/2}}, \quad \text{where} \quad k = \frac{4\sqrt{2}}{3\epsilon}\left(\tau + \frac{1}{2c}||R||^6\tau^4\epsilon^2\right)$$
$$\text{and} \quad \overline{u} = \max\left\{\frac{1}{2} + \frac{\epsilon||R||^2\tau}{4\sqrt{2}}, \frac{\epsilon||R||^2\tau}{4\sqrt{2}}\left(1+2\epsilon||R||^2\tau\right)\right\} . \tag{12}$$

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Theorem

Under the preceding assumptions, (11) in closed loop with

$$u(t-\tau) = -\epsilon\frac{R(t-\tau)^\top\xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \tag{13}$$

is ISS with respect to the set of all disturbances δ *bounded by* $\overline{\delta}$.

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where we omit the altitude subdynamics $\ddot{h} = -\alpha_h\dot{h} + \alpha_h(h^c - h)$.

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Key Model : Underactuated kino-dynamic representation that is justifiable for high-level formation flight control.

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See e.g. 2004 IEEE-TCST paper by Ren and Beard.

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We are given a C^1 reference trajectory $(x_r, y_r, \theta_r, v_r) : \mathbb{R} \to \mathbb{R}^4$, so there is a reference input $(\theta_{cr}, v_{cr}) : \mathbb{R} \to \mathbb{R}^2$ such that

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Assumption 3 : The functions $\cos(\theta_r(t))$ and $\sin(\theta_r(t))$ have period τ, there exists a constant $t_c \in [0, \tau]$ such that $\dot{\theta}_r(t_c) \neq 0$, and v_r is bounded.

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Tracking Error : $(\bar{x}, \bar{y}, \bar{\theta}, \bar{v}) = (x - x_r, y - y_r, \theta - \theta_r, v - v_r)$.

Application to UAV Dynamics

After a preliminary change of feedbacks, the tracking dynamics are

$$\begin{cases} \dot{\bar{x}} & = & \cos(\theta_r(t))\overline{v} \\ & & +[\overline{v}+v_r(t)][\cos(\overline{\theta}+\theta_r(t))-\cos(\theta_r(t))] \\ \dot{\bar{y}} & = & \sin(\theta_r(t))\overline{v} \\ & & +[\overline{v}+v_r(t)][\sin(\overline{\theta}+\theta_r(t))-\sin(\theta_r(t))] \\ \dot{\bar{v}} & = & -\alpha_v\overline{v}+u(t-\tau) \\ \dot{\bar{\theta}} & = & -\alpha_\theta\overline{\theta}\ . \end{cases} \tag{16}$$

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We apply our theory to the $(\bar{x}, \bar{y}, \bar{v})$ dynamics obtained by setting $\bar{\theta} = 0$, and then we reincorporate the $\bar{\theta}$ dynamics to get θ_c and v_c.

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$$\begin{array}{l} \theta_c(t-\tau) = \theta_{cr}(t) \text{ and} \\ v_c(t-\tau) = v_{cr}(t) - \frac{\epsilon}{\alpha_v} \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} . \end{array} \tag{17}$$

Applications and Conclusions

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- Input delays naturally occur in many engineering applications and preclude the use of standard control designs.

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- It would be interesting to extend the analysis to

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Nonlinear analogs involving PDEs would also be interesting.

What is a Lyapunov-Krasovskii Functional (LKF)?

Definition: We call $V^\sharp$ an ISS-LKF for $Y'(t) = \mathcal{G}(t, Y_t, \delta(t))$ provided there exist functions $\gamma_i \in \mathcal{K}_\infty$ such that:

1. $\gamma_1(|\phi(0)|) \leq V^\sharp(t,\phi) \leq \gamma_2(|\phi|_{[-\tau,0]})$ for all $(t,\phi) \in [0,+\infty) \times \mathcal{C}([-\tau,0],\mathbb{R}^n)$ and
2. $\frac{d}{dt}\left[V^\sharp(t, Y_t)\right] \leq -\gamma_3(V^\sharp(t, Y_t)) + \gamma_4(|\delta(t)|)$ along all trajectories of the system

Example: The function $V(Y) = \frac{1}{2}|Y|^2$ is an ISS-LKF for $Y'(t) = -Y(t) + \frac{1}{4}Y(t) + \delta(t)$ for any $\mathcal{D}$. Fix $\tau > 0$.

$$V^\sharp(Y_t) = V(Y(t)) + \frac{1}{4}\int_{t-\tau}^{t} |Y(\ell)|^2 \mathrm{d}\ell + \frac{1}{8\tau}\int_{t-\tau}^{t}\left[\int_s^t |Y(r)|^2 \mathrm{d}r\right]\mathrm{d}s$$

is an ISS-LKF for $Y'(t) = -Y(t) + \frac{1}{4}Y(t-\tau) + \delta(t)$.

Main Result

Our coordinate change $\xi(t) = x(t) + q(t)z(t)$ gave

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & R(t)u(t-\tau) \\ \dot{z}(t) & = & A(t)z(t) + B(t)u(t-\tau) \end{array} \right. \tag{19}$$

where $R(t) = q(t)B(t) + D(t)$ and q is from the lemma.

Theorem

Let Assumptions 1 and 2 hold. Then for all constants $\tau > 0$ and $\epsilon \in (0, \frac{1}{1+4\tau||R||^2})$, the controller

$$u(t-\tau) = -\epsilon \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \tag{20}$$

renders (19) UGAS.

Proof of Theorem

Show that the closed loop system (19) admits the LKF

$$V^\sharp(t,\xi_t,z(t)) \ = \ z^\top(t)P(t)z(t) + 21\beta_1 W_3(t,\xi_t), \ \ \text{where}$$

$$W_3(t,\xi_t) = W_2(t,\xi_t) + k\left[(1+2U(\xi_t))^{3/2} - 1\right],$$

$$W_2(t,\xi_t) = W_1(t,\xi_t) + \beta_0 \int_{t-\tau}^{t} \left|\frac{R(m)^\top \xi(m)}{\sqrt{1+|\xi(m)|^2}}\right|^2 \mathrm{d}m,$$

$$W_1(t,\xi_t) = \xi(t)^\top \left[\int_{t-\tau}^{t}\int_m^t R(\ell)R(\ell)^\top \mathrm{d}\ell \mathrm{d}m\right]\xi(t),$$

$$U(\xi_t) = \tfrac{1}{2}|\xi|^2 + \tfrac{1}{4\tau}\int_{t-2\tau}^{t}\int_m^t \frac{\epsilon|R(\ell)^\top\xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}}\,\mathrm{d}\ell\mathrm{d}m,$$

$$\beta_0 = \tfrac{1}{2c}||R||^6\tau^4\epsilon^2, \ \ k = \tfrac{4\sqrt{2}}{3\epsilon}(\tau+\beta_0),$$

$$\beta_1 = \max\{v_1, v_2\}, \ \ v_1 = \tfrac{2}{c}[4||P||^2\,||B||^2\,||R||^2 + 1],$$

$$\text{and} \ \ v_2 = \tfrac{16\sqrt{2}\tau}{3\epsilon k}(1 + 8\tau||P||^2\,||B||^2\,||R||^4).$$